FINITE DIMENSIONAL REPRESENTATIONS OF THE PROJECTIVE MODULAR GROUP

ARNE B. SLETSJØE

Abstract. We study finite dimensional representations of the projective modular group. Various explicit dimension formulas are given.

1. Introduction

1.1. Finite-dimensional modules. Let $k$ be an algebraically closed field of characteristic zero. An $n$-dimensional left module of a $k$-algebra $A$ is a vector space $M$ of dimension $n$ over $k$, equipped with a structure map $\rho \in \text{Hom}_{k\text{-alg}}(A, \text{End}_k(M))$. The (left) module $M$ is simple if it has no non-trivial submodules, or equivalently, if the structure map $\rho$ is surjective. A module which does not split into a direct sum of submodules is called indecomposable. To each indecomposable module $M$ we associate a unique, up to permutations of the components, semi-simple module $\overline{M}$, given as the direct sum of the composition factors of $M$.

Two structure maps $\rho, \rho'$ define equivalent modules if they are conjugates, i.e. there exists an invertible $n \times n$-matrix $P$ such that for every $a \in A$, $\rho'(a) = P^{-1}\rho(a)P$. The set of conjugation classes of modules has in general no structure of an algebraic variety, but restricting to the semi-simple modules of dimension $n$ there exist an algebraic quotient, denoted by $\mathcal{M}_n$. The variety $\mathcal{M}_n$ can also be considered as a quotient of the variety of all $n$-dimensional $A$-modules under the weaker equivalence relation, where two modules are equivalent if they have conjugate composition factors. The Zariski open subset of $\mathcal{M}_n$ of simple modules is denoted $\text{Simp}_n(A)$. This subvariety is independent of the choice of equivalence relation.

1.2. The modular group. The modular group $SL(2, \mathbb{Z})$ is generated by the two matrices

\[ u = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

where $u^3 = v^2 = I$ in the projectivised group $PSL(2, \mathbb{Z}) \simeq SL(2, \mathbb{Z})/\{\pm I\}$. There are no relations between the two generators and $PSL(2, \mathbb{Z})$ is clearly isomorphic to the free product $\mathbb{Z}_3 \ast \mathbb{Z}_2$.

Finite-dimensional representations of a group $G$ correspond bijectively to finite-dimensional modules of the group algebra $k[G]$. Thus the representation theory of $PSL(2, \mathbb{Z})$ is equivalent to the representation theory of the group algebra

\[ A = k[\mathbb{Z}_3 \ast \mathbb{Z}_2] \simeq k[\mathbb{Z}_3] \ast k[\mathbb{Z}_2] \simeq k[x, y]/(x^3 - 1, y^2 - 1) \]

The algebra $k[\mathbb{Z}_p]$ is hereditary, and free products of hereditary algebras are hereditary. Thus the algebra $A$ is hereditary.

Mathematics Subject Classification (2000): 11F06, 14A22, 14H50, 14R, 16D60, 20H05
Keywords: Modular group, representations, deformation theory.
Let $M$ be some finite dimensional representation of the non-commutative model $B = k[x, y]/(x^3 - y^2)$ of the ordinary cusp. The inclusion of the center $Z(B) \cong k[x^3 = y^2] \hookrightarrow B$ induces a fibration

$$\text{Simp}_n(B) \to \mathbf{A}^1$$

for which the fibres outside the origin are constant, isomorphic to $\text{Simp}_n(A)$.

1.3. **Low-dimensional modules.** We shall give an explicit description of the low-dimensional modules of $A = k[x, y]/(x^3 - y^2 - 1)$. For further details we refer to [12].

The variety $\text{Simp}_1(A)$ consists of 6 points, denoted $k(\omega, \alpha)$, and defined by $x \mapsto \omega, \omega^3 = 1$ and $y = \alpha = \pm 1$. We have $\text{Ext}^1_k(k(\omega, \alpha), k(\omega', \alpha')) \cong k$ if $\omega \neq \omega'$ and $\alpha \neq \alpha'$, and zero elsewhere.

The variety of two-dimensional simple modules $\text{Simp}_2(A)$ has three disjoint components, and each component is an affine line with two distinct closed points removed. We observe that there exist non-conjugate indecomposable modules $E_1$ and $E_2$, satisfying $\text{Ext}^1_k(E_1, E_2) \neq 0$. Thus it is not possible to complete $\text{Simp}_2(A)$ to an ordinary commutative scheme, under the strong conjugation relation. To illustrate, for $s \in k$ we have a 2-dimensional $A$-module

$$M_s : x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & \omega \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ s & -1 \end{pmatrix}$$

where $\omega^3 = 1, \omega \neq 1$. For $s \neq 0, 2(\omega - 1)$ the modules are simple, while $M_0$ and $M_2(\omega - 1)$ are indecomposable, non-simple, corresponding to extensions of $k(\omega, -1)$ by $k(1, 1)$, respectively $k(\omega, 1)$ by $k(1, -1)$. The trivial extensions are the respective associated semi-simple modules. In fact all simple modules have a representation of this form.

There is yet another family, given by

$$N_t : x \mapsto \begin{pmatrix} 1 & 0 \\ 1 & \omega \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix}$$

For $t \neq 0, 2(\omega - 1)$ the modules are simple and $N_t \cong M_t$. As above the two exceptional modules $N_0$ and $N_2(\omega - 1)$ are indecomposable and non-simple, but not isomorphic to any $M_t$. The module $N_0$ corresponds to an extension of $k(1, -1)$ by $k(\omega, 1)$, the opposite direction of $M_0$, and $\text{Ext}^1_k(M_0, N_0) \neq 0$.

For $n = 3$ the situation is as follows. Two matrices $P$ and $Q$ are given by

$$P = \begin{pmatrix} \lambda_1 & \frac{\lambda_1 \lambda_2}{\lambda_3} + \lambda_2 & \lambda_2 \\ 0 & \lambda_2 & \lambda_2 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad Q = \begin{pmatrix} \lambda_3 & 0 & 0 \\ -\lambda_2 & \lambda_2 & 0 \\ \lambda_2 & -\frac{\lambda_1 \lambda_2}{\lambda_3} - \lambda_2 \lambda_3 & \lambda_1 \lambda_3 \end{pmatrix}$$

where $(\lambda_1 \lambda_2 \lambda_3)^2 = 1$. The variety $\text{Simp}_3(A)$ has two components, given by $\lambda_1 \lambda_2 \lambda_3 = \pm 1$. For the component $\lambda_1 \lambda_2 \lambda_3 = 1$ we put

$$x \mapsto PQ = \begin{pmatrix} 0 & 0 & \lambda_1 \lambda_2 \\ 0 & -\lambda_1 \lambda_3 & \lambda_1 \lambda_2 \\ \lambda_2 \lambda_3 & -\frac{\lambda_1 \lambda_2}{\lambda_3} - \lambda_2 \lambda_3 & \lambda_1 \lambda_3 \end{pmatrix} \quad y \mapsto PQP = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

