

NONCOMMUTATIVE PLANE CURVES

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ABSTRACT. In this paper we study noncommutative plane curves, i.e. noncommutative k -algebras for which the 1-dimensional simple modules form a plane curve. We study extensions of simple modules and we try to enlighten the completion problem, i.e. understanding the connection between simple modules of different dimension.

1. INTRODUCTION

The study of affine algebraic plane curves goes back to Descartes and Fermat, and is still far from complete. There are many unanswered natural questions, one of which is the classification of singularities of such curves. Since an (embedded) affine algebraic plane curve is the obvious equivalence class of polynomials in two variables it is moreover clear that the complete classification of all isomorphism classes of such curves is an unrealistic task. In fact, both problems lead to extremely complex invariant problems, certainly involving noncommutative algebraic geometry, see [15].

In this paper we shall restrict our attention to a much more abordable set of problems regarding affine algebraic plane curves, equally involving noncommutative algebraic geometry.

Any affine algebraic plane curve defined on an algebraically closed field k , with coordinate k -algebra A_0 , has many noncommutative models, i.e. k -algebras A such that $A/([A, A]) = A_0$. For every such model A we may consider not only the classical variety $\text{Simp}_1(A) \subset \text{Spec}(A_0)$ of 1-dimensional simple representations (the scheme-theoretical closed points), but also the k -schemes $\text{Simp}_n(A)$ of n -dimensional simple left representations, see e.g. [13], and the relations between $\text{Simp}_n(A)$ and $\text{Simp}_m(A)$.

The structure of these new schemes, naturally associated to affine algebraic plane curves, seems to be of interest, not only to noncommutative geometry, but even to classical algebraic geometry, see [11] or [13].

The purpose of this paper is to start a systematic study of these noncommutative affine models of plane quadrics and cubics, and to relate this study to the relevant literature on noncommutative algebra, see [1].

We work over an algebraically closed field k of characteristic zero, and all k -algebras are assumed to be finitely generated.

2. LINKS TO CLASSICAL RING THEORY

In this section we consider simple modules over a PI algebra A and prove some results about extensions of non-isomorphic simple modules of finite dimension over k , for the purpose of studying finitely generated indecomposable A -modules.

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2.1. Extensions of semiprime PI algebras. A semiprime PI algebra is a k -algebra with no nilpotent ideals, and satisfying a polynomial identity. Thus the notion generalises the concept of an algebraic variety, being defined by a reduced algebra satisfying the commutation identity.

For a semiprime PI-algebra A , the degree $d = \deg(A)$ of A is the least integer such that A can be embedded in a ring of $d \times d$ -matrices over a commutative k -algebra R . The degree d is also the maximum of $\dim_k V$, where V runs through all simple A -modules (cf. [6]), and finally, d is one half of the least possible degree of a non-zero polynomial $f \in k\langle x_1, \dots, x_m \rangle$ with the property that $f(a_1, \dots, a_m) = 0$ for all $a_1, \dots, a_m \in A$ and for some $m \geq 1$.

For a simple A -module V the left annihilator of V

$$\mathfrak{m} = \text{Ann}_A(V) = \{a \in A \mid aV = 0\}$$

is a maximal two-sided ideal and $A/\mathfrak{m} \simeq \text{End}_k(V) \simeq M_l(k)$, where $l = \dim_k V$.

Recall the definition of a n -central polynomial.

Definition 2.1. A noncommutative polynomial $f \in k\langle x_1, \dots, x_m \rangle$ is said to be n -central if whenever evaluated in $M_n(R)$ for any commutative k -algebra R , it is non-vanishing on $M_n(R)$ and yields values in the center R of $M_n(R)$.

Let A be a semiprime PI algebra of degree $d = \deg(A)$ and c_d be a d -central polynomial. Then (cf. [16]) for all $a_1, \dots, a_m \in A$ the element $c = c_d(a_1, \dots, a_m)$ is central in A and non-zero for suitable choices of a_1, \dots, a_m . Therefore, if V is a simple A -module with $\dim_k V = d$, then for $v \in V$, $cv = \bar{c}v = \lambda v$ for some non-zero scalar $\lambda \in k$. Here, \bar{c} denotes the image of c in $\text{End}_k(V) \simeq M_d(k)$. Moreover, if W is a simple A -module and $\dim W < d$, then $cW = 0$ (see Chapter 13 of [16]).

Proposition 2.2. Let A be a semiprime PI algebra of degree d . Let V be a simple A -module of maximal k -dimension and M a finite length module, not containing V as a composition factor. Then $\text{Ext}_A^1(M, V) = \text{Ext}_A^1(V, M) = 0$.

Proof. We shall prove the vanishing of $\text{Ext}_A^1(V, M)$. The proof for $\text{Ext}_A^1(M, V)$ is quite similar.

Since $\text{Ext}_A^1(V, -)$ is half-exact as a functor on $A\text{-mod}$, the proposition will follow if we can prove the vanishing result for simple modules $W \not\simeq V$.

Consider a central element c defined above. Multiplication by c on V (resp. W) induces maps c^* (resp. c_*) on $\text{Ext}_A^1(V, W)$. Since c is a central element of A , $c^* = c_*$. Notice that

$$c \text{Ext}_A^1(V, W) = c^*(\text{Ext}_A^1(V, W)) = c_*(\text{Ext}_A^1(V, W))$$

Since c act as an isomorphism on V we have

$$c^*(\text{Ext}_A^1(V, W)) \simeq \text{Ext}_A^1(V, W)$$

Suppose W is a simple A -module and $\dim_k W < d$. Then $c_* = 0$, and so

$$\text{Ext}_A^1(V, W) \simeq c_*(\text{Ext}_A^1(V, W)) = 0$$

For W of maximal dimension, the element c acts as an isomorphism on W . Thus, by localisation in c we obtain the implication

$$\text{Ext}_A^1(V, M) \neq 0 \quad \Rightarrow \quad \text{Ext}_{A[c^{-1}]}^1(V, M) \neq 0$$

Since $A[c^{-1}]$ is an Azumaya algebra (see [16]), hence has no non-trivial extensions of non-isomorphic simple modules of maximal degree, the result follows. \square

2.2. Second layer link. Recall that for prime ideals \mathfrak{p} and \mathfrak{q} of a k -algebra A , a second layer link from \mathfrak{q} to \mathfrak{p} , denoted $\mathfrak{q} \rightsquigarrow \mathfrak{p}$, is defined by the existence of an ideal I such that

$$\mathfrak{q}\mathfrak{p} \subseteq I \subsetneq \mathfrak{q} \cap \mathfrak{p}$$

and $(\mathfrak{q} \cap \mathfrak{p})/I$ is a non-trivial torsionfree left A/\mathfrak{q} -module and right A/\mathfrak{p} -module. In case \mathfrak{p} and \mathfrak{q} are maximal ideals, we either have $\mathfrak{q} \rightsquigarrow \mathfrak{p}$ or $\mathfrak{q} \cap \mathfrak{p} = \mathfrak{q} \cdot \mathfrak{p}$.

If V and W are simple A -modules of finite dimension over k , the annihilators $\text{Ann}_A(V) = \mathfrak{m}$ and $\text{Ann}_A(W) = \mathfrak{n}$ are maximal ideals.

Proposition 2.3. *Let V and W be non-isomorphic finite dimensional simple A -modules with annihilators \mathfrak{m} , resp. \mathfrak{n} . Then $\text{Ext}_A^1(V, W) \neq 0$ implies that $\mathfrak{n} \rightsquigarrow \mathfrak{m}$.*

Proof. Notice that in case $\mathfrak{n} \not\rightsquigarrow \mathfrak{m}$, we have $\mathfrak{n} \cap \mathfrak{m} = \mathfrak{n} \cdot \mathfrak{m}$, $A/\mathfrak{m} \simeq V^k$ and $A/\mathfrak{n} \simeq W^l$ for suitable k and l . Since $V \not\cong W$ there is a short-exact sequence

$$0 \rightarrow \text{Hom}_A(A, A/\mathfrak{n}) \rightarrow \text{Hom}_A(\mathfrak{m}, A/\mathfrak{n}) \rightarrow \text{Ext}_A^1(V^k, W^l) \rightarrow 0$$

Observe that

$$\begin{aligned} \text{Hom}_A(\mathfrak{m}, A/\mathfrak{n}) &\simeq \text{Hom}_A(\mathfrak{m}/\mathfrak{nm}, A/\mathfrak{n}) \\ &= \text{Hom}_A(\mathfrak{m}/\mathfrak{n} \cap \mathfrak{m}, A/\mathfrak{n}) \\ &\simeq \text{Hom}_A(A/\mathfrak{n}, A/\mathfrak{n}) \\ &\simeq \text{Hom}_A(A, A/\mathfrak{n}) \end{aligned}$$

Since all modules have finite k -dimension we obtain $\text{Ext}_A^1(V, W) = 0$. \square

2.3. A non PI Result. Let $A = R[\theta; \alpha]$ be a quasipolynomial extension of a k -algebra R , where α is a k -automorphism of R . Recall that a quasipolynomial algebra is the ‘‘ordinary’’ polynomial algebra over R generated by θ with relation $\theta \cdot r = \alpha(r) \cdot \theta$ for all $r \in R$.

Proposition 2.4. *Let V and W be two simple non-isomorphic A -modules with $\theta V = \theta W = 0$. Let $\text{Ann}_R(V) = \mathfrak{m}$ and $\text{Ann}_R(W) = \mathfrak{n}$. Then the following are equivalent:*

- i) $\text{Ext}_A^1(V, W) \neq 0$
- ii) $\text{Ext}_R^1(V, W) \neq 0$ or $\alpha(\mathfrak{m}) = \mathfrak{n}$.

Proof. It is straightforward to see that for two A -modules V and W with $\theta V = \theta W = 0$, $\text{Ext}_R^1(V, W) \neq 0$ implies $\text{Ext}_A^1(V, W) \neq 0$. Therefore it suffices to show that in case $\text{Ext}_R^1(V, W) = 0$, we have $\text{Ext}_A^1(V, W) \neq 0$ if and only if $\alpha(\mathfrak{m}) = \mathfrak{n}$.

Suppose $R/\mathfrak{m} \simeq V^k$ where $k = \dim V$. By assumption the short-exact sequence

$$0 \rightarrow (\mathfrak{m}, \theta) \longrightarrow A \longrightarrow V^k \rightarrow 0$$

induces another short-exact sequence

$$0 \rightarrow \text{Hom}_A(A, W) \xrightarrow{\phi} \text{Hom}_A((\mathfrak{m}, \theta), W) \longrightarrow (\text{Ext}_A^1(V, W))^k \rightarrow 0$$

and we observe that $\text{Ext}_A^1(V, W) = 0$ if and only if ϕ is onto.

Suppose $f \in \text{Hom}_A(A, W)$, then since $\theta W = 0$ we have $f(\theta) = 0$, thus

$$\text{Ext}_A^1(V, W) \neq 0$$

if we can find an element $g \in \text{Hom}_A((\mathfrak{m}, \theta), W)$ with $g(\theta) \neq 0$.

Since $\text{Ext}_R^1(V, W) = 0$ we have an isomorphism $\text{Hom}_R(R, W) \simeq \text{Hom}_R(\mathfrak{m}, W)$ and the restriction of $g \in \text{Hom}_A((\mathfrak{m}, \theta), W)$ to \mathfrak{m} is multiplication by some element $w \in W$. Hence for any $b(\theta) \in A$ and $m \in \mathfrak{m}$

$$\begin{aligned} g(rm + b(\theta)\theta) &= rg(m) + b(\theta)g(\theta) \\ &= rmw = (rm + b(\theta)\theta)w \end{aligned}$$

and g is clearly in the image of ϕ . Consequently $Ext_A^1(V, W) \neq 0$ is equivalent to the existence of $g \in Hom_A((\mathfrak{m}, \theta), W)$ with $g(\theta) \neq 0$.

For any $g \in Hom_A((\mathfrak{m}, \theta), W)$ and $m \in \mathfrak{m}$ we have

$$0 = \theta \cdot g(m) = g(\theta \cdot m) = g(\alpha(m) \cdot \theta) = \alpha(m) \cdot g(\theta)$$

If $g(\theta) \neq 0$, then $\alpha(\mathfrak{m}) \subset Ann_R(g(\theta)) = Ann_R(W) = \mathfrak{n}$ since W is simple. But then $\alpha(\mathfrak{m}) = \mathfrak{n}$, since α is an automorphism of R and \mathfrak{m} and \mathfrak{n} are maximal ideals.

