COHOMOLOGY OF NUMERICAL MONOIDS GENERATED BY THREE ELEMENTS

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Abstract. We calculate the dimension of the tangent space of the deformation functor for numerical monoid algebras generated by three elements. Combining this with similar results of Buchweitz we obtain an implicit formula for the Frobenius number of the monoid.

0. Introduction

A numerical semigroup S is a subsemigroup of the natural numbers $\mathbf{N}$ (0 included). We shall assume that S contains $\mathbf{N}$ shifted by some $n \geq 0$, i.e. $n + \mathbf{N} \subset S$. The least number with this property is called the **conductor** of S, denoted by c. A numerical semigroup is called **symmetric** if there exists $m \in \mathbf{Z}$ such that for all $s \in \mathbf{Z}$ we have $s \in S$ if and only if $m - s \notin S$. It is easy to see that this m is the greatest **gap**, i.e. the greatest positive nonnumber of S, and that $c = m+1$. The number m is often referred to as the **Frobenius number** of S. Let $\mathcal{H} = \{n \geq 0 \,|\, n \notin S\}$ be the set of gaps. The cardinality of $\mathcal{H}$ is called the **genus** of S. All numerical semigroups $S =< p, q >$, generated by two elements are symmetric with $m = pq - p - q$. If $g = |\mathcal{H}|$ is the genus of S, a simple computation shows that for a symmetric numerical semigroup $2g = m + 1$.

In this paper we shall mainly be concerned with numerical semigruops generated by three elements, nicely described by Herzog in [H]. In this case we put $S =< g_1, g_2, g_3 >$. Let

$$R(S) = \{(z_1, z_2, z_3) \in \mathbf{Z}^3 \,|\, z_1 g_1 + z_2 g_2 + z_3 g_3 = 0\}$$

be the set of relations between the generators. A relation (v_1, v_2, v_3) is **minimal** of **type** i, $i = 1, 2, 3$, if for all $(z_1, z_2, z_3) \in R(S)$ such that either $z_i > 0$ and $z_j \leq 0$, $j \neq i$ or $z_i < 0$ and $z_j \geq 0$, $j \neq i$, we have $|v_i| \leq |z_i|$. A relation is minimal if it is minimal of any type. The degree of v is given by $deg(v) = \sum_{i=1}^{3} g_1 \max(o, v_i)$.

A symmetric numerical monoid generated by three elements g_1, g_2, g_3 has exactly to minimal relations of degree r_1, r_2 and the conductor is given by $c = r_1 + r_2 - g_1 - g_2 - g_3 + 1$. To find the moduli space of $k[S]$ we compute the tangent space $T^1(k[S])$ and the obstruction space $T^2(k[S])$. If S is generated by three elements the associated k-algebra $k[S]$ has embedding dimension 3, and $T^2(k[S]) = 0$ (see e.g. [B]). Thus the moduli space is smooth of dimension $\dim T^1(k[S]) = 2g - 1 + |\mathcal{M}|$, where $\mathcal{M}$ is the set of "maximal" gaps, i.e. $\mathcal{M} = \{h \in \mathcal{H} \,|\, h + S_+ \subset S\}$. In the symmetric case $|\mathcal{M}| = 1$ and we have

$$\dim_k T^1(k[S]) = 2g = c$$

Herzog ([H]) gives the formula $m = r_1 + r_2 - g_1 - g_2 - g_3$ for the Frobenius number of S. Thus we get the formula

$$\dim_k T^1(k[S]) = r_1 + r_2 - g_1 - g_2 - g_3 + 1$$

which is verified in this paper by quite different methods.