defining a 3-dimensional module of $A$. A consequence of the Cayley-Hamilton Theorem is that since $x^3 = y^2 = 1$, the images of the 9 monomials

$$1, x, y, x^2, xy, x^2y, xyx, x^2yx$$

form a linear basis for $M_3(k)$ if and only if the module is simple. A computation shows that this set of matrices is linearly independent if and only if

$$(\lambda_1^2 + 1)(\lambda_2^2 + 1)(\lambda_3^2 + 1) \neq 0$$
One can show that in this case the trace ring ([3]) is generated by
\[ t_{xy} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \]
\[ t_{xyz} = \lambda_1 + \lambda_2 + \lambda_3 \]
For any choice of cube root \( \lambda \) of -1 we get the relation
\[ t_{xy} - \lambda t_{xyz} + 2\lambda^2 = 0 \]
corresponding to a straight line in the \((t_{xy}, t_{xyz})\)-plane. Thus the non-simple locus
is given by the three lines
\[ t_{xy} - \lambda t_{xyz} + 2\lambda^2 = 0 \]
for \( \lambda^3 = -1 \). The three intersections of the three lines correspond to semi-simple
modules which are sums of three one-dimensional simple modules.

Similar analysis for \( Simp_n(A) \) for \( n = 4, 5 \), can be found in [12].

1.4. Main results. In this paper we investigate the finite dimensional modules of
the projective modular group \( \text{PSL}(2, \mathbb{Z}) \). Unfortunately the litterature is some-
what limited on this subject. We have mentioned the paper of Tuba and Wenzl
([12]). Other important papers are due to Westbury ([13]) and Le Bruyn and Adria-
enssen ([9]). These papers give nice descriptors of the finite-dimensional modules,
especially in low dimensions, i.e. up to dimension 5. Le Bruyn and Adriaenssen
also give modules of dimension \( 6n \). Our aim in this paper is to give more explicit
results on the varieties of modules of arbitrary dimensions.

The paper is organised as follows. In section 2 we recall a result of Wesbury [13],
proving a dimension formulae for the variety of simple \( n \)-dimensional \( A \)-modules
of given dimension vector \( \alpha = (\alpha_1, \alpha_2, \alpha_3; \alpha'_1, \alpha'_2) \), denoted \( Simp_\alpha(A) \). Suppose
\( \alpha \) is a Schur root, i.e. there exist a module \( M \) of dimension vector \( \alpha \) such that
\( \text{End}_A(M) \cong k \). Westburys result tells us that in that case
\[ \text{dim Simp}_\alpha(A) = n^2 - \sum_{i=1}^{3} \alpha_i^2 - \sum_{j=1}^{2} (\alpha'_i)^2 + 1 \]
In section 3 we look at the locus \( D_\alpha \) of weak equivalence classes of non-simple
modules, represented by the associated semi-simple module. Using deformation
theory we prove that \( E \in D_\alpha \), corresponding to an extension of two non-isomorphic
simple modules of dimension vector \( \beta \) and \( \gamma \), is a smooth point of a component of
\( D_\alpha \) of codimension \( -\langle \beta, \gamma \rangle \).

In section 4 we show that there exist \( \frac{1}{2}(n+d+1) \) distinct points of \( D_\alpha \) corresponding
to the maximally iterated extensions in the given component. Each of these
points in the weak equivalence relation corresponds to a set of strong equivalence
classes. In section 5 we study this collection more carefully. The weak equivalence
classes split up in several equivalence classes under the strong relation. The number
of strong classes depends not only on the dimension vector \( \alpha \) and the composition
factors, but also on the order in which the iterated extensions are composed, de-
scribed by the representation graph \( \Gamma \). We show that the non-commutative scheme
\( \text{Ind}_\Gamma(A) \) can be described as a quotient of a certain vector space by a group of
conjugations.

In section 6 we consider the component of \( M_n \) of highest dimension, and inside
this the subvariety of non-simple modules. It turns out that the minimal codimen-
sion of this non-simple locus is the same as the dimension of the space of modular
forms of weight \( 2n \).
2. The Quiver $K(3, 2)$

2.1. Quiver representation. Let $K(3, 2)$ be the quiver with vertices $v_1, v_2, v_3$ and $w_1, w_2$ and with an arrow from $v_i$ to $w_j$ for $i = 1, 2, 3$ and $j = 1, 2$. The variety of equivalence classes of (semi-)simple representations of $Z_3 \rtimes Z_2$ corresponds to the variety of equivalence classes of $\theta$-(semi)stable representations of $K(3, 2)$ ([13]), where $\theta$ is the character assigning to the dimension vector $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha'_1, \alpha'_2)$ the dimension difference

$$\theta(\alpha) = (\alpha'_1 + \alpha'_2) - (\alpha_1 + \alpha_2 + \alpha_3)$$

The correspondence goes via the category $C(Z_3 \rtimes Z_2)$, whose objects are triples $(M, N, \phi)$, where $M$ is a left $k[Z_3]$-module, $N$ is a left $k[Z_2]$-module and $\phi : N \rightarrow M$ is a $k$-linear map. To a semi-simple representation $M$ of $Z_3 \rtimes Z_2$ we associate the triple $(M, M, id) \in C(Z_3 \rtimes Z_2)$, where the $Z_3$- and $Z_2$-module structures of $M$ are induced by the two inclusions

$$Z_3 \hookrightarrow Z_3 \rtimes Z_2 \hookrightarrow Z_2$$

The triple $(M, M, id)$ is mapped to the $K(3, 2)$-module $M \oplus M$ where $v_i(m, m') = (e_i m, 0)$ for an idempotent $e_i = \frac{1}{3}(1 + \omega^i x + \omega^{2i} x^2) \in k[Z_3] \simeq k[x]/(x^3 - 1)$, $w_j(m, m') = (0, f_j m')$, with $f_j = \frac{1}{2}(1 + (-1)^j y) \in k[Z_2] \simeq k[y]/(y^2 - 1)$ and $\phi_{ij}(m, m') = (0, f_j e_i m)$. Here $i = 1, 2, 3$, $j = 1, 2$ and $\omega$ is a primitive cube root of unity. Thus the eigenspaces of the generators $x$ and $y$ correspond to the components of the module $M \oplus M$ over the quiver.