On the other hand, if $\alpha(\mathfrak{m}) = \mathfrak{n}$, then for any $0 \neq w \in W$, we define

$$g : (\mathfrak{m}, \theta) \rightarrow W$$

by

$$g(m + (\sum_j b_j \theta^j) \theta) = b_0 w$$

where $m \in \mathfrak{m}$ and $b_j \in R$. This is a well-defined, non-zero A -homomorphism. In fact g is R -linear and

$$\theta g(m + (\sum_j b_j \theta^j) \theta) = \theta b_0 w = 0$$

and

$$\begin{aligned} g(\theta(m + (\sum_j b_j \theta^j) \theta)) &= g(\alpha(m)\theta + \theta(\sum_j b_j \theta^j)\theta) \\ &= (\alpha(m) + \theta(\sum_j b_j \theta^j))g(\theta) \\ &= \alpha(m)g(\theta) + \theta(\sum_j b_j \theta^j)g(\theta) = 0 \end{aligned}$$

□

3. GEOMETRICAL APPROACH

In this section we explore two approaches to understanding the geometry of the scheme $Simp_n(A)$, the global approach using invariant theory, and the local approach using deformation theory.

3.1. Representations of algebras. Let k be an algebraically closed field and A a finitely presented associative k -algebra. A left A -module structure on a k -vector space M is given by a k -algebra homomorphism

$$\phi : A \longrightarrow End_k(M)$$

This homomorphism ϕ is called a representation, or equivalently, the structure map of the module M . Two structure maps give isomorphic A -module structures on M if they differ by a ring automorphism of $End_k(M)$. Such automorphisms are conjugations by an invertible linear operator $T : M \longrightarrow M$. Thus the set of left A -module structures on M is the set of structure maps $\phi : A \longrightarrow End_k(M)$, modulo conjugation.

By choosing a k -basis for M an n -dimensional representation of A is thus given by a homomorphism of k -algebras,

$$\phi : A \longrightarrow M_n(k)$$

where $M_n(k)$ is the k -algebra of $n \times n$ -matrices over k . The representation ϕ is simple if it is surjective. In the finite dimensional case, the automorphism group acting on the set $\mathcal{X}_A^{(n)} := Mor_{Alg_k}(A, M_n(k))$ of n -dimensional representations is the group $PGL_n(k)$. Since A is finitely presented it is easy to see that $\mathcal{X}_A^{(n)}$ is a well-defined affine variety.

Let $\{x_i\}_{i=1,\dots,m}$ be generators of A as a k -algebra. Associate to the representation $\phi \in \mathcal{X}_A^{(n)}$, the point of the affine space $\mathbf{A}^{m \cdot n^2}$ with coordinates $\{a_{p,q}^i\}$, $i = 1, \dots, m$, $p, q = 1, \dots, n$, given by $\phi(x_i) = (a_{p,q}^i)$. Let f_1, \dots, f_r be the relations defining A , and put $\Gamma = k[x_{p,q}^i]/(\tilde{f}_1, \dots, \tilde{f}_r)$, where $\tilde{f}_j = f_j((x_{p,q}^1), \dots, (x_{p,q}^m))$. Then $\mathcal{X}_A^{(n)} = \text{Spec}(\Gamma)$, and we have a versal family of n -dimensional A -modules with basis $\mathcal{X}_A^{(n)}$ given by the representation,

$$\Phi : A \longrightarrow M_n(\Gamma)$$

defined by,

$$\Phi(x_i) = (cl(x_{p,q}^i))$$

where $cl(x_{p,q}^i)$ denotes the class of $x_{p,q}^i$ in Γ . Clearly $PGL_n(k)$ acts on $\mathcal{X}_A^{(n)}$ and on the k -algebra Γ . Put

$$\text{Repr}_n(A) = \mathcal{X}_A^{(n)} / PGL_n(k).$$

The underlying set of this quotient is the set of orbits of $\mathcal{X}_A^{(n)}$ under the action of $PGL_n(k)$, i.e. the set of isomorphism classes of n -dimensional representations of A . The set of isomorphism classes of simple n -dimensional representations, denoted by $\text{Simp}_n(A)$ is a subset of $\text{Repr}_n(A)$.

Notice that the prime spectrum $\text{Spec}(A)$ of a ring A is the set of prime ideals with the Jacobson topology. Clearly $\text{Simp}_n(A)$ is also a subset of $\text{Spec}(A)$.

3.2. The free noncommutative algebra in m variables. Let $S = k\langle x_1, \dots, x_m \rangle$ be the free algebra in m variables. A representation $\phi : S \rightarrow M_n(k)$ of S is given by a set of $n \times n$ -matrices $\phi(x_1), \dots, \phi(x_m)$ and we identify the variety $\mathcal{X}_S^{(n)}$ with the affine space \mathbf{A}^{mn^2} of m -tuples of $n \times n$ -matrices. In general there is no way of describing $\text{Repr}_n(S)$ as an algebraic scheme, but restricting to the simple modules this is possible, as shown in [18].

Clearly, $\Gamma := \Gamma(\mathcal{O}_{\mathcal{X}_S^{(n)}}) \simeq k[x_{s,t}^i]$ is the free commutative k -algebra of polynomials in the variables $x_{s,t}^i$ for $i = 1, \dots, m$ and $s, t = 1, \dots, n$. The versal family in this case is

$$\Phi : S = k\langle x_1, \dots, x_m \rangle \longrightarrow M_n(\Gamma)$$

given by $x_i \mapsto M_i$, where M_i is a matrix whose s, t -entry is $x_{s,t}^i$. The images $\pi(S)$ of π is the ring of generic $n \times n$ -matrices and the generators M_1, \dots, M_m are in the literature called the generic matrices over k .

The automorphism group $G := PGL_n(k)$ acts on the ring Γ by conjugation. Moreover G acts on $M_n(\Gamma)$ by double conjugation, leaving Φ invariant. Therefore it is clear that the coefficients of the characteristic polynomials of the generic matrices are invariant, i.e. that they are contained in Γ^G . However, there is no versal family defined over Γ^G , extending the simple modules, see [13]. Define C_n to be the subring of $\Gamma \simeq k[x_{s,t}^i]$ generated by the coefficients of the characteristic polynomials of the generic matrices. This ring is the trace ring, or the trace ring of dimension n to be more precise.

Artin [1] conjectured and Procesi [19] proved that the trace ring C_n is precisely the subring Γ^G of $k[x_{s,t}^i]$.

Procesi has also shown [19] that the closed points of $\text{Spec}(C_n)$ are in 1-1 correspondence with equivalence classes of semisimple representations under the natural map

$$[\rho_\gamma] \longmapsto \rho_\gamma(\Phi)$$

where $\rho_\gamma : C_n \rightarrow k(\gamma) := k$ is a closed point of $\text{Spec}(C_n)$ and $\rho_\gamma(\Phi) : S \rightarrow M_n(k)$ is the corresponding n -dimensional representation.

There is a subset $F_n(S) \subset S$, see e.g. sect. 13.7 of [16], called the Formanek center of n -central polynomials such that $\phi : S \rightarrow M_n(k)$ is a simple representation if and only if $\phi(F_n(S)) \neq 0$. The set of prime ideals \mathfrak{p} in S such that $\mathfrak{p} \not\supset F_n(S)$ is an open set of $\text{Simp}_n(S)$. The subset $F_n(S) \subset S$ consists of n -central elements and by definition $\Phi(F_n(S))$ sits in the center of $M_n(\Gamma)$. The center is $\text{PGL}_n(k)$ -invariant and $\Phi(F_n(S))$, being invariant, is a subset Φ_n of the trace ring C_n . A homomorphism $\psi : C_n(S) \rightarrow k$ corresponds to a simple representation if and only if $\psi(\Phi_n) \neq 0$. Procesi has proved the following result:

Theorem 3.1. (*Procesi*) *The variety $\text{Spec}((C_n)_{\Phi_n})$ of simple n -dimensional representations of S is a smooth variety of dimension $mn^2 - (n^2 - 1) = (m - 1)n^2 + 1$*

Unluckily it is, in general, very difficult to compute $F_n(S)$, and therefore also $\text{Simp}_n(S)$, using this method.

3.3. The free noncommutative algebra in two variables. As an example, let us consider the noncommutative affine plane $S = k\langle x, y \rangle$. A left S -module M , isomorphic to k^2 as k -vector space, is given by a ring homomorphism

$$\phi : S \longrightarrow \text{End}_k(M) \simeq M_2(k)$$

The S -module M is simple if and only if ϕ is surjective, i.e. iff $X := \phi(x)$ and $Y := \phi(y)$ generate $M_2(k)$ as k -algebra. It is easily checked that this is equivalent to $\det[X, Y] \neq 0$ ([2]).

In this case the trace ring

$$C_2 = k[t_X, d_X, t_Y, d_Y, t_{XY}]$$

is the polynomial ring on the five trace elements. We use the notation t_P for the trace and d_P for the determinant of the matrix P . Notice that the determinant itself is a trace, due to the formula

$$d_P = \frac{1}{2}(t_{P^2} - (t_P)^2)$$

where we, of course have to assume $\text{char}(k) \neq 2$.

Notice that in this case, the Formanek center consists of one single element, and non-vanishing of this Formanek element, defined by

$$d_{[X, Y]} = -\frac{1}{4}((2t_{XY} - t_X t_Y)^2 - (t_X^2 - 4d_X)(t_Y^2 - 4d_Y))$$

is equivalent to simplicity of the S -module M .

Thus we have a nice description of $\text{Simp}_2(S)$ as the open subset of the affine 5-space $\text{Spec}(C_2)$, defined by $d_{[X, Y]} \neq 0$. It should be compared with the computations in section 3.8.

3.4. Simple 2-dimensional modules for noncommutative plane curves. In classical algebraic geometry the simple finite dimensional representations of a commutative k -algebra $A_0 = k[x, y]/(f_0)$ are parametrised by a plane curve given by the zero set of the polynomial f_0 . A noncommutative model of the plane curve f , is a (noncommutative) k -algebra $A = k\langle x, y \rangle/(f)$ where $f \equiv f_0([x, y])$. The 1-dimensional representations of A are the same as for the commutative curve A_0 . But, in contrast to what is the case for a commutative algebra, A may also have higher dimensional simple representations.

The purpose of this section is to give a description of the 2-dimensional simple representations of noncommutative plane curves.

Lemma 3.2. *Let R be a commutative k -algebra. For any two 2×2 -matrices $X, Y \in M_2(R)$ we have the equalities*

$$\text{i) } X^2 = t_X X - d_X$$

- ii) $X^3 = (t_X^2 - d_X)X - t_X d_X$
- iii) $YX = -XY + t_X Y + t_Y X + t_{XY} - t_X t_Y$

Proof. Equality i) and ii) follows from the Cayley-Hamilton theorem. To prove iii), notice that

$$(X + Y)^2 = t_{X+Y}(X + Y) - d_{X+Y}I$$

On the other hand we have

$$(X + Y)^2 = X^2 + XY + YX + Y^2 = t_X X - d_X + XY + YX + t_Y Y - d_Y$$

Thus, using the fact that $d_{X+Y} - d_X - d_Y = t_X t_Y - t_{XY}$ iii) follows. \square

In section 3.1 we defined a versal family

$$\Phi : k\langle x, y \rangle \rightarrow M_2(\Gamma)$$

where $\Gamma = k[x_{p,q}^i]/(\tilde{f})$. Let $X := \Phi(x)$ and $Y := \Phi(y)$. Then for $f \in k\langle x, y \rangle$ we have $\Phi(f) = c_1 XY + c_2 X + c_3 Y + c_4 I$, where the coefficients $c_i \in \Gamma$, $i = 1, \dots, 4$ are uniquely determined by f . This follows from the Cayley-Hamilton theorem and from the linear independence of the set $\{I, X, Y, XY\}$.

The simplicity criterion for a representation is given by the non-vanishing of the Formanek element

$$d_{[X,Y]} = -\frac{1}{4}((2t_{XY} - t_X t_Y)^2 - (t_X^2 - 4d_X)(t_Y^2 - 4d_Y))$$

Now, let $\phi : k\langle x, y \rangle \rightarrow M_2(k)$ be a 2-dimensional representation of $k\langle x, y \rangle$, corresponding to a closed point $\gamma \in \text{Spec}(\Gamma)$. Then ϕ induces a simple representation of $A = k\langle x, y \rangle/(\tilde{f})$ if and only if $c_i(\gamma) = 0$ for $i = 1, \dots, 4$ and $d_{[X,Y]}(\gamma) \neq 0$.

Example. For $\delta \in k$ we put $f_\delta = x^2 + y^2 - 1 + \delta[x, y]$ and $A_\delta = k\langle x, y \rangle/(\tilde{f}_\delta)$. Using lemma 3.2 we find

$$\begin{aligned} \Phi(x^2 + y^2 - 1 + \delta[x, y]) = \\ 2\delta XY + (t_X - \delta t_Y)X + (t_Y - \delta t_X)Y + (-d_X - d_Y - 1 - \delta t_{XY} + \delta t_X t_Y) \end{aligned}$$

Thus $\text{Simp}_2(A_\delta) = \emptyset$ unless $\delta = 0$. For $\delta = 0$ we have

$$\text{Spec}(\Gamma(A_{\delta=0})) = V(t_X, t_Y, d_X + d_Y + 1) \subset \mathbf{A}^4$$

and $\text{Simp}_2(A_{\delta=0})$ is the open subscheme of

$$\text{Spec}(\Gamma(A_{\delta=0})) \subset \mathbf{A}^4$$

given by the non-vanishing of the Formanek element

$$t_{XY}^2 - 4d_X(-1 - d_X) = t_{XY}^2 + (2d_X + 1)^2 - 1$$

3.5. The general case. Now for a general finitely generated algebra $A = S/I$, $I = (f_1, \dots, f_r)$ we may, as above, consider the trace ring $C_n \subseteq \Gamma^G$. One might hope that $\text{Spec}(C_n) = \text{Spec}(\Gamma^G)$, and that this space parametrises the semisimple n -dimensional representations. However, this is in general not true, as we shall see in example 3.6.