The non-symmetric case is somewhat more complicated. Then we have three minimal relations and the formula for the Frobenius number is no more valid. Buchweitz ([B]) gives a formula for the dimension of the moduli space

$$\dim_k T^1(k[S]) = 2g - 1 + |\mathcal{M}|$$

We prove that in the non-symmetric, three-generator case $|\mathcal{M}| = 2$ and it follows that this dimension is $2g + 1$. On the other hand we also prove the formula

$$\dim_k T^1(k[S]) = r_1 + r_2 - g_1 - g_2 - g_3 + |\mathcal{M}| - |N|$$

where r_1, r_2 are the smallest minimal relations and $N \subset \mathcal{H}$ is a set of gaps, defined via the order relation of S. We also prove the formula

$$2g = m_1 + 1 + (2m_2 - m_1)\hat{}$$

where $\mathcal{M} = \{m_2 < m_1\}$ and $\hat{\gamma} = (\gamma - S) \cap \mathbf{N}$. This gives a formula involving the Frobenius number

$$m_1 + (2m_2 - m_1)\hat{} + |N| = r_1 + r_2 - g_1 - g_2 - g_3$$

The paper is divided into two parts. In the first we recall some basic and important facts about cohomology of monoid-like sets and in the second part we do the computations in the three-generator case.

1. Cohomology

A numerical semigroup S has the structure of an ordered set given by

$$s_1 \leq s_2 \in S \quad \text{if} \quad \exists t \in S \quad \text{such that } s_1 + t = s_2$$

Let $L \subset S$ be some sub-ordered set.

DEFINITION **(1.1)**. *$L \subset S$ is said to be a* **monoid-like ordered set** *if for all (S-)relations $s_1 \leq s_2$ in L there exists $t \in L$ such that $s_1 + t = s_2$.*

Let $L \subset S$ be a monoid-like set and let k be a field of characteristic zero. Let $C_n(L)$ be the vector space on the set $\{(\lambda_1, \ldots, \lambda_n) \in L^n \mid w(\underline{\lambda}) \in L\}$ where w is the weight-function given by $w(\underline{\lambda}) = \sum_{i=1}^n \lambda_i \in S$. The symmetric group Σ_n acts on the basis of $C_n(L)$ by

$$\sigma(\lambda_1, \ldots, \lambda_n) = (\lambda_{\sigma^{-1}(1)}, \ldots, \lambda_{\sigma^{-1}(n)})$$

The group Σ_n also acts on the dual groups $C^n(L) = \mathrm{Hom}_k(C_n(L), k)$, by$(\phi \cdot \sigma)(\underline{\lambda}) = \phi(\sigma(\underline{\lambda}))$ for $\phi \in C^n(L)$ and $\sigma \in \Sigma_n$.

Denote by $Sh_n(L)$ the subspace of $C_n(L)$ generated by all shuffle-products, i.e. the submodule $s_n \cdot C_n(L)$, where $s_n = \sum_{i=1}^{n-1} s_{i,n-i}$ is the sum of all $(i, n-i)$-shufflings and let $C_S^n(L) = Hom_k(C_n(L)/Sh_n(L), k)$.
The inhomogenous differential

$$\delta^n : C^{n-1}(L) \longrightarrow C^n(L)$$

is defined by

$$\begin{aligned}\delta^n \xi(\lambda_1, \dots, \lambda_n) =& \xi(\lambda_2, \dots, \lambda_n) + \sum_{i=1}^{n-1} (-1)^i \xi(\lambda_1, \dots, \lambda_i + \lambda_{i+1}, \dots \lambda_n) \\ &+ (-1)^n \xi(\lambda_1, \dots, \lambda_{n-1})\end{aligned}$$

For n=1 we put $\delta^1 \xi = 0$.
Notice that the differential acts as a graded derivation with respect to the shuffle product.

DEFINITION **(1.2)**. *The inhomogenous Harrison cohomology $HA^n(L, k)$ of the ordered set L is the cohomology of the cocomplex $C_S^\bullet(L)$ with the inhomogenous differential δ.*

In this paper we have fixed a ground field k and without any confusion we may skip the k in the expression $HA^n(L, k)$.
There is also a relative version of Harrison cohomology. Let $L_0 \subset L \subset S$ and suppose L_0 is **full** in L, i.e. if $\gamma \in L$, $\gamma_0 \in L_0$ and $\gamma \geq \gamma_0$, then $\gamma \in L_0$. Then $S - L_0$ is a monoid-like set. The relative Harison cocomplex is given by

$$C_S^n(L - L_0, L) = \{\phi \in C^n(L - L_0, L) \mid \phi(Sh_n(L)) = 0\}$$

where

$$C^n(L - L_0, L) = \{\phi \in C^n(L) \mid \phi(\underline{\lambda}) = 0 \text{ for } \omega(\underline{\lambda}) \in L - L_0\}$$

and relative Harrison cohomology, $HA^n(L - L_0, L)$ is cohomology of the relative complex.