Let $M$ be an indecomposable $A$-module with dimension vector denoted by $\alpha = (\alpha_1, \alpha_2, \alpha_3; \alpha'_1, \alpha'_2)$ such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = \alpha'_1 + \alpha'_2 = n$. The dimension of $Ext^1_A(M, M)$ is given by the formula ([11])

$$\dim Ext^1_A(M, M) = \dim Hom_A(M, M) - \langle \alpha, \alpha \rangle$$

where

$$\langle \alpha, \beta \rangle = \sum_{i=1}^{3} \alpha_i \beta_i + \sum_{j=1}^{2} \alpha'_j \beta'_j - \sum_{i=1}^{3} \alpha_i \sum_{i=1}^{3} \beta_i$$

Thus if $End_A(M) \simeq k$, we get

$$d = \dim Ext^1_A(M, M) = 1 + n^2 - \sum_{i=1}^{3} \alpha_i^2 - \sum_{j=1}^{2} (\alpha'_j)^2$$

2.2. Westburys theorem. The following theorem is due to Westbury ([13]).

**Theorem 2.1.** Let $\alpha = (\alpha_1, \alpha_2, \alpha_3; \alpha'_1, \alpha'_2)$ be a dimension vector of non-negative integers, satisfying

i) $\alpha_1 + \alpha_2 + \alpha_3 = \alpha'_1 + \alpha'_2 = n$

ii) $\alpha_i + \alpha'_j \leq n$ for all $i = 1, 2, 3$ and $j = 1, 2$

Then the component $Simp_\alpha(A)$ of $Simp_n(A)$ corresponding to the dimension vector $\alpha$ is a non-empty smooth affine variety of dimension

$$d = \dim Simp_\alpha(A) = 1 + n^2 - \sum_{i=1}^{3} \alpha_i^2 - \sum_{j=1}^{2} (\alpha'_j)^2$$

**Corollary 2.2.** The dimension of $Simp_n(A)$, i.e. the largest dimension of its components is given by

$$\dim Simp_n(A) = \begin{cases} 
6m^2 + 2sm + s - 1 & n = 6m + s, s = 1, \ldots, 5, m \geq 0 \\
6m^2 + 1 & n = 6m, m \geq 1 
\end{cases}$$
Proof. It is easy to see that maximal dimension is obtained when the partition of \( n \) is most evenly distributed. Using this general principle the computation of the dimensions is straightforward. \( \square \)

**Remark 2.3.** The generating function of the sequence of maximal dimensions is given by
\[
\sum_{n=1}^{\infty} \dim (\text{Simp}_{\alpha}(A))t^n = \frac{t^2 + 2t^6 - 2t^7 + t^8}{(1 - t)^2(1 - t^9)}
\]

3. **Deformation theory**

3.1. **Extensions for the modular group.** The variety \( M_\alpha \) of finite dimensional modules of \( A \) of given dimension vector \( \alpha \), up to weak equivalence, where two modules are equivalent if they have conjugate composition factors, is a coarse moduli space. The \( \alpha \)-component of the variety of simple modules \( \text{Simp}_{\alpha}(A) \subset M_\alpha \) is a Zariski open subset of \( M_\alpha \). The complement in \( M_\alpha \) of weak equivalence classes of non-simple modules is denoted \( D_\alpha \). The dimension of the tangent space of \( M_\alpha \) at an indecomposable module \( M \) is given in section 2 by the formula for the dimension of \( \text{Ext}_A^1(M, M) \). The ring \( A \) is hereditary so there are no obstructions and the coarse moduli is a smooth space of the given dimension.

Let \( E \) be an indecomposable, non-simple representation and suppose
\[
0 \rightarrow W \rightarrow E \rightarrow V \rightarrow 0
\]
is a non-split exact sequence, presenting the indecomposable module \( E \) as an extension of simple \( A \)-modules \( V \) and \( W \). At the point \( E \in D_\alpha \) the tangent space of \( M_\alpha \) contains directions corresponding to simple deformations as well as directions corresponding to non-simple modules. To study the non-simple deformation we consider deformations of \( E \) under the extension constrain, i.e. liftings of \( E \) that preserves the exact sequence presentation. This is the topic of the next subsections.

3.2. **Deformations of presheaves.** Let \( a \) be the category of commutative pointed artinian \( k \)-algebras, i.e. objects are commutative artinian \( k \)-algebras \( R \) with a fixed quotient \( \pi : R \rightarrow k \) and morphisms preserving the point.

Let \( R \in \text{obj}(a) \) and let \( F : \Lambda \rightarrow \text{A-mod} \) be a presheaf of left \( A \)-modules on a poset \( \Lambda \), i.e. \( F \) is a contravariant functor on \( \Lambda \) with values in the category of left \( A \)-modules. A lifting of the presheaf \( F \) to \( R \) is a presheaf \( F_R : \Lambda \rightarrow A \otimes_k R^{\text{op}} \text{-mod} \) together with an isomorphism of presheaves
\[
\eta : F_R \otimes_R k \xrightarrow{\sim} F
\]
and such that \( F_R \simeq F \otimes_k R \) as presheaves of right \( R \)-modules.

Two liftings \( F_R' \) and \( F_R'' \) are said to be equivalent if there exists an isomorphism
\[
\phi : F_R' \rightarrow F_R''
\]
of presheaves such that \( \eta'' \circ (\phi \otimes \text{id}_k) = \phi \circ \eta' \). Denote by
\[
\text{Def}_F(R) = \{ \text{liftings of } F \text{ to } R \} / \sim
\]
the set of equivalence classes of liftings of \( F \) to \( R \). We shall refer to such equivalence classes as deformations of \( F \) to \( R \). This construction is functorial, defining a covariant functor
\[
\text{Def}_F : a \rightarrow \text{sets}
\]
One can show that there exist an element \( \psi = (\psi^2, \psi^3, \psi^0) \) representing a class of a certain cohomology group \( \text{Ext}_A^2(\mathcal{F}, F \otimes \ker \pi) \) such that \([\psi] = 0\) in \( \text{Ext}_A^3(\mathcal{F}, F \otimes \ker \pi) \).
$\ker \pi$) is equivalent to the existence of a lifting of $F$ to $R$. If $[\psi] = 0$, then the set of deformations is given as a principal homogenous space over $\text{Ext}_A^1(F, F \otimes \ker \pi)$.

Applying the deformation functor to the $k$-algebra $R = k[x]/(x^2)$, we have $\ker \pi \simeq k$, and the tangent space of the deformation functor $\text{Def} F$ is given by $\text{Ext}_A^1(F, F)$. In the next section we shall study $\text{Ext}_A^i(F, G)$ for $i \geq 0$ in more details.

3.3. A double complex. Let $F, G : \Lambda \to A$-mod be presheaves of left $A$-modules on a poset $\Lambda$. Define a covariant functor

$$\text{Hom}_k(F, G) : \text{Mor}(\Lambda) \to A-bimod$$

by

$$\text{Hom}_k(F, G)(\lambda' \leq \lambda) = \text{Hom}_k(F(\lambda), G(\lambda')).$$

There is a double complex

$$K^{\bullet \bullet} = D^\bullet(\Lambda, C^\bullet)$$

given by

$$K^{p,q} = \prod_{\Lambda_0 \leq \cdots \leq \lambda_p} C^q(A, \text{Hom}_k(F(\lambda_p), G(\lambda_0)))$$

where $C^q(A, \mathcal{H})$ is the Hochschild cochain complex, with Hochschild differential

$$d : K^{p,q} \to K^{p,q+1}$$

and where

$$\delta : K^{p,q} \to K^{p+1,q}$$

is the differential of the $D^\bullet$-complex given by

$$\delta \psi(\lambda_0 \leq \cdots \leq \lambda_{p+1}) = G(\lambda_0 \leq \lambda_1) \psi(\lambda_1 \leq \cdots \leq \lambda_{p+1})$$

$$+ \sum_{i=1}^p (-1)^i \psi(\lambda_0 \leq \cdots \leq \lambda_i \leq \cdots \leq \lambda_{p+1})$$

$$+ (-1)^{p+1} \psi(\lambda_0 \leq \cdots \leq \lambda_p) F(\lambda_p \leq \lambda_{p+1}).$$

Let $\text{Tot}(K^{\bullet \bullet})$ be the total complex with total differential $\partial = d + (-1)^p \delta$, satisfying $\partial^2 = 0$. Define

$$\text{Ext}_A^n(F, G) = H^n(\text{Tot}(K^{\bullet \bullet})) \quad n \geq 0$$

As indicated in the last section we are interested in $\text{Ext}_A^n(F, F)$ for $n = 0, 1, 2$. Our main tool will be the spectral sequence of the next proposition. Denote by $s, t$ the source and target functors on $\text{Mor}(\Lambda)$.