In the paper ([3]) De Concini and Procesi handle the problem by restricting to algebras with trace. This limitation excludes some interesting examples and weakens the results substantially.

One of the weak points of this approach is related to the embedding of $\text{Simp}_n(A)$ in the representation space. The closure of $\text{Simp}_n(A)$ in this space consist of semisimple modules, which “do not deform back into simple modules”.

We feel that this picture should be extended to include completions of $\text{Simp}_n(A)$ and therefore include the indecomposable modules, see section 4. To achieve this we need to take a different point of view.

3.6. Example. The set of 2-dimensional representations of the polynomial ring $S = k[x]$ is parametrised by the variety $\mathcal{X}_S^{(2)} = \text{Spec}(\Gamma_S)$ where

$$\Gamma_S = k[x_{11}, x_{12}, x_{21}, x_{22}]$$

Let $A = k[x]/(x^2)$. Then

$$\mathcal{X} = \mathcal{X}_A^{(2)} = \text{Spec}(\Gamma)$$

where, as we have seen in section 3.1,

$$\Gamma = k[x_{11}, x_{12}, x_{21}, x_{22}]/J$$

and J is the two-sided ideal

$$J = (x_{11}^2 + x_{12}x_{21}, (x_{11} + x_{22})x_{12}, (x_{11} + x_{22})x_{21}, x_{22}^2 + x_{12}x_{21})$$

Thus \mathcal{X} is a 2-dimensional subvariety of $\mathbf{A}^4 = \text{Spec}(\Gamma_S)$. The versal family is given by the subring of the image of the map

$$\pi : A \longrightarrow \Gamma \otimes M_2(k)$$

given by $x \mapsto \sum_{i,j} \bar{x}_{ij} \otimes e_{ij}$. The trace ring C_2 is generated by the traces of the matrices in the versal family. It is easily seen that $C_2 \simeq k[t]$ where $t = \bar{x}_{11} + \bar{x}_{22}$. We call $\text{Spec}(C_2) = \mathbf{A}^1$ the trace space.

The action of the group $G = PGL_2$ on \mathcal{X} has two orbits, the trivial semi-simple module V_0 , corresponding to the origin, and V_1 , corresponding to the indecomposable module given by $x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus \mathcal{X}/G consists of two points, one in the closure of the other.

The ideal J is G -invariant and G acts on the short-exact sequence

$$0 \rightarrow J \longrightarrow \Gamma_S \longrightarrow \Gamma \rightarrow 0$$

inducing an long-exact sequence

$$0 \rightarrow H^0(G, J) \longrightarrow H^0(G, \Gamma_S) \longrightarrow H^0(G, \Gamma) \longrightarrow H^1(G, J) \rightarrow \dots$$

But $H^1(G, J) = 0$ since J is a graded k -vector space and the reductive group G acts on each finite dimensional graded component. Therefore the sequence reduces to a short-exact sequence

$$0 \rightarrow J \cap \Gamma_S^G \longrightarrow \Gamma_S^G \longrightarrow \Gamma^G \rightarrow 0$$

where $\Gamma_S^G = k[x_{11} + x_{22}, x_{11}x_{22} - x_{12}x_{21}]$. Since $J \cap \Gamma_S^G$ is precisely the ideal of Γ_S^G generated by the determinant of the generic matrix, we have

$$\Gamma^G \simeq k[t]$$

and consequently $\text{Spec}(\Gamma^G) \not\simeq \text{Spec}(\Gamma)/G$.

In section 3.8 we shall discuss this example a bit further.

3.7. Local approach using deformation theory. A different approach to understanding the geometry of $\text{Simp}_n(A)$ is to analyse the local structure of $\text{Simp}_n(A)$ using deformation theory, see [11],[12] of which we briefly recall the main content. The formal moduli of an n -dimensional simple A -module V is a complete (noncommutative) k -algebra $H^A(V)$ ([11]). Using the structure map of V and the basic properties of the formal moduli one can show that there is a natural, topological surjective k -algebra homomorphism (see [13])

$$A \rightarrow H^A(V) \otimes_k \text{End}_k(V) \simeq M_n(H^A(V))$$

Recall that a standard n -commutator relation in a k -algebra A is a relation of the type,

$$[a_1, a_2, \dots, a_{2n}] := \sum_{\sigma \in \Sigma_{2n}} \text{sign}(\sigma) a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(2n)} = 0$$

where $\{a_1, a_2, \dots, a_{2n}\}$ is a subset of A . Let $I(n)$ be the two-sided ideal of A generated by the subset,

$$\{[a_1, a_2, \dots, a_{2n}] \mid \{a_1, a_2, \dots, a_{2n}\} \subset A\}.$$

Any k -algebra homomorphism $\phi : A \rightarrow M_n(k)$ factors through $A(n) := A/I(n)$, and $\text{Simp}_m(A) = \text{Simp}_m(A(n))$ for $m \leq n$, see e.g. [5]. For $m > n$ we have $\text{Simp}_m(A(n)) = \emptyset$. Notice that $A(1)$ is the commutativisation of A .

A usefull fact is that the formal moduli of $V \in \text{Simp}_n(A)$ considered as an $A(n)$ -module is isomorphic to the commutativisation of the formal moduli of V over A ;

$$H^{A(n)}(V) \simeq H^A(V)_{\text{com}}$$

Consequently $H^{A(n)}(V)$ is a commutative k -algebra, easily computed when we know $H^A(V)$.

For a free algebra $S = k\langle x_1, \dots, x_m \rangle$ on m variables and $V \in \text{Simp}_n(S)$ the formal moduli is a noncommutative power series algebra generated by $m \cdot n^2 - (n^2 - 1)$ elements, thus in that case we have

$$H^{S(n)}(V) \simeq k[[t_1, \dots, t_{(m-1)n^2+1}]]$$

which should be compared to the Theorem 3.1 of Procesi given above.

Now consider the product morphism

$$\mu : A(n) \rightarrow \prod_{V \in \text{Simp}_n(A)} H^{A(n)}(V) \otimes_k \text{End}_k(V)$$

In general this map is not injective. Put

$$B = \prod_{V \in \text{Simp}_n(A)} H^{A(n)}(V)$$

Let $x_i \in A$, $i = 1, \dots, m$ be generators of A , and consider the images

$$\mu(x_i) = (x_{p,q}^i) \in B \otimes_k \text{End}_k(k^n) \simeq M_n(B)$$

obtained by choosing bases in all $V \in \text{Simp}_n(A)$. Now, since B is commutative, the k -subalgebra $C(n) \subset B$ generated by the elements $\{x_{p,q}^i\}_{i=1, \dots, d; p, q=1, \dots, n}$ is commutative. One can show (see [13]) that there is an open embedding $\text{Simp}_n(A) \subset \text{Simp}_n(C(n))$. Thus any simple, n -dimensional A -module corresponds to a closed point $v \in \text{Simp}_1(C(n))$.

The space $\text{Simp}_1(C(n))$ is in a rather straightforward sense, a *compactification* of $\text{Simp}_n(A)$, as is $\text{Simp}_1(C_n)$, but neither are good completions of $\text{Simp}_n(A)$.

The complement of the simple locus of $\text{Simp}_1(C_n)$ is in [20] identified with the set of semisimple modules. However, for a non-simple, semisimple A -module M we have

$$\dim_k(\text{End}_A(M)) \geq 2$$

while for an indecomposable module E which is an extension of non-isomorphic 1-dimensional representations $\text{End}_A(E) = k$. Any modular infinitesimal deformation preserves the dimension of the endomorphism ring, see [11]. Therefore there is no modular deformation of a decomposable module M into a simple module, i.e. a non-simple semi-simple module M is not in the completion of any $\text{Simp}_n(A)$.

Let us return to our example from the previous section.

Continuation of example 3.6. Recall that we considered the 2-dimensional representations of the k -algebra $A = k[x]/(x^2)$. The tangent space of the modular part of the deformations of the semi-simple representation V_0 is given by

$$\text{Ext}_A^1(V_0, V_0)^{\text{End}_A(V_0)} = \{\xi \in \text{Ext}_A^1(V_0, V_0) \mid \phi^* \xi = \xi \phi_* \ \forall \phi \in \text{End}_A(V_0)\} \simeq k$$

Since the cup product $\xi \cup \xi \in Ext_A^2(V_0, V_0)$ is non-zero the formal moduli is given by $k[y]/(y^2)$, where y corresponds to the tangent direction ξ . So the moduli space \mathcal{X}/G consists of two points, the semi-simple module V_0 , and the indecomposable V_1 . The G -orbit of V_1 has V_0 in its closure.

3.8. The free noncommutative algebra in two variables, again. Let us once more consider the noncommutative affine plane $S = k\langle x, y \rangle$ and its 2-dimensional representations. This time we use the local approach via deformation theory.

In [13] Laudal has computed a versal family of 2-dimensiona representations of S , as deformations of the simple module $V \in Simp_2(S)$, given by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The formal moduli is the smooth, formal polynomial k -algebra

$$H^S(V)_{com} \simeq k[[t_1, t_2, t_3, t_4, t_5]]$$

and the versal family is defined by

$$x \mapsto \begin{pmatrix} 0 & 1 + t_3 \\ t_5 & t_4 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} t_1 & t_2 \\ 1 + t_3 & 0 \end{pmatrix}$$

The algebra $C(2)$ is therefore given by

$$C(2) \simeq k[t_1, t_2, t_3, t_4, t_5]$$

Clearly $C_2 \subsetneq C(2)$, where C_2 , given in section 3.3, is generated by

$$\begin{aligned} t_X &= t_4, & t_Y &= t_1 \\ d_X &= -t_5(1 + t_3), & d_Y &= -t_2(1 + t_3) \\ t_{XY} &= (1 + t_3)^2 + t_2t_5 \end{aligned}$$

with Formanek element

$$d_{[X,Y]} = -((1 + t_3)^2 - t_2t_5)^2 + (t_1(1 + t_3) + t_2t_4)(t_4(1 + t_3) + t_1t_5)$$

Notice that the indecomposable, non-simple modules obtained by e.g. letting $t_5 = 1 + t_3 = 0$, are not visible in $Spec(C_2)$. The family of non-isomorphic indecomposable modules, parametrised by t_2 , collapses in the trace ring to one semi-simple module.

4. EXTENSIONS

The simple A -modules of dimension n are parametrised by an open set in some affine space. One of the main objectives of this paper is to construct natural completions of the simple locus. The completions are formed by adding indecomposable modules obtained as iterated extensions of simple modules of lower dimension.

4.1. The Noncommutative Jacobi matrix. Let $S = k\langle x_1, \dots, x_m \rangle$ be a free k -algebra on m non-commuting variables. Let $f \in S$ be a polynomial. Let $\phi_p : S \rightarrow k(p)$ be a 1-dimensional representation corresponding to a point $p \in Simp_1(S) \simeq \mathbf{A}^m$, such that $\phi_p(f) = f(p) = 0$. The representation is given by $\phi_p(x_i) = a_i$ for some $a_i \in k$ and $i = 1, 2, \dots, m$, i.e. $p = (a_1, \dots, a_m) \in \mathbf{A}^m$.

Definition 4.1. Let $f \in S$ and let $\phi_p : S \rightarrow k(p)$ be a 1-dimensional representation, corresponding to the point $p = (a_1, \dots, a_m) \in \mathbf{A}^m$. An equality

$$f = \sum_{k=1}^m f_{k,p}(x_k - a_k)$$

is called a left decomposition of f with respect to $k(p)$.

Proposition 4.2. *For any $f \in S$ and any $\phi_p : S \rightarrow k(p)$ satisfying $\phi(f) = 0$ there exists a unique left decomposition of f with respect to $k(p)$.*

Proof. For any monomial \underline{m} in f of positiv degree, we can write

$$\underline{m} = \underline{m}'(x_i - a_i) + a_i \underline{m}'$$

for suitable $a_i \in k$, and where \underline{m}' is of strictly less degree than \underline{m} . Thus by an inductive procedure we can write

$$f(\underline{x}) = f(\underline{a}) + \sum_{k=1}^m f_{k,p}(x_k - a_k)$$

The existence of a left decomposition follows from the fact that $f(\underline{a}) = \phi_p(f) = 0$.

