Base change, transitivity and Künneth formulas for the Quillen decomposition of Hochschild homology

by

Christian Kassel and Arne B. Sletsjøe

Let A be any commutative algebra over a commutative ring k and let M be any symmetric A-bimodule. In [Q], §8, Quillen proved that the Hochschild groups

$$H_\star(A,M) = Tor_\star^{A\otimes\ A}(M,A)$$

have a natural decomposition, called the Quillen decomposition,

$$H_n(A,M) \cong \underset{p+q=n}{\oplus} D_q^{(p)}(A/k,M)$$

under the hypothesis that A is flat over k, containing the field $\mathbf{Q}$ of rational numbers. The right-hand side is defined in terms of exterior powers of the cotangent complex of A over k For $p=1$, the groups $D_\star^{(1)}(A/k,M)$ are isomorphic to the André-Quillen homology groups $D_\star(A/k,M)$.
The purpose of this note is to prove base change, transitivity and Künneth formulas for all $D_\star^{(p)}(A/k,M)$ - and hence for Hochschild homology in characteristic zero - extending analogous formulas established by André [A] and Quillen [Q] for $D_\star(A/k,M)$.
Lately M. Ronco [R] proved that the Quillen decomposition coincides with a decomposition introduced by combinatorial methods on the level of Hochschild standard complex by Gerstenhaber-Schack [GS]. The latter decomposition coincides with another one due to Feigin-Tsygan [FT] and Burghelea-Vigué [BV][V]. In the notation of [L], M. Ronco‘s result can be written as follows (for all p and n)

$$D_{n-p}^{(p)}(A/k,M) \simeq H_n^{(p)}(A,M)$$

We assume all rings to be commutative with unit.

1. Definition of $D_\star^{(p)}(A/k,M)$

For any map of rings $u : k \to A$ and any nonnegative integer p, we define the simplicial A-module

$$\mathbf{L}_{A/k}^p = \Omega_{P/k}^p \otimes_P A$$

where P is a simplicial cofibrant k-algebra resolution of A in the sense of [Q]. By [Q], the simplicial A-module $\mathbf{L}^p_{A/k}$ is independent, up to homotopy equivalence, of the choice of P. In Quillen's notation

$$\mathbf{L}^p_{A/k} = \Lambda^p_A \mathbf{L}^1_{A/k}$$

where $\mathbf{L}^1_{A/k}$ is the cotangent complex. Thus we define

$$D^{(p)}_\star(A/k, M) = H_\star(\mathbf{L}^p_{A/k} \otimes_A M) \quad \text{and} \quad D^\star_{(p)}(A/k, M) = H^\star(Hom_A(\mathbf{L}^p_{A/k}, M))$$

for any A-module M.

REMARK **(1.1)**.

a) If $p = 0$, then $\mathbf{L}^p_{A/k} \simeq A$ and

$$D^{(0)}_n(A/k, M) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

b) If $p = 1$, $D^{(1)}_\star(A/k, M) = D_\star(A/k, M)$ where the right-hand side was defined by André [A] and Quillen [Q]. These groups coincide with the Harrison groups [H] in characteristic zero.

We derive now some properties of the group $D^{(p)}_\star(A/k, M)$ which are immediate consequences of Quillen's formalism.

LEMMA **(1.2)**. *$\mathbf{L}^p_{A/k}$ is a free simplicial A-module.*

Proof. This follows from the fact that if P is free over k, say $P = S_k(V)$, then

$$\Omega^p_{P/k} \otimes_P A \simeq (\Lambda_k(V) \otimes_k P) \otimes_P A \simeq \Lambda_k(V) \otimes_k A$$

•

COROLLARY **(1.3)**. *For any exact sequence of A-modules*

$$0 \to M' \to M \to M'' \to 0$$

there are long exact sequences

$$\ldots \to D^{(p)}_n(A/k, M') \to D^{(p)}_n(A/k, M) \to D^{(p)}_n(A/k, M'') \to D^{(p)}_{n-1}(A/k, M') \to \ldots$$

and

$$\ldots \to D^n_{(p)}(A/k, M') \to D^n_{(p)}(A/k, M) \to D^n_{(p)}(A/k, M'') \to D^{n+1}_{(p)}(A/k, M') \to \ldots$$

•

The module $\mathbf{L}^p_{A/k}$ has the following vanishing property.