**Proposition 3.1.** Let $A$ be a $k$-algebra, $\Lambda$ a poset and $F, G$ two presheaves of left $A$-modules on $\Lambda$. Then there exists a first quadrant spectral sequence

$$E_2^{p,q} = \lim_{\text{Mor}(\Lambda)}^p \text{Ext}_A^q(F(t), G(s))$$

converging to the total cohomology $\text{Ext}_A^{p+q}(F, G)$.

**Proof.** The spectral sequence is obtained by computing cohomology of the double complex $K^{\bullet \bullet}$ with respect to the Hochschild differential and then with respect to the $D^\bullet$-differential. □
3.4. Application to the poset $\Lambda_2 = \{0 \leq 1 \leq 2\}$. A short exact sequence

$$\mathcal{F}: \quad 0 \to W \xrightarrow{\epsilon} E \xrightarrow{\pi} V \to 0$$

of left $A$-modules can be considered as a presheaf on $\Lambda_2$ with certain obvious properties. We put $\mathcal{F}(2) = W$, $\mathcal{F}(1) = E$, $\mathcal{F}(0) = V$, $\mathcal{F}(0 \leq 1) = \pi$ and $\mathcal{F}(1 \leq 2) = \epsilon$.

The smooth moduli of deformations of the $A$-module $E$ has a tangent space $T_{M,E}$ given by $\text{Ext}_A^1(E,E)$ and the subspace $T^F_{M,E} \subset T_{M,E}$ corresponding to directions in $T_{M,E}$ subject to the extension constrain, is given by the image of the projection map

$$\text{Ext}_A^1(\mathcal{F},\mathcal{F}) \to \text{Ext}_A^1(E,E)$$

Denote this image by $\text{Ext}_A^1(E,E)_0$. Observe that the longest sequence of non-trivial elements of $\text{Mor}(\Lambda_2)$ has length 2, inducing vanishing of $\text{Ext}_A^{p,q}$ for $p \geq 3$. On the other extreme we have $\text{Ext}_A^{p,q} = 0$ for $q \geq 2$ since $A$ is hereditary.

Furthermore, the higher derivatives of the invers limit functor can be computed as cohomology of the complex

$$\text{End}(\mathcal{F}(0)) \oplus \text{End}(\mathcal{F}(1)) \oplus \text{End}(\mathcal{F}(2))$$

$$\xrightarrow{\begin{pmatrix} \pi^* & -\pi_* & 0 \\ (\pi \circ \epsilon)^* & 0 & (\pi \circ \epsilon)_* \\ 0 & \epsilon^* & -\epsilon_* \end{pmatrix}}$$

$$\text{Hom}(\mathcal{F}(1),\mathcal{F}(0)) \oplus \text{Hom}(\mathcal{F}(2),\mathcal{F}(0)) \oplus \text{Hom}(\mathcal{F}(2),\mathcal{F}(1))$$

$$\xrightarrow{\begin{pmatrix} \epsilon^* & -1 & \pi_* \end{pmatrix}}$$

$$\text{Hom}(\mathcal{F}(2),\mathcal{F}(0))$$

We have $\pi \circ \epsilon = 0$ and in fact $\ker \pi = \text{im} \epsilon$. It is obvious that the second cohomology group vanishes,

$$\lim_{\text{Mor}(\Lambda_2)} \text{(2)} \text{Hom}_A(\mathcal{F}(t),\mathcal{F}(s)) = 0$$

Similar argument proves that

$$\lim_{\text{Mor}(\Lambda_2)} \text{(2)} \text{Ext}_A^1(\mathcal{F}(t),\mathcal{F}(s)) = 0$$

Thus the spectral sequence reduces to the exact sequence

$$0 \to E_2^{1,0} \to \text{Ext}_A^1(\mathcal{F},\mathcal{F}) \to E_2^{0,1} \to 0$$

and the isomorphism

$$0 \to E_2^{1,1} \to \text{Ext}_A^2(\mathcal{F},\mathcal{F}) \to 0$$

Proposition 3.2. Let $A$ be as above and let

$$\mathcal{F}: \quad 0 \to W \xrightarrow{\epsilon} E \xrightarrow{\pi} V \to 0$$

be a non-split short exact sequence of $A$-modules. Suppose $\text{Hom}_A(V,W) = \text{Hom}_A(W,V) = 0$ and that $\text{End}_A(V) \simeq \text{End}_A(W) \simeq k$ Then

$$\text{Ext}_A^1(E,E)_0 \simeq \ker \{ \epsilon^* \pi_* : \text{Ext}_A^1(E,E) \to \text{Ext}_A^1(W,V) \}$$

Proof. The map $\text{Ext}_A^1(E,E) \to \text{Ext}_A^1(E,E)_0$ factors through the surjective map $\text{Ext}_A^1(\mathcal{F},\mathcal{F}) \to E_2^{0,1}$ and it is enough to consider the image of the map $E_2^{0,1} \to \text{Ext}_A^1(E,E)$. Pick $(a,b,c) \in \text{Ext}_A^1(W,W) \oplus \text{Ext}_A^1(E,E) \oplus \text{Ext}_A^1(W,V)$ such that $\epsilon_*(a) = \epsilon^*(b) \in \text{Ext}_A^1(W,E)$ and $\pi_*(b) = \pi^*(c) \in \text{Ext}_A^1(E,V)$. Thus $\pi_*\epsilon^*(b) = \epsilon^*\pi_*(b) = 0 \in \text{Ext}_A^1(W,V)$ and the image of $E_2^{0,1}$ lies in the given kernel.
On the other hand, for \( b \in \ker \{ \text{Ext}^1_A(E, E) \to \text{Ext}^1_A(W, V) \} \), we have \( \epsilon^*(b) \in \ker \pi_* \) and \( \pi_*(b) \in \ker \epsilon^* \). By exactness there exists \( a \in \text{Ext}^1_A(W, W) \) such that \( \epsilon_*(a) = \epsilon^*(b) \) and \( c \in \text{Ext}^1_A(V, V) \) such that \( \pi^*(c) = \pi_*(b) \). This proves that \( E_{2,1}^0 \) maps surjectively onto the kernel.