Since the set $\{x_k - a_k\}_{k=1,\dots,m}$ is a free generating set for $k\langle x_1, \dots, x_m \rangle$ the decomposition is obviously unique. \square

There exists an algorithm for computing the components of this decomposition. Let $\phi_p : S \rightarrow k(p)$ and let $D_i(-; p)$ be the linear form defined on S such that $D_i(a; p) = 0$ for $a \in k$, $D_i(x_j; p) = \delta_{ij}$ and for a product fg

$$D_i(fg; p) = fD_i(g; p) + D_i(f; p)g(p)$$

So $D_i(-; p)$ act as a partial derivation, satisfying the Leibniz rule where we evaluate in p on the right hand side. In fact

$$D_i(-; p) \in \text{Der}_k(S, \text{Hom}_k(k(p), S)),$$

which proves that $D_i(-; p)$ is well-defined.

Definition 4.3. *The element $D_k(f; p)$ is called the noncommutative (left) k -th partial derivative of f with respect to the 1-dimensional representation $k(p)$.*

Proposition 4.4. *The left components of an element $f \in S$ with respect to $k(p)$, where $p = (a_1, \dots, a_m)$, is precisely the left partial derivatives of f , i.e. the equality*

$$f = \sum_{k=1}^m D_k(f; p)(x_k - a_k)$$

holds.

Proof. If one applies $D_k(-; p)$ to the equality

$$f = \sum_{k=1}^m f_{k,p}(x_k - a_k)$$

and use the Leibniz rule, then it follows that $f_{k,p} = D_k(f; p)$. \square

We can extend the definition of noncommutative partial derivative to a two-sided ideal $I \subset S$. Let $I \subset S$ be a two-sided ideal generated by f^1, \dots, f^r and ϕ_p be a representation of S corresponding to a point $p \in \text{Simp}_1(S/I)$. Consider the left ideal of $A = S/I$ generated by the image of the i -th partial derivatives of the generators f^1, \dots, f^r . Denote this ideal $J_i(I; f^1, \dots, f^r; p)$.

Lemma 4.5. *The ideal $J_i(I; f^1, \dots, f^r; p)$ is independent of the choice of generator set for the ideal I .*

Proof. Let

$$f = \sum_{j,k} \alpha_{jk} f^j \beta_{jk}$$

be an arbitrary element of I , with $\alpha_{jk}, \beta_{jk} \in S$. Then we have

$$\begin{aligned} D_i(f; p) &= D_i\left(\sum_{j,k} \alpha_{jk} f^j \beta_{jk}; p\right) \\ &= \sum_{j,k} (D_i(\alpha_{jk}; p) \phi_p(f^j \beta_{jk}) + \alpha_{jk} D_i(f^j; p) \phi_p(\beta_{jk}) + \alpha_{jk} f^j D_i(\beta_{jk}; p)) \\ &\in J_i(I; f^1, \dots, f^m; p) + I \end{aligned}$$

□

We write $J_i(I; p)$ for this ideal.

Definition 4.6. *The ideal $J_k(I; p) \subset S$ is called the k -th noncommutative Jacobi ideal of I with respect to $k(p)$. The matrix $J(I; p) = (J_k(f^j; p))_{k,j}$ is called the noncommutative Jacobi matrix of the presentation $I = (f^1, \dots, f^r)$.*

Notice that for any element $g = \sum_i r_i(x_i - a_i) \in S$ we have $r_i = D_i(g; p)$ by the uniqueness of partial derivatives. If $g \in I$ the proof of the lemma shows that

$$r_i = \sum_j \left(\sum_k \alpha_{jk} \phi_p(\beta_{jk}) \right) D_i(f^j; p) + \sum_{j,k} \alpha_{jk} f^j D_k(\beta_{jk}; p)$$

Modulo the ideal I we can write

$$r_i = \sum_j s_j D_i(f^j; p)$$

where

$$s_j = \sum_k \alpha_{jk} \phi_p(\beta_{jk})$$

A consequence of this is that

$$A^r \xrightarrow{J(I;p)} A^m \xrightarrow{(\underline{x}-\underline{a})} A \xrightarrow{\phi_p} k(p) \rightarrow 0$$

gives a truncated resolution of the representation $k(p)$ as a left A -module.

The noncommutative Jacobi matrix $J(I; p)$ is an $r \times m$ -matrix over S . For a representation $\phi_2 : S \rightarrow k(p_2)$ we write $J(I; p_1)(p_2)$ for the evaluation of $J(I; p_1)$ in p_2 .

Proposition 4.7. *Let $A = k\langle x_1, \dots, x_m \rangle / I$ be a k -algebra and let ϕ_1, ϕ_2 be 1-dimensional representations corresponding to $k(p_1), k(p_2) \in \text{Simp}_1(A)$. Suppose $p_1 \neq p_2$, then*

$$\dim_k \text{Ext}_A^1(k(p_1), k(p_2)) = m - 1 - \text{rk} J(I; p_1)(p_2).$$

If $p_1 = p_2 =: p$, then

$$\dim_k \text{Ext}_A^1(k(p), k(p)) = m - \text{rk} J(I; p)(p).$$

Proof. A direct consequence of the above discussion. □

For $A = S/I$ commutative the commutators $x_i x_j - x_j x_i$ are part of a generator set for the ideal $I \subset S$. Notice that for $p = (a_1, \dots, a_m)$

$$D_k(x_i x_j - x_j x_i; p) = \begin{cases} 0 & k \neq i, j \\ a_j - x_j & k = i \\ x_i - a_i & k = j \end{cases}$$

Thus $J_k(I; p_1) \subset \mathfrak{m}$, the maximal ideal of S corresponding to p_1 . Therefore, for $p_1 \neq p_2$ we have

$$\text{rk} J_k(I; p_1)(p_2) = m - 1$$

proving the well-known vanishing of $\text{Ext}_A^1(k(p_1), k(p_2))$ for A commutative and $p_1 \neq p_2$. For A noncommutative this is no longer true, and a natural question to ask is, given a representation ϕ_p of A , corresponding to a point $k(p) \in \text{Sim}_1(A)$, describe the variety of simple 1-dimensional representations ϕ_q with non-vanishing $\text{Ext}_A^1(k(p), k(q))$. Under certain finiteness conditions this gives a correspondance on the set of simple 1-dimensional representations. In the next section we shall explore this in the case of plane curves.

4.2. Noncommutative Taylor series expansion. What we did in the previous section is a special case of a more general set-up, see [13]. Let m be some natural number and consider the free k -algebra on $3m$ variables,

$$k\langle \underline{x}, \underline{v}, \underline{u} \rangle = k\langle x_1, \dots, x_m, v_1, \dots, v_m, u_1, \dots, u_m \rangle$$

Definition 4.8. Let $f \in S = k\langle x_1, \dots, x_m \rangle$. Denote by

$$D_{x_i}(f; \underline{x}, \underline{v}, \underline{u}) \in k\langle \underline{x}, \underline{v}, \underline{u} \rangle$$

the linear function in f , defined for $f = x_i$ and $f = \underline{m}x_j$, \underline{m} any monomial, by

$$\begin{aligned} D_{x_i}(x_j; \underline{x}, \underline{v}, \underline{u}) &= \delta_{i,j}v_i \\ D_{x_i}(\underline{m}x_j; \underline{x}, \underline{v}, \underline{u}) &= D_{x_i}(\underline{m}; \underline{x}, \underline{v}, \underline{u})u_j + \delta_{i,j}\underline{m}v_i \end{aligned}$$

The $D_{x_i}(f)$ is a general noncommutative differentiation symbol. We can as well define higher order differentiation by iteration of the function D , considering the $(\underline{v}, \underline{u})$ -variables as constants. Inductively we use the notation

$$D_{x_{i_n}, \dots, x_{i_2}, x_{i_1}}(f; \underline{x}, \underline{v}, \underline{u}) = D_{x_{i_n}}(D_{x_{i_{n-1}}, \dots, x_{i_1}}(f; \underline{x}, \underline{v}, \underline{u}); \underline{x}, \underline{v}, \underline{u})$$

Proposition 4.9. For $f \in S$ we have the equality in $k\langle \underline{u}, \underline{v} \rangle$, given by

$$f(\underline{u} + \underline{v}) = f(\underline{u}) + \sum_n \sum_{1 \leq i_1, \dots, i_n \leq d} D_{x_{i_n}, \dots, x_{i_2}, x_{i_1}}(f; \underline{u}, \underline{v}, \underline{u})$$

Proof. Reduce to the case where f is a monomial and use a straightforward combinatorial argument, see [13]. \square

If we put $\underline{v} = \underline{x} - \underline{u}$ into this formula we get a noncommutative Taylor expansion

$$f(\underline{x}) = f(\underline{u}) + \sum_n \sum_{(i_1, \dots, i_n) \in \{1, \dots, m\}^n} D_{x_{i_n}, \dots, x_{i_2}, x_{i_1}}(f; \underline{u}, \underline{x} - \underline{u}, \underline{u})$$

Notice that if the variable set \underline{u} is assumed to be in the center of the algebra, i.e. if we work over $k[\underline{u}]\langle \underline{x} \rangle$, the Taylor expansion takes the more familiar form,

$$f(\underline{x}) = f(\underline{u}) + \sum_n \sum_{1 \leq i_1, \dots, i_n \leq d} D_{x_{i_n}, \dots, x_{i_2}, x_{i_1}}(f; \underline{u}, \underline{u})(x_{i_n} - u_{i_n}) \dots (x_{i_1} - u_{i_1})$$

where we use the notation

$$D_{x_{i_n}, \dots, x_{i_2}, x_{i_1}}(f; \underline{x}, \underline{u}) = D_{x_{i_n}, \dots, x_{i_2}, x_{i_1}}(f; \underline{x}, (1, \dots, 1), \underline{u})$$

The connection with $D_k(f; p)$ as defined in the previous section is given by the next proposition.

Proposition 4.10. For any $f \in k\langle \underline{x} \rangle$ we have the following equality in the extended k -algebra $k[\underline{u}]\langle \underline{x} \rangle$,

$$f(\underline{x}) = f(\underline{u}) + \sum_{i=1}^m D_{x_i}(f; \underline{x}, \underline{u})(x_i - u_i)$$

Proof. Applying the Taylor formula to the polynomial $D_{x_{i_1}}(f; \underline{x}, \underline{u})$ we get the equality

$$\begin{aligned} D_{x_{i_1}}(f; \underline{x}, \underline{u}) &= D_{x_{i_1}}(f; \underline{u}, \underline{u}) \\ &+ \sum_n \sum_{1 \leq i_2, \dots, i_n \leq d} D_{x_{i_n}, \dots, x_{i_2}, x_{i_2}}(D_{x_{i_1}}(f; \underline{u}, \underline{u}); \underline{u}, \underline{u})(x_{i_n} - u_{i_n}) \dots (x_{i_2} - u_{i_2}) \end{aligned}$$

Multiplying from right by $(x_{i_1} - u_{i_1})$, adding up over all $i = 1, 2, \dots, d$ and using the Taylor formula once more gives the result. \square

Since the left decomposition is unique we it follows that

$$D_k(f; p) = D_{x_k}(f; \underline{x}, p)$$

4.3. Extension relations on plane curves. Let $A = k\langle x, y \rangle / (f)$ be a noncommutative model for the reduced algebraic plane curve,

$$C = \text{Spec}(k[x, y] / (f_0)) := \text{Simp}_1(A)$$

and define a relation \mathcal{R} on the affine plane \mathbf{A}^2 by

$$\mathcal{R} = V(D_1(f; \underline{u})(\underline{v}), D_2(f; \underline{u})(\underline{v})) \subset \mathbf{A}^2 \times \mathbf{A}^2$$

where the product is parametrised by $(\underline{u}, \underline{v})$.

Theorem 4.11. *Let $A = k\langle x, y \rangle / (f)$ and $C = \text{Simp}_1(A) = \text{Simp}_1(A_0)$.*

i) *The restriction of \mathcal{R} to C induces a relation on C , i.e.*

$$\mathcal{R} \cap (C \times \mathbf{A}^2) = C \times C$$

Moreover, $\mathcal{R} \cap \Delta C = \text{Sing}(C)$, the singular locus of C .

ii) *Let $p_1, p_2 \in C$ be two points corresponding to the simple modules $k(p_1)$ and $k(p_2)$. Then there exist a noncommutative model $A = k\langle x, y \rangle / (\tilde{f})$ of f_0 such that $\text{Ext}_A^1(k(p_1), k(p_2)) \neq 0$.*

iii) *Consider models given by $\tilde{f}_t = f + t[x, y]$. Then for generic t the projection $\text{pr}_1 : \mathcal{R} \cap (C \times C) \rightarrow C$ is dominant.*

Proof. i) By Prop. 4.4 we have for $p = (\alpha, \beta) \in C$

$$f = D_1(f; p)(x - \alpha) + D_2(f; p)(y - \beta).$$

If for $q \in \mathbf{A}^2$ we have

$$D_1(f; p)(q) = D_2(f; p)(q) = 0$$

Then this implies $f(q) = 0$, i.e. $q \in C$.