PROPOSITION **(1.4)**. *If A is a free k-algebra, then $\mathbf{L}^p_{A/k}$ has the homotopy type of $\Omega^p_{A/k}$. Consequently, for any A-module M*

$$D_n^{(p)}(A/k, M) = D^n_{(p)}(A/k, M) = 0 \quad \textit{if } n \geq 1$$

and

$$D_0^{(p)}(A/k, M) = \Omega^p_{A/k} \otimes_A M \quad \textit{and} \quad D^0_{(p)}(A/k, M) = Hom_A(\Omega^p_{A/k}, M)$$

Proof. Take $P = A$.

•

2. Base change and Künneth formulas

The following result states how $\mathbf{L}^p$ behaves under tensor products.

THEOREM **(2.1)**. *If A and B are k-algebras such that $Tor_q^k(A, B) = 0$ for $q > 0$, then we have the following isomorphisms*

a) Base change

$$\mathbf{L}^p_{A\otimes\ B/B} \simeq A \otimes_k \mathbf{L}^p_{B/k}$$

b) Künneth-type formula

$$\mathbf{L}^p_{A\otimes\ B/k} \simeq \bigoplus_{q+r=p} (\mathbf{L}^q_{A/k} \otimes_k \mathbf{L}^r_{B/k})$$

Proof. Under the hypothesis of the theorem, if P (resp. Q) is a cofibrant k-resolution of A (resp. of B), then $A \otimes_k Q$ (resp. $P \otimes_k Q$) is a cofibrant resolution of $A \otimes_k B$ over B (resp. over k). Now

$$\begin{aligned}\Omega^p_{A\otimes\ Q/k} \otimes_{A\otimes\ Q} (A \otimes_k B) &\simeq (A \otimes_k \Omega^p_{Q/k}) \otimes_{A\otimes\ Q} (A \otimes_k B) \\ &\simeq A \otimes_k (\Omega^p_{Q/k} \otimes_Q B)\end{aligned}$$

For the Künneth formula, we have

$$\begin{aligned}\Omega^p_{P\otimes\ Q/k} \otimes_{P\otimes\ Q} (A \otimes_k B) &= \bigoplus_{q+r=p} ((\Omega^q_{P/k} \otimes_k \Omega^r_{Q/k}) \otimes_{P\otimes\ Q} (A \otimes_k B) \\ &\simeq \bigoplus_{q+r=p} ((\Omega^q_{P/k} \otimes_P A) \otimes_k (\Omega^r_{Q/k} \otimes_Q B))\end{aligned}$$

•

COROLLARY **(2.2)**. *Under the same hypothesis as Theorem 2.1, and for any $A \otimes_k B$-module M, we have the following isomorphisms of graded modules*

$$D_\star^{(p)}(A \otimes_k B/B, M) \simeq D_\star^{(p)}(B/k, M)$$

and

$$D_\star^{(p)}(A \otimes_k B/k, M) \simeq \bigoplus_{q+r=p} D_\star^{(q)}(A/k, M) \otimes_k D_\star^{(r)}(B/k, M)$$

•

In characteristic zero the corresponding isomorphism for $HH_\star^{(p)}(A\otimes_k B)$ and for the cyclic groups $HC_\star^{(p)}(A\otimes_k B)$ are also proved in [K].

3. Transitivity

Suppose we have maps $k\xrightarrow{u}A\xrightarrow{v}B$ of commutative rings. We start by defining a filtration of $\Omega^p_{B/k}$. Let $F^i_A = F^i_A(\Omega^p_{B/k})$ be the sub-A-module of $\Omega^p_{B/k}$ generated by $b_0db_1\dots db_p$ where at least i elements among $b_1,\dots,b_p$ lie in A. We have the following sequence of inclusions of A-modules,

$$\Omega^p_{B/k} = F^0_A \supset F^1_A \supset \dots \supset F^p_A = \Omega^p_{A/k}\otimes_k B$$

LEMMA **(3.1)**. *If B is A-free and A is k-free, then the map*

$$\psi_i : \Omega^i_{A/k}\otimes_A \Omega^{p-i}_{B/A} \longrightarrow F^i_A/F^{i+1}_A$$

given by

$$\psi(a_0\, da_1\dots da_i\otimes b_0\, db_{i+1}\dots db_p) = a_0b_0\, da_1\dots da_i\cdot db_{i+1}\dots db_p$$

is an isomorphism.

Proof. First check that ψ_i is well-defined without any hypothesis on A and B. If $A = S_k(V)$ and $B = S_A(A\otimes W) = S_k(V)\otimes S_k(W) = S_k(V\oplus W)$ one computes easily both source and target of ψ_i.