Similarly the obstruction space for deformations under the extension constrain is given by the image of the map

\[
\text{Ext}^2_A(\mathcal{F}, \mathcal{F}) \to \text{Ext}^2_A(E, E)
\]

which is trivial since \( A \) is hereditary.

Notice that if \( \text{Ext}^1_A(W, V) = 0 \) and \( E \) is a non-trivial extension of \( V \) by \( W \), then by the completion theorem [3] there are no simple deformations of \( E \), all deformations of \( E \) preserve the extension structure. This fits with the fact that \( \text{Ext}^1_A(E, E)_0 \simeq \text{Ext}^1_A(E, E) \) and there are no obstructions. The map \( \epsilon^* \pi_* \) of Proposition 3.2 is a composition of two surjective maps, and therefore itself surjective. It follows that the moduli space of extensions is smooth of dimension

\[
\dim \text{Ext}^1_A(E, E) - \dim \text{Ext}^1_A(W, V)
\]

**Proposition 3.3.** Let \( \mathcal{F} \) be an non-split exact sequence as above, with dimension vector \( \beta \) and \( \gamma \) for \( W \) and \( V \) respectively. Then

\[
\dim \text{Ext}^3_A(\mathcal{F}, \mathcal{F}) = 1 - \langle \beta + \gamma, \beta + \gamma \rangle + \langle \beta, \gamma \rangle
\]

**Proof.** Follows immediately from the dimension formula and

\[
\dim \text{Ext}^3_A(\mathcal{F}, \mathcal{F}) = \dim \text{Ext}^1_A(E, E)_0 = \dim \text{Ext}^1_A(E, E) - \dim \text{Ext}^1_A(W, V)
\]

Notice that the codimension of the non-simple locus at \( \mathcal{F} \) is given by \(-\langle \beta, \gamma \rangle\) where \( \beta + \gamma = \alpha \).

### 3.5. An example.
Consider the dimension vector \( \alpha = (2, 2, 2, 3, 3), n = 6 \). Then \( \dim \text{Simp}_n(A) = 1 + 6^2 - (2^2 + 2^2 + 2^2) - (3^2 + 3^2) = 7 \). The minimum value of the codimension \(-\langle \beta, \gamma \rangle\) is obtained for \( \beta = (2, 2, 1, 3, 2), \gamma = (0, 0, 1; 0, 1) \) which gives \(-\langle \beta, \gamma \rangle = 2 \). This shows that \( \mathcal{M}_\alpha \) in this case is a 7-dimensional smooth variety with a 5-dimensional closed subvariety of non-simple modules.

**Remark 3.4.** Notice that the non-simple locus does not have to be a smooth subvariety, even if the obstruction space of deformations under the extension constrain is trivial. A non-simple module can be written as an extension in many different ways.

### 4. Maximally iterated extensions

**4.1. Maximally iterated extensions.** Let \( M \) be a left \( A \)-module. The structure map \( \rho : A \to \text{End}_A(M) \) of the module \( M \) is determined by two \( n \times n \)-matrices \( X = \rho(x) \) and \( Y = \rho(y) \), up to simultaneous conjugation, satisfying \( X^3 = Y^2 = I \). The eigenspaces of \( X \) and \( Y \) corresponding to the three cube roots and the two square roots of unity, denoted \( X_i, i = 1, 2, 3 \), resp. \( Y_j, j = 1, 2 \), have dimensions given by the dimension vector \( \alpha = (\alpha_1, \alpha_2, \alpha_3; \alpha'_1, \alpha'_2) \).

Let

\[
M = M_r \supset M_{r-1} \supset \cdots \supset M_1 \supset M_0 = 0
\]

be any composition series of \( M \). We say that \( M \) is a maximally iterated extension if all factors \( \overline{M}_i = M_i/M_{i-1} \) have dimension 1.
4.2. Counting Maximally iterated extensions. There exists a set of distinct points in $\mathcal{M}_0$, corresponding to the weak classes of maximally iterated extensions of $A$-modules.

**Proposition 4.1.** Let $\alpha$ be a Schur root. The set of maximally iterated extensions of dimension vector $\alpha$ is finite of cardinality $N = \frac{1}{2}(d + n + 1)$.

**Proof.** The associated semi-simple module $\overline{M} = \bigoplus_{i=1}^{n} M_i$ of a maximally iterated extension $M$ can be written as a sum

$$\overline{M} = k(1,1)^{a_1} \oplus k(\omega,1)^{a_2} \oplus k(\omega^2,1)^{a_3} \oplus k(1,-1)^{a_1-a_3} \oplus k(\omega,-1)^{a_2-a_3} \oplus k(\omega^2,-1)^{a_3-a_3}$$

for non-negative integers $a_1, a_2, a_3$, satisfying $a_i \leq \alpha_i$, $i = 1, 2, 3$. The generating function for the problem of finding all integers fulfilling this condition is given by

$$g(x) = (1 + x + x^2 + \ldots + x^{\alpha_1})(1 + x + x^2 + \ldots + x^{\alpha_2})(1 + x + x^2 + \ldots + x^{\alpha_3})$$

$$= \frac{1 - x^{\alpha_1+1}}{1 - x} \cdot \frac{1 - x^{\alpha_2+1}}{1 - x} \cdot \frac{1 - x^{\alpha_3+1}}{1 - x}$$

$$= (1 - x^{\alpha_1+1})(1 - x^{\alpha_2+1})(1 - x^{\alpha_3+1}) \frac{1}{(1 - x)^3}$$

$$= (1 - x^{\alpha_1+1})(1 - x^{\alpha_2+1})(1 - x^{\alpha_3+1}) \sum_{i=0}^{\infty} \left(\frac{i + 2}{2}\right) x^i$$

Let $m = a_1 + a_2 + a_3$. The coefficient of the $x^m$-term is given by

$$N = \left(\frac{m + 2}{2}\right) - \left(\frac{m - a_1 + 1}{2}\right) - \left(\frac{m - a_2 + 1}{2}\right) - \left(\frac{m - a_3 + 1}{2}\right)$$

$$+ \left(\frac{m - a_1 - n_2}{2}\right) + \left(\frac{m - a_1 - a_3}{2}\right) + \left(\frac{m - a_2 - a_3}{2}\right)$$

$$- \left(\frac{m - a_1 - a_2 - a_3}{2}\right)$$

By definition $m = \alpha'_i$ and $n - m = \alpha'_2$. The conditions for existence of a Schur root give

$$\alpha_i + m = \alpha_i + \alpha'_i \leq \alpha_i + \alpha_j + \alpha_k \implies m \leq \alpha_j + \alpha_k$$

and

$$\alpha_i + \alpha_1 + \alpha_2 + \alpha_3 - m = \alpha_i + \alpha'_2 \leq \alpha_i + \alpha_j + \alpha_k \implies m \geq \alpha_i$$

for $\{i, j, k\} = \{1, 2, 3\}$.