Now use Prop. 4.7 and see that for $p = q$ the above equations imply $J((f); p)(p)$ has rank zero. So $p \in C$ is a singular point.

ii) Let $\tilde{f} = f + t[x, y]$ for some $t \in k$ and let $p_1, p_2 \in C$ be two points corresponding to the simple modules $k(p_1)$ and $k(p_2)$. By Prop. 4.7 $\text{Ext}_A^1(k(p_1), k(p_2)) \neq 0$ if and only if there exists a scalar t such that

$$D_1(\tilde{f}; p_1)(p_2) = D_1(f; p_1)(p_2) + t \cdot (b_1 - b_2) = 0$$

$$D_2(\tilde{f}; p_1)(p_2) = D_2(f; p_1)(p_2) + t \cdot (a_2 - a_1) = 0$$

or equivalently

$$D_1(f; p_1)(p_2)(a_2 - a_1) + D_2(f; p_1)(p_2)(b_2 - b_1) = 0$$

Using Prop. 4.4 we identify the left hand side of this equality as $f(p_2) - f(p_1)$ and the result follows.

iii) Since $\mathcal{R} \cap \Delta = \text{Sing}(C)$ is finite, we just need to show that the k -algebra homomorphism

$$pr_1^* : k[u_1, u_2]/(f_0) \longrightarrow k[u_1, u_2, v_1, v_2]/(D_1(\tilde{f}_t; \underline{v})(\underline{v}), D_2(\tilde{f}_t; \underline{v})(\underline{v}), f_0(\underline{u}), f_0(\underline{v}))$$

is injective. Suppose it is not injective. Then there must exist

$$h_1, h_2 \in k[u_1, u_2, v_1, v_2]$$

such that

$$\begin{aligned} & h_1 D_1(\tilde{f}_t; \underline{v})(\underline{v}) + h_2 D_2(\tilde{f}_t; \underline{v})(\underline{v}) \\ &= h_1 (D_1(f; \underline{v})(\underline{v}) + t(u_2 - v_2)) + h_2 (D_2(f; \underline{v})(\underline{v}) + t(v_1 - u_1)) \\ &= h_1 D_1(f; \underline{v})(\underline{v}) + h_2 D_2(f; \underline{v})(\underline{v}) + t(h_1(u_2 - v_2) + h_2(v_1 - u_1)) \\ &\in (u_1, u_2) \end{aligned}$$

For this to be true for generic t , we must have

$$h_1(u_2 - v_2) + h_2(v_1 - u_1) \in (u_1, u_2)$$

The indeterminants v_1 and v_2 are linearly independent over $k[u_1, u_2]$, hence there must exist $h \in k[\underline{u}, \underline{v}]$ such that $h_1 = h \cdot (v_1 - u_1)$, $h_2 = h \cdot (v_2 - u_2)$. But then, Prop. 4.10 implies

$$\begin{aligned} & h_1 D_1(\tilde{f}_t; \underline{v})(\underline{v}) + h_2 D_2(\tilde{f}_t; \underline{v})(\underline{v}) \\ &= h(D_1(\tilde{f}_t; \underline{v})(\underline{v})(v_1 - u_1) + D_2(\tilde{f}_t; \underline{v})(\underline{v})(v_2 - u_2)) = h \cdot f_0(u_1, u_2) \end{aligned}$$

proving the injectivity of the map. \square

5. LIFTING FACTORISATIONS TO $k\langle x, y \rangle$

There is no general unique factorisation theorem in $k\langle x, y \rangle$, e.g. the two factorisations

$$(xy + 1)x = x(yx + 1)$$

are different. Moreover, a factorisation $f = gh$ in $k[x, y]$ of an element $f \in k\langle x, y \rangle$ cannot always be lifted back to $k\langle x, y \rangle$.

Let $A = k\langle x, y \rangle / (f)$ and $A_0 = k[x, y] / (f_0)$ the natural commutativisation. Put $C_f := \text{Simp}_1(A)$.

Proposition 5.1. *Let $f \in k\langle x, y \rangle$ be reducible with factorisation $f = gh$. Then for all $p_1 \in C_h$ and $p_2 \in C_g$ we have*

$$\text{Ext}_A^1(k(p_1), k(p_2)) \neq 0$$

where $A = k\langle x, y \rangle / (f)$.

Proof. Let $p_1 = (\alpha_1, \beta_1)$ and $p_2 = (\alpha_2, \beta_2)$. Then

$$D_i(f; p_1)(p_2) = g(p_2)D_i(f; p_1)(p_2) + D_i(g; p_1)(p_2)h(p_1) = 0$$

for $i = 1, 2$. As shown in section 4.1 this implies that $\text{Ext}_A^1(k(p_1), k(p_2)) \neq 0$. \square

We are interested in some sort of converse to this statement. Can we from geometrical data connected to the curve C_f decide whether or not there exists a factorisation $f = gh \in k\langle x, y \rangle$? We shall not try to give a complete solution to this problem, but in the following theorem we give a partial answer.

Theorem 5.2. *Let $f \in k\langle x, y \rangle$ be such that f_0 is reduced and has a proper factorisation $f_0 = gh$ in $k[x, y]$. Assume*

$$\text{Ext}_A^1(k(p_1), k(p_2)) \neq 0$$

for all $p_1 \in C_h$, $p_2 \in C_g$ and where $A = k\langle x, y \rangle / (f)$. Then there exist non-commutative models \hat{g} and \hat{h} of g resp. h , such that

$$f - \hat{g}\hat{h} \in ([x, y])^2$$

Proof. Initially we choose non-commutative models \hat{g} and \hat{h} for g and h such that the degree of the models do not exceed the degree of the original elements. Then we have

$$f - \hat{g}\hat{h} = \sum_{i=1}^l g_i[x, y]h_i$$

for some $g_i, h_i \in k\langle x, y \rangle$ and such that $\deg(g_i) + \deg(h_i) < \deg(f) - 1$ for all $i = 1, \dots, l$. For $p_1 = (\alpha_1, \beta_1) \in C_h$ and $p_2 = (\alpha_2, \beta_2) \in C_g$ we have

$$\begin{aligned} D_1(f; p_1)(p_2) &= \sum g_i(\alpha_2, \beta_2)(\beta_1 - \beta_2)h_i(\alpha_1, \beta_1) \\ D_2(f; p_1)(p_2) &= \sum g_i(\alpha_2, \beta_2)(\alpha_2 - \alpha_1)h_i(\alpha_1, \beta_1) \end{aligned}$$

By assumption $D_1(f; p_1)(p_2) = D_2(f; p_1)(p_2) = 0$. Since C_g and C_h are non-empty curves with no common component this implies that

$$(*) \quad \sum_{i=1}^l g_i(\alpha_2, \beta_2)h_i(\alpha_1, \beta_1) = 0, \quad (\alpha_2, \beta_2) \in C_g, (\alpha_1, \beta_1) \in C_h$$

Let us first treat the case $l = 1$. Then we have $g(\alpha_2, \beta_2)h(\alpha_1, \beta_1) = 0$ for all $(\alpha_2, \beta_2) \in C_g, (\alpha_1, \beta_1) \in C_h$ and $\deg(h_1) < \deg(h)$ or $\deg(g_1) < \deg(g)$. If $\deg(g_1) < \deg(g)$ there exists $(\alpha_2, \beta_2) \in C_g$ such that $g_1(\alpha_2, \beta_2) \neq 0$ since f is reduced. Hence $h_1(\alpha_1, \beta_1) = 0$ for all $(\alpha_1, \beta_1) \in C_h$ and $h_1 - k_1\hat{h} \in ([x, y])$ for some $k_1 \in k\langle x, y \rangle$ and

$$f = \hat{g}\hat{h} + g_1[x, y]k_1\hat{h} + ([x, y])^2$$

hence the claim holds, since $\hat{g} + \hat{g}_1[x, y]k_1$ is a non-commutative model for g .

Assume

$$\sum_{i=1}^l g_i(\alpha_2, \beta_2)h_i(\alpha_1, \beta_1) = 0$$

for all $(\alpha_1, \beta_1) \in C_h$ and for all $(\alpha_2, \beta_2) \in C_g$. By symmetry we can assume $\deg(h_1) < \deg(h)$ and choose (α_0, β_0) such that $h_1(\alpha_0, \beta_0) \neq 0$. Then for all α_2 and β_2

$$g_1(\alpha_2, \beta_2) = \sum_{i=2}^l \lambda_i g_i(\alpha_2, \beta_2)$$

for some $\lambda_i \in k$. Inserting this in (*) we get

$$\sum_{i=2}^l (h_i(\alpha_1, \beta_1) + \lambda_i h_1(\alpha_1, \beta_1))g_i(\alpha_2, \beta_2) = 0$$

for all α_2 and β_2 . If we have started with “ h_1 ” of minimal degree we get

$$\deg(h_i + \lambda_i h_1) \leq \deg(h_i)$$

This procedure will of course lead us to the case $l = 1$ which was treated above. \square

6. THE COMPLETION PROBLEM

The main result of this section gives a partial answer to the following problem, which indecomposable modules deform into simple modules?

6.1. Graphs with no complete cycles.

Definition 6.1. Let A be a k -algebra and $\mathcal{V} = \{V_1, \dots, V_r\}$ a finite set of simple A -modules. The directed extension graph $Q\mathcal{V}$ of \mathcal{V} has the modules V_i as its vertices and there is an arrow from V_i to V_j if and only if $\text{Ext}_A^1(V_i, V_j) \neq 0$.

By a cycle in the extension graph $Q\mathcal{V}$ we shall mean a path, starting and ending at the same vertex. A cycle is said to be complete if it contains all the vertices of $Q\mathcal{V}$. The successor set $E(\nu)$ of a vertex ν is the set of vertices ω of $Q\mathcal{V}$ such that there is a path from ν to ω , and the precursor set $P(\omega)$ consists of all vertices $\nu \in \mathcal{V}$ such that there is a path from ν to ω . Notice that we always include ν both in the successor and the precursor set of ν and that $E(\nu) = \mathcal{V}$ for all vertices $\nu \in \mathcal{V}$ if and only if there exists a complete cycle in \mathcal{V} .

Lemma 6.2. Let $Q\mathcal{V}$ be an extension graph with no complete cycles. Then there exists a disjoint union $\mathcal{V} = \mathcal{M} \cup \mathcal{N}$ such that there are no arrows (in $Q\mathcal{V}$) from $Q\mathcal{M}$ to $Q\mathcal{N}$.

Proof. Since there are no complete cycles in $Q\mathcal{V}$ there exists a vertex ν such that $E(\nu) \neq \mathcal{V}$. Let $\mathcal{M} = E(\nu)$ and $\mathcal{N} = \mathcal{V} - \mathcal{M}$. Then $\mathcal{V} = \mathcal{M} \cup \mathcal{N}$ is a disjoint union and since there are no arrows out of a successor set, there are no arrows from $Q\mathcal{M}$ to $Q\mathcal{N}$. \square

6.2. Homomorphisms and extensions of indecomposable modules. Let E be a non-simple, indecomposable A -module with a finite composition series. Let the cofiltration

$$E = E_r \rightarrow E_{r-1} \rightarrow \dots \rightarrow E_1 \rightarrow E_0 = 0$$

be a cocomposition series of E and let $V_i = \ker\{E_i \rightarrow E_{i-1}\}$, $i = 1, 2, \dots, r$ be the simple kernels. The set $\mathcal{V} = \{V_1, \dots, V_r\} =: \text{Supp}(E)$ is called the support of E . It only depends on E and not on the cofiltration. Let Γ be an ordering of the elements of \mathcal{V} ,

$$\Gamma = (V_{i_1}, V_{i_2}, \dots, V_{i_r})$$

An A -module F with a finite cofiltration

$$F = F_r \rightarrow F_{r-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 = 0$$

such that $\ker\{F_j \rightarrow F_{j-1}\} = V_{i_j}$, $j = 1, 2, \dots, r$ is called an **iterated extension** of $\text{Supp}(E)$ defined by the ordering Γ . The set of all iterated A -module extensions defined by Γ is denoted $\text{Ind}_\Gamma(A)$. It has an affine non-commutative scheme structure (see [13]).

Lemma 6.3. Let M, N be finite dimensional A -modules with disjoint support. Then

$$\text{Hom}_A(M, N) = 0$$

Proof. The proof is by induction on the cardinality of $\text{Supp}(N)$, resp. $\text{Supp}(M)$.

Assume first that M and N are simple modules. By Schurs lemma

$$\text{Hom}_A(M, N) = 0$$

since M and N are non-isomorphic. Let M be simple and $V \subseteq N$. Then there is a short-exact sequence

$$0 \rightarrow V \rightarrow N \rightarrow N/V \rightarrow 0$$

inducing a long-exact sequence

$$0 \rightarrow \text{Hom}_A(M, V) \rightarrow \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, N/V) \rightarrow \text{Ext}_A^1(M, V) \rightarrow \dots$$

By induction hypothesis $\text{Hom}_A(M, V) = \text{Hom}_A(M, N/V) = 0$ and therefore also $\text{Hom}_A(M, N) = 0$.