•

THEOREM **(3.2)**. *Let $k\xrightarrow{u}A\xrightarrow{v}B$ be maps of commutative rings and let M be a B-module. Then there is a spectral sequence (E^r,d^r) converging to $D^{(p)}_\bullet(B/k,M)$. The k-modules $E^1_{i,j}$ have the following properties:*

a) $E^1_{i,j}=0$ for $i>0$ or $i<-p$.

b) $E^1_{0,j} = D^{(p)}_j(B/A,M)$ and $E^1_{-p,j} = D^{(p)}_{j-p}(A/k,M)$.

c) Fix any p. For every i there is a first quadrant spectral sequence $({}^{(i)}E^r,d^r)$ converging to $E^1_{-i,i+\star}$ such that

$${}^{(i)}E^2_{k,\ell} = D^{(i)}_k(A/k, D^{(p-i)}_\ell(B/A,M))$$

REMARK **(3.3)**.

a) The edge homomorphisms

$$D^{(p)}_j(B/k,M)\longrightarrow E^1_{0,j} = D^{(p)}_j(B/A,M)$$

and

$$E^1_{-p,p+j} = D^{(p)}_j(A/k,M)\longrightarrow D^{(p)}_j(B/k,M)$$

are the natural homomorphisms. For $p = 1$, the first spectral sequence reduces to two columns, so that one recovers the well-known long exact sequence

$$\ldots \to D_j(A/k, M) \to D_j(B/k, M) \to D_j(B/A, M) \to D_{j-1}(A/k, M) \to \ldots$$

b) Applying Theorem 3.2 to the map of rings $k \to A \to A \otimes_k B$, one sees that the spectral sequences degenerate and one recovers the Künneth formula of Corollary 2.2.

Proof of Theorem 3.2. Let P be a simplicial cofibrant k-resolution of A. Consider the composite map $P \to A \to B$ and choose a simplicial cofibrant P-resolution Q of B. Let us consider the following commutative diagram

$$\begin{array}{ccccc} & & & & Q \\ & P & & A \otimes_P Q & \\ k & & A & & B \end{array}$$

Then it follows from [Q] that $A \otimes_P Q$ is a simplicial cofibrant A-resolution of B.
We apply the construction of Lemma 3.1 to the map of rings $k \to P \to Q$. Then we get a filtration of $\Omega^p_{Q/k} \otimes_Q M$ such that the associated graded is $\Omega^i_{P/k} \otimes_P \Omega^{p-i}_{Q/P} \otimes_Q M$. This yields the first spectral sequence with

$$E^1_{i,j} = H_{i+j}(\Omega^i_{P/k} \otimes_P (\Omega^{p-i}_{Q/P} \otimes_Q M))$$

converging to $H_{i+j}(\Omega^p_{Q/k} \otimes_Q M)$ which is $D^{(p)}_{i+j}(B/k, M)$ because Q is also a simplicial cofibrant k-resolution of B.
To compute the homology of $\Omega^i_{P/k} \otimes_P \Omega^{p-i}_{Q/P} \otimes_Q M$ we use the fact that it has a double simplicial structure. Therefore it gives rise to a spectral sequence with E^2-term of the form

$$\begin{aligned} {}^{(i)}E^2_{k,\ell} &= H_k(\Omega_{P/k} \otimes_P H_\ell(\Omega^{p-i}_{Q/P} \otimes_Q M)) \\ &= D^{(i)}_k(A/k, H_\ell(\Omega^{p-i}_{Q/P} \otimes_Q M)) \end{aligned}$$

Now we use the base change formula of Theorem 2.1 to get the following isomorphism of P-modules

$$\begin{aligned} D^{(p-i)}_\ell(B/A, M) &= H_\ell(\Omega^{(p-i)}_{A\otimes\ Q/A} \otimes_{A\otimes\ Q} M) \\ &= H_\ell(\Omega^{(p-i)}_{Q/P} \otimes_Q M) \end{aligned}$$

•

4. Applications

The following is an extension of Quillen's Theorem 5.4 [Q].

PROPOSITION **(4.1)**. *Assume that $k \supset \mathbf{Q}$ and $\Omega^1_{A/k}$ is A-flat.*

i) If Spec $A \to$ Spec k is étale, then $\mathbf{L}^p_{A/k}$ is acyclic for $p \geq 1$.

ii) If Spec $A \to$ Spec k is smooth, then $\mathbf{L}^p_{A/k} \simeq \Omega^p_{A/k}$.

Proof. i) Let P be a simplicial cofibrant k-resolution of A. By [Q], if A is étale over k, then $\Omega^1_{P/k} \otimes_P A = \mathbf{L}^1_{A/k}$ is acyclic. Hence

$$\mathbf{L}^p_{A/k} = \Lambda^p_A \mathbf{L}^1_{A/k}$$

which is a direct summand (in characteristic zero) of $(\mathbf{L}^1_{A/k})^{\otimes p}$ is acyclic.
ii) We have the following isomorphisms

$$\mathbf{L}^p_{A/k} = \Lambda^p_P \Omega^1_{P/k} \otimes_P A \simeq \Lambda^p_A \Omega^1_{A/k} \otimes_A A \simeq \Omega^p_{A/k}$$

in the derived category of A-modules.