Thus $m - \alpha_i - \alpha_j \leq 0$ and $m - \alpha_i + 1 \geq 1$ and we get

$$N = \left(\frac{m + 2}{2}\right) - \left(\frac{m - a_1 + 1}{2}\right) - \left(\frac{m - a_2 + 1}{2}\right) - \left(\frac{m - a_3 + 1}{2}\right)$$

$$= \frac{1}{2} \left(\frac{m^2 + 3m + 2}{2} - \frac{m^2 - 2ma_1 + m + \alpha_1^2 - a_1}{2}\right)$$

$$- \frac{1}{2} \left(\frac{m^2 - 2ma_2 + m + \alpha_2^2 - a_2}{2}\right) - \frac{1}{2} \left(\frac{m^2 - 2ma_3 + m + \alpha_3^2 - a_3}{2}\right)$$

$$= \frac{1}{2} \left(\frac{m^2 + 4m + 2 + 2mn - \sum_i \alpha_i^2 + n}{2}\right)$$

$$= m(n - m) + \frac{1}{2}(n - \sum_i \alpha_i^2) + 1$$
Changing back to \( m = \alpha_1', n - m = \alpha_2' \), this takes the form
\[
N = \alpha_1' \alpha_2' + \frac{1}{2} (n - \sum \alpha_i^2) + 1
\]
But \( n = \alpha_1' + \alpha_2' \), so \( \alpha_1' \alpha_2' = \frac{1}{2} (n^2 - \sum (\alpha_i')^2) \) and we get
\[
N = \frac{1}{2} (n + n^2) - \frac{1}{2} \sum \alpha_i^2 - \frac{1}{2} \sum (\alpha_j')^2 + 1
\]
From Theorem 2.1 we know that
\[
d = \dim \text{Ext}_A^1(E, E) = n^2 - \sum \alpha_i^2 - \sum (\alpha_j')^2 + 1
\]
showing that
\[
N = \frac{1}{2} (n + n^2) - \frac{1}{2} n^2 - \frac{1}{2} + \frac{d}{2} + 1 = \frac{1}{2} (n + d + 1)
\]
□

Notice that a consequence of this formula is that the number \( n + d \) must be odd for any dimension vector \( \alpha \).

5. Parametrisation of Maximally iterated extensions

5.1. Strong vs. weak equivalence relation. A point in the subvariety \( D_\alpha \) corresponds to an equivalence class of indecomposable representations, all with conjugate composition factors. Under the stronger conjugation equivalence relation this class splits up in several conjugacy classes. Since indecomposable modules deform into simples, but not vice versa, the most extreme modules are the maximally iterated extensions, where all composition factors are of dimension 1. Let \( E \in D_\alpha \) be the weak equivalence class of a maximally iterated extension \( E \). The problem is to understand how this equivalence class splits into different classes under the stronger conjugation equivalence relation.

More precisely, given a dimension vector \( \alpha \) we choose one of the \( \frac{1}{2} (n + d + 1) \) points of \( M_\alpha \) of the last section, corresponding to a maximally iterated extension \( E \). The corresponding semi-simple module \( E \) defines a set of 1-dimensional simple representations, possible of multiplicity greater than 1. Let \( \Gamma \) be a representation graph for this set, i.e. an ordering of the given simple factors. Following Laudal ([8]) we define \( \text{Ind}_\Gamma(A) \) as the non-commutative scheme of indecomposable \( \Gamma \)-representations. Our task is to describe this non-commutative scheme for the modular group ring.

5.2. Some useful lemmas. For an upper triangular \( n \times n \)-matrix \( Z = (z_{ij}) \) we introduce the notation
\[
i \sim j
\]
\( Z \) to indicate that the diagonal elements \( z_{ii} = z_{jj} \). This is obviously an equivalence relation on the set \( \{1, \ldots, n\} \). The subscript \( Z \), referring to the matrix, will be omitted if the reference to the matrix is unambiguous.

**Lemma 5.1.** Let \( X \) be an upper triangular \( n \times n \)-matrix on Jordan block form, satisfying the equation \( X^3 = I \). Then \( X \) is diagonal.

**Proof.** Suppose \( X = (x_{ij}) \) is on Jordan form, i.e. the three eigenvalues on the diagonal is sequentially ordered and all off-diagonal entries are zero, except possible for \( x_{i,i+1} \) if \( x_{ii} = x_{i+1,i+1} \). But \((X^3)_{i,i+1} = 3(x_{ii})^2 x_{i,i+1} = 0\), showing that \( X \) is diagonal. □
Lemma 5.2. Let \( Z \) be a diagonalisable upper triangular \( n \times n \)-matrix. Then there exists an upper triangular invertible matrix \( U \) such that \( U^{-1}ZU \) is diagonal.

Proof. Let \( P \) be an invertible \( n \times n \)-matrix. By the LU-decomposition Theorem \( P = \Gamma L \) where \( \Gamma \) is a permutation matrix, \( L \) is lower triangular and \( U \) is upper triangular. Suppose \( P^{-1} \) diagonalize \( Z \), i.e. \( PZP^{-1} = D, D \) is diagonal. Then

\[
UZU^{-1} = L^{-1}\Gamma^{-1}DL^{-1}
\]

The left-hand side of this equation is an upper triangular matrix, while the right-hand side is lower triangular since conjugation of a diagonal matrix by a permutation matrix again is diagonal. Thus the matrix \( UZU^{-1} \) is diagonal. \( \square \)

The outcome of these two lemmas is that given a pair of upper triangular matrices \((X, Y)\), where \( X \) satisfies the equation \( X^2 = I \), we can always diagonalise \( X \), without disturbing the upper triangular form of \( Y \).

The next lemma gives an explicit form for the upper triangular matrices \( Y \), satisfying \( Y^2 = I \).

Let \( \mathcal{P}(i, k) \), \( i < k \) be given by
\[
\{ i = i_0 < i_1 < \cdots < i_m < i_{m+1} = k \} \in \mathcal{P}(i, k)
\]
if \( m \) is odd and \( y_{i,i+r} = (-1)^ry_{i,i} \) for \( r = 0, 1, 2, \ldots, m + 1 \). Define the sequence \( \chi : \mathbb{Z}_+ \to \mathbb{Z}_+ \) recursively by \( \chi(1) = 1 \) and \( \chi(n) = \sum_{i=1}^{n-1} \chi(i)\chi(n-i) \).

Lemma 5.3. Let \( Y = (y_{ij}) \) be an upper triangular \( n \times n \)-matrix, satisfying \( Y^2 = I \). Then \( y_{ii} = \pm 1 \) and for \( y_{ii} = y_{kk}, i < k \) we have
\[
y_{ik} = \sum_{\mathcal{P}(i,k)} (-y_{ii})^{\chi(i)} \frac{1}{2m} \prod_{s=0}^{m} y_{i,s+1}
\]
where \( m = 2\nu + 1 \). The entry \( y_{ik} \) can be choosen freely whenever \( y_{ii} \neq y_{kk} \).