A similar argument gives the induction step for $\text{Hom}_A(-, N)$. \square

Lemma 6.4. *Let E be a non-simple, indecomposable A -module given by a short-exact sequence*

$$0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0$$

with $\text{Supp}(M) \cap \text{Supp}(N) = \emptyset$. Suppose $\text{End}_A(M) \simeq \text{End}_A(N) \simeq k$. Then $\text{End}_A(E) \simeq k$.

Proof. We have the following diagram of k -vector spaces

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Hom}_A(N, M) & \longrightarrow & \text{Hom}_A(N, E) & \longrightarrow & \text{Hom}_A(N, N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Hom}_A(E, M) & \longrightarrow & \text{Hom}_A(E, E) & \longrightarrow & \text{Hom}_A(E, N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Hom}_A(M, M) & \longrightarrow & \text{Hom}_A(M, E) & \longrightarrow & \text{Hom}_A(M, N) \end{array}$$

By Lemma 6.3

$$\text{Hom}_A(N, M) = \text{Hom}_A(M, N) = 0$$

and there are inclusions

$$\begin{aligned} 0 &\rightarrow \text{Hom}_A(N, E) \rightarrow \text{Hom}_A(N, N) \simeq k \\ 0 &\rightarrow \text{Hom}_A(E, M) \rightarrow \text{Hom}_A(M, M) \simeq k \end{aligned}$$

Since the given short-exact sequence is not split, this implies that

$$\text{Hom}_A(N, E) = \text{Hom}_A(E, M) = 0$$

Thus $\text{End}_A(E) \simeq k$. □

We have the following characterisation of indecomposable modules.

Proposition 6.5. *Let the A -module E be an iterated extension such that $\text{Supp}(E)$ contains no multiple simple modules. Then E is indecomposable if and only if $\text{End}_A(E) \simeq k$.*

Proof. The only if part follows from Lemma 6.4 and a straightforward induction argument. If E is decomposable we can write $E \simeq E' \oplus E''$ as a proper sum, proving that $\dim_k \text{End}_A(E) \geq 2$. □

In addition to these general facts about homomorphisms we also need some basic results about infinitesimal homomorphisms, i.e. about the various ext-groups.

Lemma 6.6. *Let $\mathcal{V} = \{V_1, \dots, V_r\}$ be a family of simple A -modules and let $Q\mathcal{V}$ be the corresponding extension graph. Suppose there are no complete cycles in $Q\mathcal{V}$, and choose $\nu \in Q\mathcal{V}$ such that $E(\nu) \neq \mathcal{V}$. If M an A -module with $\text{Supp}(M) = E(\nu)$ and N an A -module such that $\text{Supp}(N) \cap E(\nu) = \emptyset$, then*

$$\text{Ext}_A^1(M, N) = 0$$

Proof. Let

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_m = M$$

be a filtration of M such that

$$M_i/M_{i-1} \simeq V_{j_i}, \quad i = 1, 2, \dots, m$$

Let

$$0 = N_0 \subset N_1 \subset N_2 \subset \dots \subset N_n = N$$

be a similar filtration of N . We claim that

$$\text{Ext}_A^1(M_i, N_k) = 0 \quad i = 1, 2, \dots, m \quad k = 1, 2, \dots, n$$

In fact there are long-exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_A(V_{j_i}, N_k) &\longrightarrow \text{Hom}_A(M_i, N_k) \longrightarrow \text{Hom}_A(M_{i-1}, N_k) \rightarrow \\ \text{Ext}_A^1(V_{j_i}, N_k) &\longrightarrow \text{Ext}_A^1(M_i, N_k) \longrightarrow \text{Ext}_A^1(M_{i-1}, N_k) \end{aligned}$$

Now by Lemma 6.3 $\text{Hom}_A(M_{i-1}, N_k) = 0$ since $\text{Supp}(M_{i-1}) \cap \text{Supp}(N_k) = \emptyset$. For any $V_s \in \text{Supp}(M)$ and $V_t \in \text{Supp}(N)$ we have by assumption $\text{Ext}_A^1(V_s, V_t) = 0$ and an induction argument using the long-exact sequence above and similar sequences for the extensions in N gives the result. \square

6.3. A completion theorem. Let $\mathcal{V} = \{V_1, \dots, V_r\}$ be a finite family of simple A -modules such that the extension graph $Q\mathcal{V}$ has no complete cycles. By Lemma 6.2 there is a disjoint union $\mathcal{V} = \mathcal{M} \cup \mathcal{N}$ with no arrows in $Q\mathcal{V}$ from $Q\mathcal{M}$ to $Q\mathcal{N}$.

Definition 6.7. Let \mathcal{V} be a finite family of simple A -modules and let $\mathcal{V} = \mathcal{M} \cup \mathcal{N}$ be a disjoint union with no arrows in $Q\mathcal{V}$ from $Q\mathcal{M}$ to $Q\mathcal{N}$. An ordering $\Gamma = (V_{j_1}, \dots, V_{j_r})$ of \mathcal{V} is said to respect the union $\mathcal{V} = \mathcal{M} \cup \mathcal{N}$ if for some integer l $V_{j_1}, \dots, V_{j_l} \in \mathcal{N}$ and $V_{j_{l+1}}, \dots, V_{j_r} \in \mathcal{M}$.

The set of indecomposable A -modules defined as iterated extensions of the simple modules V_1, \dots, V_r , in accordance with the ordering Γ , is denoted $\text{Ind}_\Gamma(A)$ (see [13] for details).

The next proposition is crucial for the Completion Theorem 6.11. It will be proved in section 6.5.

Proposition 6.8. Let E be an indecomposable A -module with $\text{Supp}(E) = \mathcal{V}$ and $\text{End}_A(E) \simeq k$ and suppose there are no complete cycles in $Q\mathcal{V}$. Then there exists a short-exact sequence

$$0 \rightarrow M \longrightarrow E \longrightarrow N \rightarrow 0$$

such that there are no arrows in $Q\mathcal{V}$ from $Q\mathcal{M}$ to $Q\mathcal{N}$, where $\mathcal{M} = \text{Supp}(M)$ and $\mathcal{N} = \text{Supp}(N)$.

By choosing any cofiltration of M and N we see that there is an equivalent conclusion for this proposition, saying that with the same assumptions, there exists a Γ which respects the union $\mathcal{V} = \mathcal{M} \cup \mathcal{N}$ and such that $E \in \text{Ind}_\Gamma(A)$.

Let Γ be some ordering of \mathcal{V} . Denote by

$$\text{Simp}_\Gamma(A) = \text{Ind}_\Gamma(A) \cup \text{Simp}_n(A)$$

where n is the k -dimension of the A -modules in $\text{Ind}_\Gamma(A)$. Laudal has shown (see [13]) that $\text{Simp}_\Gamma(A)$ has a non-commutative scheme structure, called a completion of $\text{Simp}_n(A)$, represented by a sheaf of k -algebras. There are of course several completions of $\text{Simp}_n(A)$, but we do not know whether or not there exists a non-commutative scheme which at the same time parametrises all possible completions of $\text{Simp}_n(A)$. Nevertheless, for any indecomposable A -module with $\text{End}_A(E) \simeq k$ there exists a formal moduli $H(E)$ for the deformation functor Def_E .

For a short-exact sequence

$$\xi : \quad 0 \rightarrow M \longrightarrow E \longrightarrow N \rightarrow 0$$

we consider the functor Def_ξ , containing all liftings of E which preserves the extension structure, i.e. liftings of E which are extensions of liftings of N by liftings of M . A

more precise description can be found in section 6.4. The formal moduli for Def_ξ is denoted $H(\xi)$ and there is a map of formal k -algebras

$$\tilde{\xi} : H(E) \longrightarrow H(\xi)$$

The kernel of this map is our candidate for deformations pointing into the simple locus. But even if $\tilde{\xi}$ induces an injection on tangent level, E may deform into simple modules. To avoid such “singularities” in the completion of $Simp_n(A)$ at E we make the following definition.

Definition 6.9. *An indecomposable A -module E with $End_A(E) \simeq k$ is called a smooth completion point of $Simp_n(A)$ if for all non-split exact sequences*

$$\xi : \quad 0 \rightarrow M \longrightarrow E \longrightarrow N \rightarrow 0$$

the map $\tilde{\xi}$ is not a monomorphism on the tangent level.

The last ingredient of the completion theorem is the bridge between deformation theory and the homological properties of the extension graph $Q\mathcal{V}$. It is built in the next proposition. The proof is postponed to section 6.4.

Proposition 6.10. *Suppose the indecomposable A -module E is a smooth completion point of $Simp_n(A)$, and let*

$$\xi : \quad 0 \rightarrow M \longrightarrow E \longrightarrow N \rightarrow 0$$

be some exact sequence presenting E . Then

$$Ext_A^1(M, N) \neq 0$$

The main result of this sections is the completion theorem, giving a criterion for when the indecomposable A -module E deforms into simple modules.

Theorem 6.11. *Let A be an associative k -algebra and E an indecomposable A -module, such that $End_A(E) \simeq k$. Let $\mathcal{V} = Supp(E) = \{V_1, \dots, V_r\}$ be the associated simple modules and let $Q\mathcal{V}$ be the extension graph. Suppose E is a smooth completion point of $Simp_n(A)$. Then there exist a complete cycle in $Q\mathcal{V}$.*

Proof. Assume for a contradiction that there are no complete cycles in $Q\mathcal{V}$. Then by Lemma 6.2 there is a disjoint union $\mathcal{V} = \mathcal{M} \cup \mathcal{N}$ with no arrows in $Q\mathcal{V}$ from $Q\mathcal{M}$ to $Q\mathcal{N}$. Then there exist a short-exact sequence

$$\xi : \quad 0 \rightarrow M \longrightarrow E \longrightarrow N \rightarrow 0$$

There are no arrows in $Q\mathcal{V}$ from $Q\mathcal{M}$ to $Q\mathcal{N}$ and by Proposition 6.6 this implies that

$$Ext_A^1(M, N) = 0$$

contradicting the assertion of Proposition 6.10. □

Notice that the existence of a complete cycle is not a sufficient condition for existence of simple modules of the given dimension. Consider e.g. the plane curve given by the equation $f = x^2 - 1 + [x, y]^2 y$. It has complete 2-cycles, but there are no 2-dimensional simple modules. In fact it is easy to construct examples of two k -algebras A and A' such that $Simp_1(A) \simeq Simp_1(A')$ and with isomorphic extension graphs, but such that the higher dimensional simple structures are completely different.

6.4. Proof of Proposition 6.10. Let N and M be left A -modules and consider an extension E of N by M , given by the short-exact sequence

$$\xi : \quad 0 \rightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} N \rightarrow 0$$

Let \underline{l} denote the category of commutative local artinian k -algebras with residue field k . We associate to the exact sequence ξ a deformation functor

$$Def_{\xi} : \underline{l} \rightarrow \underline{Sets}$$

defined as follows. For any pointed local artinian k -algebra $\rho : R \rightarrow k$ the set $Def_{\xi}(R)$ consists of isomorphism classes of short-exact sequences of left $A \otimes R^{op}$ -modules

$$\xi_R : \quad 0 \rightarrow M \otimes R \xrightarrow{\iota_R} E \otimes R \xrightarrow{\pi_R} N \otimes R \rightarrow 0$$

such that the A -module structure of $E \otimes R$, given by a k -algebra homomorphism

$$\eta'_E : A \rightarrow End_R(E \otimes R)$$

is compatible with some A -module structure of $N \otimes R$ and $M \otimes R$, and such that $\xi_R \otimes_R k \simeq \xi$.

There is a forgetful morphism of deformation functors

$$Def_{\xi} \rightarrow Def_E$$

inducing a k -linear map of tangent spaces

$$Def_{\xi}(k[\epsilon]) \rightarrow Def_E(k[\epsilon]) \simeq Ext_A^1(E, E)$$

The image of the map is denoted $Ext_A^1(E, E)_{\xi} \subset Ext_A^1(E, E)$. Our concern is to determine when the map is surjective.

An element $\alpha \in Def_{\xi}(k[\epsilon])$ is given by a triple $\alpha = (\beta, \eta, \mu)$, where

$$\beta \in Ext_A^1(E, E) \quad \eta \in Ext_A^1(M, M) \quad \mu \in Ext_A^1(N, N)$$

and such that $\iota_*(\eta) = \iota^*(\beta)$ and $\pi^*(\mu) = \pi_*(\beta)$.

$$\begin{array}{ccccc} Ext_A^1(M, M) & & Ext_A^1(E, E) & & Ext_A^1(N, N) \\ \downarrow \iota_* & \swarrow \iota^* & \downarrow \chi & \searrow \pi_* & \downarrow \pi^* \\ Ext_A^1(M, E) & & & & Ext_A^1(E, N) \\ & \searrow \pi_* & & \swarrow \iota^* & \\ & & Ext_A^1(M, N) & & \end{array}$$

Referring to the diagram we put $\chi := \iota^* \circ \pi_* = \pi_* \circ \iota^*$.