•

COROLLARY **(4.2)**. *Under the hypothesis of Proposition 4.1 and if A is smooth over k, then for all p*

$$D^{(p)}_n(A/k, M) = \begin{cases} \Omega^p_{A/k} \otimes_A M & \textit{if } n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

SPECIAL CASES **(4.3)**.

Let $k \to A \to B$ be maps of rings such that $k \supset \mathbf{Q}$ and let M be a B-module.

a) If A is smooth over k, then by Theorem 3.2 and Corollary 4.2 the spectral sequence converging to $D^{(p)}_\star(B/k, M)$ has E^1-term given by

$$E^1_{-i,i+j} = \Omega^i_{A/k} \otimes_A D^{(p-i)}_j(B/A, M)$$

b) If A/k is étale, we get: $D^{(p)}_\star(B/k, M) = D^{(p)}_\star(B/A, M)$ from Theorem 3.2 and Prop. 4.1.i. The resulting isomorphism for Hochschild homology

$$H_\star(B/k, M) \simeq H_\star(B/A; M)$$

was proved by Gerstenhaber-Schack [GES].

c) If B is smooth over A, then the E^1-terms are given by

$$E^1_{-i,i+j} = D^{(i)}_j(A/k, \Omega^{(p-i)}_{B/A} \otimes_B M)$$

If moreover B is étale over A, then $\Omega^p_{B/A} = 0$ for $p > 0$. From Theorem 3.2 we get the following isomorphism:

$$D^{(p)}_\star(B/k, M) \simeq D^{(p)}_\star(A/k, M)$$

If the B-module M is extended from A, i.e. is of the form $B \otimes_A N$ where N is an A-module, then we have the following étale descent isomorphism

$$D^{(p)}_\star(B/k, M) \simeq D^{(p)}_\star(A/k, N) \otimes_A B$$

When $N = A$, we thus recover Theorem 0.1 of [WG] stating that

$$H_\star(B, B) \simeq H_\star(A, A) \otimes_A B$$

REFERENCES

[A] M. André, Homologie des algèbres commutatives
Springer Gru. 206, Berlin-Heidelberg-New York (1974).

[BV] D.Burghelea, M.Vigué-Poirrier, Cyclic homology of commutative algebras I,
Lect. Notes in Math. 1318 (1988) pp.51-72.

[FT] B.L.Feigin, B.L.Tsygan, Additive K-theory and crystalline cohomology,
Funct. Anal. Appl. 19 (1985) pp.124-132.

[GES] M.Gerstenhaber, S.D.Schack, Relative Hochschild cohomology, rigid algebras and the Bockstein,
J. Pure Appl. Alg. 43 (1986) pp.53-74.

[GS] M.Gerstenhaber, S.D.Schack, A Hodge-type decomposition for commutative algebra cohomology,
J. Pure Appl. Alg. 48 (1987) pp.229-247.

[H] D.K.Harrison, Commutative algebras and cohomology,
Trans. Amer. Math. Soc. 104 (1962) pp.191-204.

[K] C.Kassel, Une formule de Künneth pour la décomposition de l'homologie cyclique des algèbres commutatives,
Preprint, Strasbourg (1989).

[L] J.-L.Loday, Opérations sur l'homologie cyclique des algèbres commutatives,
Invent. Math. 96 (1989) pp.205-230.

[Q] D.Quillen, On the (co)homology of commutative rings,
Proc. Symp. Pure Math. 17 (1970) pp.65-87.

[R] M.Ronco, Sur l'homologie d'André-Quillen,
Preprint IRMA, Strasbourg (1990).

[V] M.Vigué-Poirrier, Décomposition de l'homologie cyclique des algèbres différentielles graduées commutatives,
To appear in K-Theory.

[WG] C.A.Weibel, S.C.Geller, Étale descent for Hochschild and cyclic homology,
Preprint (1990).

CHRISTIAN KASSEL
UNIVERSITÉ LOUIS PASTEUR
DÉPT. DE MATHÉMATIQUE
7, RUE RENÉ DESCARTES
F-67084 STRASBOURG CEDEX
FRANCE

ARNE B. SLETSJØE
MATEMATISK INSTITUTT
UNIVERSITETET I OSLO
PB. 1053 BLINDERN
N-0316 OSLO 3
NORGE