An immediate consequence of this lemma is that the space of upper triangular \( n \times n \)-matrices, satisfying \( Y^2 = I \) with eigenspaces of dimension \( \alpha'_1, \alpha'_2 \) is an affine space of dimension \( \alpha'_1 + \alpha'_2 = \frac{1}{2}(n^2 - (\alpha'_1)^2 - (\alpha'_2)^2) \).

Proof. Since \( Y \) is upper triangular and \( Y^2 = I \) the diagonal entries obviously satisfy the equation \( y'_{ii} = 1 \).

The relation \( Y^2 = I \) can be written as a system of \( \binom{n}{2} \) quadratic equations in the \( \binom{n}{2} \) off-diagonal entries \( y_{ik}, 1 \leq i < k \leq n \)

\[
(y_{ii} + y_{kk})y_{ik} + \sum_{j=i+1}^{k-1} y_{ij}y_{jk} = 0
\]

(1)

The proof of the lemma has two parts. First we show that the given vector \( (y_{ik}) \) is a solution of the system (1). Then we show that all solutions are of this form. We proceed by induction on the difference \( \delta = k - i \). If \( \delta = 1 \) the defining equation reduces to
\[
(y_{ii} + y_{i+1,i+1})y_{i,i+1} = 0
\]
If \( y_{ii} + y_{i+1,i+1} = 0 \) there are no conditions on \( y_{i,i+1} \). If \( y_{ii} + y_{i+1,i+1} \neq 0 \) it follows that \( y_{i,i+1} = 0 \). This fits with the given formula since in that case \( \mathcal{P}(i, i + 1) = \emptyset \). Thus the lemma is true for \( \delta = 1 \).
Now suppose the formula is valid for \( \delta < t \) and let \((i, k)\) satisfy \( k - i = t \). If \( y_{ii} + y_{kk} = 0 \) we have to show that
\[
\sum_{j=i+1}^{k-1} y_{ij} y_{jk} = 0
\]
Since \( y_{ii} \neq y_{kk} \) the sum splits into two parts
\[
\sum_{j=i+1}^{k-1} y_{ij} y_{jk} = \sum_{i < j} y_{ij} y_{jk} + \sum_{j < k} y_{ij} y_{jk}
\]
\[
= \sum_{i < j} \left( \sum_{p(i,j)} (-y_{ii}) \frac{\chi(p)}{2^p} \prod_{s=0}^{p} y_{i,js} \right) y_{jk}
\]
\[
+ \sum_{j < k} y_{ij} \left( \sum_{p(j,k)} (-y_{jj}) \frac{\chi(q)}{2^q} \prod_{s=0}^{q} y_{jj+1} \right)
\]
For each sequence
\[
i = i_0 < i_1 < \ldots < i_{m+1} = k
\]
there are two appearances of the term \( y_{i_0 i_1} y_{i_1 i_2} \ldots y_{i_{m-1} k} \), one in each sum. In the first sum \( j = i_m \) and in the second sum \( j = i_1 \). Thus the two terms add up to
\[
(-y_{ii}) \frac{\chi(m-1)}{2^{m-1}} \prod_{s=0}^{m-1} y_{i,i+s} y_{jk} + y_{ij} (-y_{jj}) \frac{\chi(m-1)}{2^{m-1}} \prod_{t=1}^{m} y_{jj+1}
\]
\[
= (-y_{i0i_0} - y_{i_0 i_1}) \frac{\chi(m-1)}{2^m} \prod_{t=0}^{m-1} y_{i_1 i+1} = 0
\]
For \( y_{ii} = y_{kk} = \pm 1 \) the equation (1) takes the form
\[
y_{ik} = -\frac{1}{2} y_{ii} \sum_{j=i+1}^{k-1} y_{ij} y_{jk}
\]
and we have to show that
\[
\frac{1}{2} y_{ii} \sum_{j=i+1}^{k-1} y_{ij} y_{jk} = \sum_{p(i,k)} (-y_{ii}) \frac{\chi(m-1)}{2^{m-1}} \prod_{s=0}^{m} y_{i,i+s}
\]
We need a lemma.

**Lemma 5.4.** Let \( i < j < k \) and suppose \( y_{ii} = y_{kk} \). Then
\[
\left\{ \cup_{i < j} (P(i,j) \times P(j,k)) \right\} \cup \{ i < j < k \mid y_{ii} \neq y_{jj} \} = P(i,k)
\]

**Proof.** Let \( \mathcal{I} = \{ i = i_0 < i_1 < \ldots < i_p = j \} \in P(i,j) \) and \( \mathcal{J} = \{ j = j_0 < j_1 < \ldots < j_q = k \} \in P(j,k) \). Define the join
\[
\mathcal{I} \cdot \mathcal{J} = \{ i = i_0 < i_1 < \ldots < i_p = j_0 < j_1 < \ldots j_q = k \}
\]
\[
= \{ i = i_0 < i_1 < \ldots < i_q < i_{p+1} < \ldots i_{p+q} = k \}
\]
where we have put \( i_{p+t} = j_t \) for \( t = 0, 1, \ldots, q \). If \( p, q \) are even, so is \( p + q \), and for \( r > p \) we have
\[
y_{i_r r} = y_{j_{r-p} j_{r-p}} = (-1)^{r-p} y_{j_{r-p}} = (-1)^{r-p} y_{i_{p+1}} = (-1)^{r-p} y_{i_{p+1}} = (-1)^r y_{ii}
\]
Thus \( \mathcal{I} \cdot \mathcal{J} \in P(i,k) \). The set \( \{ i < j < k \mid y_{ii} \neq y_{jj} \} \) is easily seen to be in \( P(i,k) \).
Now pick some $K \in \mathcal{P}(i,k)$ and choose $j \in K$ such that $y_{ii} = y_{jj} = y_{kk}$. Then $K = I \cdot J$ where

$$I = \{i = k_0 < \cdots < k_p = j\} \quad \text{and} \quad J = \{j = k_p < k_{p+1} \cdots < k_{p+q} = k\}$$

and $y_{ii} = y_{kk}, k_p = (-1)^{p}y_{ii}$ is possible only if $p = j - i$ is an even number. \hfill \square

**Proof of Lemma 5.3 continued.**

Our assumption is that

$$y_{ik} = \sum_{p(i,k)} (-y_{ii}) \frac{\chi(n)}{2^{m}} \prod_{s=0}^{m} y_{is_{i+1}}$$

for $k - i = \delta < t$. For $k - i = t$ we get

$$(y_{ii} + y_{kk})y_{ik} + \sum_{j=i+1}^{k-1} y_{ij}y_{jk}$$

$$= (y_{ii} + y_{kk})y_{ik} + \sum_{i \neq j} y_{ij}y_{jk}$$

$$+ \sum_{j=i+1}^{k-1} \left( \sum_{p(i,j)} (-y_{ii}) \frac{\chi(n-1)}{2^{n-1}} \prod_{s=0}^{n} y_{is_{i+1}} \right) \left( \sum_{p(j,k)} (-y_{jj}) \frac{\chi(n-1)}{2^{n-1}} \prod_{t=0}^{m} y_{ts_{t+1}} \right)$$

$$= (y_{ii} + y_{kk})y_{ik} + \sum_{i \neq j} y_{ij}y_{jk}$$

$$+ \sum_{j=i+1}^{k-1} \left( \sum_{p(i,j)} (-y_{ii}) \frac{\chi(n-1)}{2^{n-1}} \prod_{s=0}^{n} y_{is_{i+1}} \right)$$

where $s_{n+1} = t_{j}$ for $j = 0, 1, \ldots, m$.