Lemma 6.12. *If $Ext_A^1(E, E)_{\xi} \neq Ext_A^1(E, E)$, then $Ext_A^1(M, N) \neq 0$.*

Proof. Assume $Ext_A^1(M, N) = 0$. Then the above diagram reduces to

$$\begin{array}{ccccc} Ext_A^1(M, M) & & Ext_A^1(E, E) & & Ext_A^1(N, N) \\ \downarrow \iota_* & \swarrow \iota^* & & \searrow \pi_* & \downarrow \pi^* \\ Ext_A^1(M, E) & & & & Ext_A^1(E, N) \end{array}$$

where ι_* and π^* are surjections. Thus for any $\beta \in Ext_A^1(E, E)$ there exist $\eta \in Ext_A^1(M, M)$ and $\mu \in Ext_A^1(N, N)$ such that $\iota_*(\eta) = \iota^*(\beta)$ and $\pi^*(\mu) = \pi_*(\beta)$. Consequently $Ext_A^1(E, E)_{\xi} = Ext_A^1(E, E)$. \square

Proof of Proposition 6.10. The condition of the lemma says that the map

$$\tilde{\xi} : H(E) \longrightarrow H(\xi)$$

is not a monomorphism on tangent level, i.e. E is a smooth completion point of $\text{Simp}_n(A)$. The conclusion is the non-vanishing of $\text{Ext}_A^1(M, N)$. \square

6.5. Proof of Proposition 6.8. In this section we shall give a proof of Proposition 6.8. We start with the so-called Moving Lemma. It says that when certain ext-group vanishes, we can “move” components of the cofiltration of a given indecomposable module.

Lemma 6.13. *Consider the diagram of short-exact sequences of A -modules*

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & N & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & N' & & \\ 0 & \longrightarrow & W & \longrightarrow & N' & \longrightarrow & W' \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Suppose $\text{Ext}_A^1(W, N) = 0$. Then there exists a similar diagram with W and N interchanged:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & W & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow \\ & & & & \downarrow & & \\ & & & & N' & & \\ 0 & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & W' \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Proof. Let L_1 be the kernel of the composed map

$$M'' \longrightarrow N' \longrightarrow W'$$

A simple diagram chasing argument shows that L_1 is an extension of W by N , i.e. there is a short-exact sequence

$$0 \rightarrow N \rightarrow L_1 \rightarrow W \rightarrow 0$$

By assumption this sequence splits and there is a section $s : W \rightarrow L_1$ which composed with the surjection $L_1 \rightarrow W$ gives the identity on W . The composed map

$$W \xrightarrow{s} L_1 \rightarrow M''$$

is clearly injective. Let N'' be the cokernel. The composed map

$$N \longrightarrow M'' \longrightarrow N''$$

is again injective, with cokernel W' and we are done. \square

Let $\mathcal{V} = \{V_1, \dots, V_r\}$ be any set of simple A -modules with extension graph $Q\mathcal{V}$. Let E be any A -module with $\text{Supp}(E) = \mathcal{V}$. Let $\mathcal{V}_0 \subset \mathcal{V}$ be a subfamily such that there are no arrows in \mathcal{V} from any element of \mathcal{V}_0 to any element outside \mathcal{V}_0 . i.e. if $V_s \in \mathcal{V}_0$ and $V_t \notin \mathcal{V}_0$, then $\text{Ext}_A^1(V_s, V_t) = 0$.

Lemma 6.14. *Let E be an A -module with $\text{Supp}(E) = \mathcal{V}$. Let $\mathcal{V}_0 \subset \mathcal{V}$ be a subfamily as described above, with no arrows out of \mathcal{V}_0 . Then there exists a submodule $E' \subset E$ such that $\text{Supp}(E') = \mathcal{V}_0$.*

Proof. Let

$$E = E_r \xrightarrow{f_r} E_{r-1} \longrightarrow \dots \xrightarrow{f_1} E_1$$

be a cofiltration of E , where f_i is surjective with kernel

$$\ker(f_i) = V_i \quad i = 1, 2, \dots, r$$

and where we put $K_i = \ker\{E \rightarrow E_i\}$, $i = 1, \dots, r$.

Suppose $V_{i-1} \in \mathcal{V}_0$ and $V_i \notin \mathcal{V}_0$. Applying Lemma 6.13 to the diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & V_i & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & K_i & \longrightarrow & E & \longrightarrow & E_i & \longrightarrow & \\ & & & & \downarrow & & \\ & & & & 0 & \longrightarrow & V_{i-1} & \longrightarrow & E_{i-1} & \longrightarrow & E_{i-2} & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

we obtain a new cofiltration corresponding to Γ' derived from Γ by interchanging V_{i-1} and V_i . Repeat this procedure until there are no members of \mathcal{V}_0 preceding any non-members of \mathcal{V}_0 . Let Γ'' be the corresponding ordering. Then $E' = \ker\{E \rightarrow E''_j\}$ for some specific j will do the job. \square

Notice that the same argument proves the “dual” lemma:

Lemma 6.15. *Let E be a A -module with $\text{Supp}(E) = \mathcal{V} = \{V_1, \dots, V_r\}$. Let $\mathcal{V}_1 \subset \mathcal{V}$ be a subfamily, with no arrows into \mathcal{V}_1 , i.e. for any $V_s \notin \mathcal{V}_1$ and $V_t \in \mathcal{V}_1$, we have $\text{Ext}_A^1(V_s, V_t) = 0$. Then there exists a quotient module $E \rightarrow E''$ such that $\text{Supp}(E'') = \mathcal{V}_1$.*

Proof. A dual, but quite similar argument as in the proof of lemma 6.14. \square

Proof of Proposition 6.8. Let

$$0 \rightarrow M \longrightarrow E \longrightarrow N \rightarrow 0$$

be some extension defining E . If there are no arrows in $Q\mathcal{V}$ from QM to QN we are done.

Assume in contrary that there exist an arrow in $Q\mathcal{V}$ from $Q\mathcal{M}$ to $Q\mathcal{N}$. Let $\nu \in Q\mathcal{N}$ be the target of this arrow and consider the successor set $E(\nu)$. By Lemma 6.14 there is a submodule $N' \hookrightarrow N$ with $Supp(N') = E(\nu)$ and a diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & N' & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & N/N' & & \\
 & & & & \downarrow & & 0 & &
 \end{array}$$

The kernel of the composed map

$$K = \ker\{E \longrightarrow N \longrightarrow N/N'\}$$

is an extension of N' by M

$$0 \rightarrow M \rightarrow K \rightarrow N' \rightarrow 0$$

and there is of course an exact sequence

$$0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$$

where $Q = N/N'$ is the quotient module. By construction

$$Supp(K) = Supp(M) \cap E(\nu)$$

We have “moved” the successor graph $E(\nu)$ from the epi side to the mono side of certain extension sequences. Repeat this procedure, i.e. construct new extension sequences

$$0 \rightarrow K_j \rightarrow E \rightarrow Q_j \rightarrow 0 \quad j = 1, 2, \dots, l$$

until there are no more arrows in the extension graph from the mono side to the epi side. The precise meaning of the last statement is that for any $V_s \in Supp(K_i)$ and $V_t \in Supp(Q_i)$, we have $Ext_A^1(V_s, V_t) = 0$.

There is a dual procedure to the one described, moving a precursor subgraph $P(\omega)$ from the mono side to the epi side.

The procedures work as long as $E(\nu) \neq \mathcal{N}$ or $P(\omega) \neq \mathcal{M}$. If $E(\nu) = \mathcal{N}$ and $P(\omega) = \mathcal{M}$ the procedures would in fact collapse the exact sequence. So suppose $E(\nu) = \mathcal{N}$ and $P(\omega) = \mathcal{M}$ for all $\nu \in Q\mathcal{N}$ and $\omega \in Q\mathcal{M}$. There are two possibilities:

1) There are only arrows from mono to epi side, i.e. for any $V_s \in Supp(N)$ and $V_t \in Supp(M)$, we have $Ext_A^1(V_s, V_t) = 0$. By lemma 6.6 this implies that $Ext_A^1(N, M) = 0$ and the sequence splits. Thus the short-exact sequence

$$0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$$

will do the job.

2) There are arrows in both directions. But then it is easily seen that there exists a complete cycle, in contradiction to our assumptions.

□

7. THE GEOMETRY OF PLANE NONCOMMUTATIVE CURVES

To illustrate the theory of the previous sections we consider non-commutative models of plane quadrics and cubics, their finite-dimensional simple modules and the relations defined in the previous sections.

7.1. Classification of quadrics. Up to a linear shift of basis a general plane noncommutative quadric can be written

$$A = k\langle x, y \rangle / (\lambda_1 x^2 + \lambda_2 y^2 + \delta[x, y] + ex + fy + g)$$

with coefficients in k . Going through all possibilities for the coefficients we achieve the following classification of plane noncommutative quadrics, up to isomorphism.

- 1) The smooth case, $x^2 + y^2 - 1 + \delta[x, y]$. This curve is isomorphic to the quantised Weyl algebra $xy - qyx - 1$, whith $q = \frac{\delta+i}{\delta-i}$ for $\delta \neq i$.
- 2) The singular case $x^2 + y^2 + \delta[x, y]$ which is better known as the quantum plane $xy - qyx$ with $q \neq 1$ and the same relation between q and δ as above.
- 3) The degenerate cases, including two parallell lines $x^2 + ex + \delta[x, y]$, a double line ($e = 0$) and a simple line $x + \delta[x, y]$.
- 4) The first Weyl algebra $1 + \delta[x, y]$, $\delta \neq 0$.
- 5) The affine plane $[x, y]$

The Weyl algebra has no finite-dimensional representations at all and the geometry of the commutative polynomial ring in two variables is well-known. The remaining cases are 1)-3) and we shall treat each case separately.

7.2. The quantum plane. For the quantum plane $xy = qyx$, $q \neq 1$, the following result holds.

Proposition 7.1. *Let $A = k\langle x, y \rangle / (xy - qyx)$, where we assume $q \neq 1$. If q is a primitive m 'th root of unity, the only simple modules of A of dimension strictly greater than one, are of dimension m and they correspond to $(\lambda, \gamma) \in k^* \times k^*$, given by*

$$x \mapsto \begin{bmatrix} \lambda & & & \\ & q\lambda & & \\ & & \ddots & \\ & & & q^{m-1}\lambda \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & & & \gamma \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix}$$

For other q all finite dimensional simple representations are 1-dimensional.

Proof. If $q \neq 0$ the simple modules are well known (see e.g. [7] or [9]). For $q = 0$ the defining relation is

$$xy = 0$$

Now for a simple left $k\langle x, y \rangle / (xy)$ -module V the submodule yV is either 0 or V . Therefore

$$\text{Simp}_1(A) = k \cup k \quad \text{and} \quad \text{Simp}_n(A) = \emptyset \quad n > 1$$

□

The relation for this curve is the variety $\mathcal{R} \subset C \times C \subset \mathbf{A}^4$ given by the k -algebra $(k[x, y]/(f) \otimes k[u, v]/(f)) / (qy - v, x - qu)$. The relation is of degree 1 and given by the map

$$\Phi(u, v) = \begin{cases} (0, q^{-1}v) & \text{if } u = 0 \\ (qu, 0) & \text{if } v = 0 \end{cases}$$

It is of finite order m if and only if q is a primitive m 'th root of unity. In that case the completion of the simple locus $\text{Simp}_n(A)$ is precisely the multi-extensions given by the finite cycles in the extension graphs. Thus $\text{Simp}_1(A)$ is singular and $\text{Simp}_m(A)$ is smooth (if it is non-empty).

7.3. The smooth quadric. For the smooth case we have a similar situation. The noncommutative “conic section ”

$$x^2 + y^2 - 1 + \delta[x, y] = 0$$

is isomorphic to the quantized Weyl algebra

$$xy - qyx = 1$$

with $q = \frac{\delta-i}{\delta+i}$. Substituting $\tilde{y} = y - \frac{1}{1-q}x^{-1}$ we can transform the quantum plane into the quantised Weyl algebra (the cases where x does not act as an automorphism are treated in [9]). Thus for q an m 'th root of unity we have simple modules of dimension m . In this case both $Simp_1(A)$ and $Simp_m(A)$ are smooth.

For $\delta^2 \neq -1$ the relation \mathcal{R} is well-defined and of degree 1 on C . It defines a linear map

$$\Phi \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{\delta^2-1}{\delta^2+1} & \frac{-2\delta}{\delta^2+1} \\ \frac{2\delta}{\delta^2+1} & \frac{\delta^2-1}{\delta^2+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Suppose the ground field $k = \mathbf{C}$ is the complex numbers. Then for δ real the relation is precisely rotation by a fixed angle on a circle. For δ imaginary with $\delta \neq \pm i$ the relation is just hyperbolic rotation by a certain hyperbolic angle.

Notice also that if q is a primitive root of 1, $Ext_A^1(V, W) = 0$ if V or W are non-isomorphic m -dimensional simple modules, by Proposition 2.2.

As a brief remark we like to pay attention to the algebra given by the relation $xy + q[x, y]^2yx = 1$ with $q \neq 0$. This is an other model for the smooth affine curve $xy - 1$. In this case there are no non-trivial extensions between 1-dimensional representations.

Let

$$\phi : k\langle x, y \rangle / (xy + q[x, y]^2yx - 1) \rightarrow M_2(\Gamma)$$

be a 2-dimensional representation. By the Cayley-Hamilton theorem we can write

$$\phi(xy + q[x, y]^2yx - 1) = XY - qd_{[X, Y]}YX - I$$

where $\phi(x) = X$ and $\phi(y) = Y$. Now X and Y with the relation $XY - qd_{[X, Y]}YX - I = 0$ generates $M_2(k(p))$ for $p \in Simp_1(\Gamma)$ if and only if $-qd_{[X, Y]} = 1$. Using this fact and lemma 3.2 we get

$$\begin{aligned} XY - qd_{[X, Y]}YX - 1 &= XY - qd_{[X, Y]}(-XY + t_X Y + t_Y X + t_{XY} - t_X t_Y) - 1 \\ &= (1 + qd_{[X, Y]})XY - qd_{[X, Y]}(t_X Y + t_Y X + t_{XY} - t_X t_Y) - 1 \end{aligned}$$

and it follows that $qd_{[X, Y]} = -1$, $t_X = t_Y = 0$ and $t_{XY} = 1$. Substituting these values into the Formanek element we obtain the equality $4d_X d_Y = 1 - \frac{1}{q}$. Thus for $q \neq 1$ $Simp_2(R)$ is a smooth curve. In the singular case $q = 1$ $Simp_2(R)$ consists of two lines with a normal crossing.

7.4. The degenerate case. For the simple line $x + \delta[x, y]$ we have [7]

$$Simp_n(A) = \begin{cases} \mathbf{A}^1 & n = 1 \\ \emptyset & n > 1 \end{cases}$$

In the defining relation

$$x^2 + \delta xy - \delta yx + ex = 0, \quad \delta \neq 0$$

for the double line, x is a normal element ($xA = Ax$), and acts as $x = 0$ or as an automorphism on a simple module. The 1-dimensional simple modules correspond to $x = 0$ or $x = -e$, i.e. 2 lines. If x acts as an automorphism we write z for x^{-1} and the defining relation becomes

$$1 + \delta yz - \delta zy + ez = 0$$

This is either isomorphic to the Weyl algebra ($e = 0$) or a simple line as described above ($e \neq 0$). In both cases we have $\text{Simp}_n(A) = \emptyset$ for $n > 1$.

Notice that for the simple as well as for the double line, there exist non-trivial extensions of simple 1-dimensional representations: If the defining relation is a simple line $x + \delta[x, y]$ the non-trivial extensions are given by ordered pairs

$$[(0, \beta), (0, \beta + \frac{1}{\delta})]$$

In the two-line case $x^2 + ex + \delta[x, y]$ the non-trivial extensions are located in the ordered pairs

$$[(0, \beta), (0, \beta + \frac{e}{\delta})]$$

and

$$[(e, \beta), (e, \beta + \frac{3e}{\delta})]$$

For the degenerate case $x^2 + \delta[x, y] = 0$ there are no non-trivial extensions outside the diagonal.

In the remaining case, where $\delta = 0$, the equation is just

$$x^2 + ex = 0$$

and we have a completely different situation. In this case there are simple modules of all dimensions. Examples of such are

$$x \mapsto -ee_{11} \quad y \mapsto \sum_{i=1}^{n-1} e_{i,i+1} + e_{n,1}$$

when $e \neq 0$ and

$$x \mapsto e_{n1} \quad y \mapsto \sum_{i=1}^{n-1} e_{i,i+1}$$

when $e = 0$.

7.5. The cusp. Consider the ordinary cusp, i.e. the k -algebra

$$A = k\langle x, y \rangle / (y^2 - x^3)$$

A 2-dimensional representation $\phi : A \rightarrow M_2(\Gamma)$ maps $y^2 - x^3$ to

$$t_Y Y - (t_X^2 - d_X)X - (d_Y - t_X d_X)I$$

where $X = \phi(x)$ and $Y = \phi(y)$. If the representation is simple then $t_Y = t_X^2 - d_X = d_Y - t_X d_X = 0$, i.e. $t_Y = 0$, $d_X = t_X^2$ and $d_Y = t_X^3$. The 2-dimensional simple modules forms a subvariety of the affine 2-space $\text{Spec}(k[t_X, t_{XY}])$, given as the complement of the hypercusp $t_{XY}^2 - 3t_X^5 = 0$, corresponding to the Formanek element.

Let V be a 1-dimensional simple module, represented by the closed point (a, b) on the curve, and let W be another 1-dimensional simple A -module. Then $\text{Ext}_A^1(V, W) \neq 0$ if and only if W is represented by a closed point $(\omega a, -b)$ on the curve, with ω any primitive cube root of 1.

Notice that the center $Z(A) \subset A$ is the subalgebra of A generated by $t := x^3 = y^2$. It is clear that any surjective homomorphism of k -algebras,

$$\rho_v : A \longrightarrow \text{End}_k(V)$$

will map $Z(A) = k[t]$ into $Z(\text{End}_k(V)) \simeq k$, inducing a point $v \in \text{Simp}(k[t]) = \mathbf{A}^1$. Thus $\text{Simp}_n(A)$ is fibred over the affine line $\text{Spec}(k[t]) = \mathbf{A}^1$. Let $\rho_v(x)^3 = \rho_v(y)^2 =$

$\kappa(v)I_n$, where $\kappa(v)$ is a function on the curve, such that $\kappa(v) = 0$ if and only if $v = \text{origin} =: \underline{0}$. Consider the diagram:

$$\begin{array}{ccc}
 k[t = x^3 = y^2] & & \\
 \downarrow & \searrow & \\
 A & \xrightarrow{\rho_v} & \text{End}_k(V) \\
 \downarrow & \nearrow & \\
 k[x]/(x^3 - \kappa(v)) * k[y]/(y^2 - \kappa(v)) & &
 \end{array}$$

Clearly, if $\kappa(v) \neq 0$ the simple representations of A are fibered on the cusp with fibres being the simple representations of $U := k[x]/(x^3 - \kappa(v)) * k[y]/(y^2 - \kappa(v))$, isomorphic to the group algebra of the projective modular group $PSL(2, \mathbf{Z})$.

For the 2-dimensional representations as described above, this correspond to simple representations

$$x \mapsto \begin{pmatrix} x_{11} & x_{12} \\ 0 & \omega x_{11} \end{pmatrix} \quad y \mapsto \begin{pmatrix} y_{11} & 0 \\ y_{21} & -y_{11} \end{pmatrix}$$

where $x_{11}^3 = y_{11}^2 = \kappa(v)$ and ω is a primitive third root of unity. The fibre of $\text{Simp}_2(A)$ above $v \in \mathbf{A}^1$ is an open set of (the three lines) $t_X^3 - \kappa(v) = 0$.

For $\kappa(v) = 0$ the matrices, $\rho_{\underline{0}}(x)$ and $\rho_{\underline{0}}(y)$ must both be nilpotent 2×2 -matrices, and generating $M_2(k)$. It turns out that we may assume,

$$\rho_{\underline{0}}(x) = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad \rho_{\underline{0}}(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

where $\lambda \in k$.

A formal versal family is given by,

$$\tilde{\rho}(x) = \begin{pmatrix} t & 1 + \mu \\ 0 & t \end{pmatrix}, \quad \tilde{\rho}(y) = \begin{pmatrix} 0 & 0 \\ 1 + \mu & 0 \end{pmatrix},$$

defined over the complete k -algebra,

$$H^A(V) = k[[t, \mu]]/(t^3, (\mu + 1)t^2)$$

For $\mu \neq -1$ this determines a simple representation.

7.6. Elliptic curve. In this section we shall study the geometry of a noncommutative version of an elliptic curve given by the k -algebra

$$A = k\langle x, y \rangle / (y^2 - x^3 - ax - b + q[x, y])$$

The 1-dimensional simple modules are represented by the elliptic curve

$$C = \text{Simp}_1(A) = \text{Spec}(k[x, y]/(y^2 - x^3 - ax - b))$$

A 2-dimensional representation

$$\phi : A \longrightarrow M_2(\Gamma)$$

is determined by two 2×2 -matrices $X = \phi(x)$ and $Y = \phi(y)$, satisfying the equation $Y^2 - X^3 - aX - b + q[X, Y] = 0$. Using elementary arithmetic for 2×2 -matrices, stated in Lemma 3.2, it is easily seen that this is equivalent to

$$\begin{aligned}
 2qXY + (t_Y - qt_X)Y + (-t_X^2 + d_X - a - qt_Y)X \\
 + (-d_Y + t_X d_X - b - qt_{XY} + qt_X t_Y) = 0
 \end{aligned}$$

If ϕ corresponds to a simple module the set $\{1, X, Y, XY\}$ is linearly independent over k and the relations

$$2q = t_Y - qt_X = -t_X^2 + d_X - a - qt_Y = -d_Y + t_X d_X - b - qt_{XY} + qt_X t_Y = 0$$

holds. Substituting $q = 0$ into the rest of the equations gives

$$t_Y = -t_X^2 + d_X - a = -d_Y + t_X d_X - b = 0$$

and the trace ring is clearly generated by t_X and t_{XY} . The simple locus is given by the non-vanishing of the Formanek element

$$\begin{aligned} d_{[X,Y]} &= -t_{XY}^2 + (t_X^2 - 4t_X^2 - 4a)(-t_X^3 - at_X + b) \\ &= -t_{XY}^2 + 3t_X^5 + 7at_X^3 - 3bt_X^2 + 4a^2t_X - 4ab \end{aligned}$$

corresponding to the complement of a hyperelliptic curve of genus 2.

The main tool for studying the indecomposable modules of A is the noncommutative Jacobian of the curve. Let $P = (u, v)$ be a generic point on the elliptic curve. Then vanishing of the Jacobian matrix

$$J((f); p) = \begin{pmatrix} -u^2 - ux - x^2 - a + qv - qy & v + y + qx - qu \end{pmatrix} = (0)$$

produces a quadratic relation in x . There are two different solutions outside of the zero set of the discriminant

$$\mathcal{D} = (u - q^2)^2 - 4(u^2 + q^2u + a - 2qv) = 0$$

The two solutions of the quadratic equation can be written in the form

$$x_1 = -\frac{1}{2}u + \frac{1}{2}q^2 + \frac{1}{2}\sqrt{\mathcal{D}} \quad x_2 = -\frac{1}{2}u + \frac{1}{2}q^2 - \frac{1}{2}\sqrt{\mathcal{D}}$$

In both cases we get

$$y = -v - q(x - u)$$

Thus the two solutions of the equation $J((f); p) = (0)$ are given by

$$Q_i = (x_i, y_i) = (u, -v) + (x_i - u)(1, -q) \quad i = 1, 2$$

Notice that, with respect to the addition rule on the elliptic curve, $-P = (u, -v)$ and it is easily seen that the three points $-P, Q_1$ and Q_2 are collinear.

Thus we have the following theorem.

Theorem 7.2. *Let $P = (u, v)$ be a point on an affine elliptic curve, as given above. Let $(P, Q_i) \in \mathcal{R}$ for $i = 1, 2$, then*

$$P = Q_1 + Q_2$$

Moreover, the slope of the line through the points Q_1, Q_2 equals $-q$, and this number is independent of choice of the point P on the elliptic curve.

Proof. Follows immediately from the above discussion. \square

Notice the following observation. Suppose there exist points P and Q on the elliptic curve with non-vanishing extension groups

$$\text{Ext}_A^1(P, Q) \neq 0, \quad \text{Ext}_A^1(Q, P) \neq 0$$

According to Theorem 6.11 this is a necessary condition for the existence of simple 2-dimensional modules close to any indecomposable modules given by the unordered pair (P, Q) .

The non-vanishing criterion says that there exist points Q_2 and P_2 such that

$$P = Q + Q_2 \quad Q = P + P_2$$

and such that $\text{Ext}_A^1(P, Q_2), \text{Ext}_A^1(Q_1, P_2) \neq 0$. According to theorem 7.2 we have $Q = -P + t_1(1, -q)$ and $P = -Q + t_2(1, -q)$ or $-P = Q + t_2(1, q)$. Thus we must have $t_1(1, -q) + t_2(1, q) = 0$ which is impossible unless $q = 0$.

Notice that, by a similar argument, there are no 3-cycles P_1, P_2, P_3 such that $\text{Ext}_A^1(P_i, P_{i+1}) \neq 0$, $i \in \mathbf{Z}/(3)$ for the elliptic curve. In fact the existence of such a cycle gives

$$\begin{aligned} P_2 &= -P_1 + t_1(1, -q) \\ P_3 &= -P_2 + t_2(1, -q) = P_1 + t_1(i, q) + t_2(1, -q) \\ P_1 &= -P_3 + t_3(1, -q) \end{aligned}$$

which is easily seen to be impossible.

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