For $y_{ii} + y_{kk} \neq 0$ we get

$$y_{ik} = -\sum_{j=i+1}^{k-1} \left( \sum_{p(i,k)} (-y_{ii}) \frac{\chi(n-1)}{2^{n-1}} \prod_{s=0}^{n} y_{is_{i+1}} \right)$$

and for $y_{ii} + y_{kk} = 0$ this relation gives no new condition on $y_{ik}$. \hfill \square

In the next lemma we establish the explicit form for the stabilising $GL_n$-subgroup of the diagonal matrix $X$, satisfying $X^3 = I$, with eigenspaces of dimension $\alpha_1, \alpha_2, \alpha_3$.

**Lemma 5.5.** *The stabiliser subgroup $G_X \subset GL_n$, stabilising the diagonal matrix $X$, is given by*

$$G_X = \{(g_{ij}) \in GL_n \mid g_{ij}(x_{ii} - x_{jj}) = 0\}$$

*It is a linear group of dimension*

$$\dim G_X = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$

*isomorphic to the product group $M_{\alpha_1}(k) \times M_{\alpha_2}(k) \times M_{\alpha_3}(k)$.*

**Proof.** Let $g \in G_X$. The relation $gX = Xg$ gives the entries $g_{ij}x_{jj} = x_{ii}g_{ij}$ or $g_{ij}(x_{jj} - x_{ii}) = 0$. Thus we must have $g_{ij} = 0$ for all pairs $(i,j)$ such that $x_{jj} - x_{ii} \neq 0$, i.e. $g_{ij} = 0$ whenever $x_{ii} \neq x_{jj}$. There are $n^2$ entries in the whole matrix, $n$ are on the diagonal and $\sum_{i=1}^{3} \alpha_i^2 - \alpha_i$ entries correspond to off-diagonal pairs $(i,j)$ with $x_{ii} = x_{jj}$, adding up to the given dimension.
Since $X$ is diagonal with at most three different eigenvalues, conjugation by a certain permutation matrix shows that the stabiliser subgroup is isomorphic to the product of three full matrix groups, of given dimension.

Denote by

$$\mathcal{Y}(\Gamma) = \{ Y \in M_\alpha(k) \mid Y \text{ upper triangular}, Y^2 = I, Y \in \Gamma_Y \}$$

By $Y \in \Gamma_Y$ we mean that $y_{11}, \ldots, y_{nn}$ is chosen in accordance with the representation graph $\Gamma$.

We have $\dim \mathcal{Y}(\Gamma) = \alpha'_1\alpha'_2 = \frac{1}{2}(n^2 - \sum_j (\alpha'_j)^2)$ with a natural basis given by $\{(i, j) \mid i < j, i \neq j \}$. The subgroup of $G_X$ for $X \in \Gamma_X$ stabilising $\mathcal{Y}(\Gamma)$ is denoted $G_{X, \mathcal{Y}(\Gamma)}$.

Recall that the original problem of this section was the following: Given a weak equivalence class $\widetilde{E} \in D_\alpha$ of a maximally iterated extension $E$. This equivalence class splits up in a bunch of equivalence classes under the stronger conjugation relation, described by the non-commutative scheme $Ind_\Gamma(A)$. Then we have the following

**Proposition 5.6.** Then there is a one-to-one correspondence

$$Ind_\Gamma(A) \longleftrightarrow \mathcal{Y}(\Gamma)/G_{X, \mathcal{Y}(\Gamma)}$$

**Proof.** A representation in $Ind_\Gamma(A)$ is given by two upper triangular matrices $X$ and $Y$, satisfying $X^3 = Y^2 = I$. By Lemma 5.1 we can assume that $X$ is diagonalisable and by Lemma 5.2 we can in fact assume that $X$ is diagonal and $Y$ is upper triangular. The parameter space $\mathcal{Y}(\Gamma)$ is described in Lemma 5.3. \qed

6. Relation to modular forms

The algebra of modular forms is freely generated as a commutative ring by the two forms $E_4$ and $E_6$. Let $f(n)$ be the dimension of the space of modular forms of dimension $2n$, $n \geq 2$. The generating function for $f(n)$ is given by

$$\sum_{n=0}^{\infty} f(n) x^n = \frac{1}{1-x^2}, \frac{1}{1-x^3}$$

We have seen that $dim Simp_{\omega}(\alpha) = dim Simp_{\omega}(\alpha)$ for $|\alpha| = n$, where the partitions $n = \alpha_1 + \alpha_2 + \alpha_3 = \alpha'_1 + \alpha'_2$ are at most evenly distributed. Let $\alpha = \beta + \gamma$. We are interested in the number $-\langle \beta, \gamma \rangle = |\beta| \cdot |\gamma| - \sum_{i=1}^{3} \beta_i \gamma_i - \sum_{j=1}^{2} \beta'_j \gamma'_j$. It can be proven that the minimal value of this number is obtained when $|\gamma| = 1$ (or $|\beta| = 1$).

For fixed $|\beta|$ and $|\gamma|$, the smallest value of $-\langle \beta, \gamma \rangle$ is again achieved when $\beta$ and $\gamma$ are at most evenly distributed. On the other hand, $|\beta| \cdot |\gamma|$ has its minimal positive value when one of the factors equals 1. Thus the minimal value for $-\langle \beta, \gamma \rangle$ is given by $n - 1 - \beta_i - \beta'_j$ for maximal $\beta_i = \alpha_i - 1$ and $\beta'_j = \alpha'_j - 1$. Since $\alpha$ is at most evenly distributed we have $\beta_i = [\frac{n-1}{2}]$ and $\beta'_j = [\frac{n-1}{2}]$, where $[q]$ denotes the greatest integer, less than $q$.

The generating function for the number $[\frac{n-1}{2}]$ is given by

$$(x + x^2 + x^3 + \ldots)(x^k + x^{2k} + x^{3k} + \ldots) = \frac{x^{k+1}}{(1-x)(1-x^k)}$$

Thus the generating function for the sequence $n - 1 - [\frac{n-1}{2}] - [\frac{n-1}{2}]$ is given by
Thus except for the one of weight 0, the number of modular forms of weight $2n$ equals the codimension of the non-simple stratum of the highest-dimensional component of the variety of semi-simple representations of $A$ of dimension $n$.

REFERENCES

[1] Eriksen, E., Noncommutative deformations of sheaves and presheaves of modules, math. AG/04052344
[8] Le Bruyn, L., Adriaenssens, J., Non-commutative covers and the modular group, math.RA/0307139