There is a long-exact sequence

$$\begin{aligned}0 \longrightarrow HA^1(L - L_0, L) \longrightarrow & HA^1(L) \longrightarrow HA^1(L - L_0) \\ & \longrightarrow HA^2(L - L_0, L) \longrightarrow HA^2(L) \longrightarrow \dots\end{aligned}$$

relating Harrison cohomology of the ordered sets L and $L - L_0$ with relative cohomology.

There is a close relation between Harrison cohomology of a numerical semigroup and the graded parts of algebra cohomology of the associated semigroup algebra. The associated semigroup algebra is a curve with an isolated singularity at the origin and the cohomology groups $\mathrm{Harr}^n(S, k[S])$ for $n \geq 2$ have finite dimension. Thus we have an isomorphism

$$\sum_{\lambda \in \mathbf{Z}} \mathrm{Harr}^{n,\lambda}(S, k[S]) \xrightarrow{\sim} \mathrm{Harr}^n(S, k[S])$$

for all $n \geq 1$. The graded cohomology group $\mathrm{Harr}^{n,\lambda}(S, k[S])$ is obtained from the subcomplex of homogenous cochains of degree λ. (See [Sl] for details.)
The relation between graded Harrison cohomology groups and Harrison cohomology of ordered sets is stated in the next theorem and proved in [Sl].

THEOREM **(1.3)**. *With the notation as above there is an isomorphism in cohomology*

$$\mathrm{Harr}^{n,\lambda}(S, k[S]) \xrightarrow{\simeq} HA^n(S_+ - S_{-\lambda}, S_+) \quad n \geq 1$$

where $S_\lambda = (\lambda + S) \cap S_+$, *and* $S_+ = S - \{0\}$.

Proof. See [Sl]. $\square$

For $n = 1$ it is easy to see that

$$HA^1(S_+ - S_{-\lambda}, S_+) \simeq \mathrm{Harr}^{1,\lambda}(S, k[S]) \simeq \mathrm{Der}^{1,\lambda}(S, k[S])$$

The dual of the cohomology group $\mathrm{Harr}^2(S, k[S])$ equals the tangent space $T^1(k[S])$ of the deformation functor for the k-algebra $k[S]$. Thus we have

$$\dim_k T^1(k[S]) = \sum_{\lambda \in \mathbf{Z}} HA^2(S_+ - S_{-\lambda}; S_+)$$

In the next section we shall give an explicit formula for the dimension of this group for an arbitrary numerical monoid generated by three elements.

2. The case of three generators

In this section we study numerical monoids generated by three elements as described in the introduction.

PROPOSITION **(2.1)**. *The first relative Harrison cohomology group is given by*

$$HA^1(S_+ - S_\lambda, S_+) \simeq \begin{cases} k & \textit{when } -\lambda \in S \cup \mathcal{M} \\ 0 & \textit{when } -\lambda \notin S \cup \mathcal{M}, \end{cases}$$

where $\mathcal{M} = \{\lambda \in \mathcal{H} \,|\, \lambda + S_+ \subset S\}$ *is the set of "maximal" gaps.*

Proof. Since $S \subset \mathbf{N}_+$ it is easy to see that $\dim \mathrm{Der}^{1,\lambda}(S, k[S]) \leq 1$. It follows that $\mathrm{Der}^{1,\lambda}(S, k[S]) \neq 0$ if and only if there exists a graded derivation D such that $D(g_i) \neq 0$ for all $i = 1, 2, 3$, i.e. $g_i + \lambda \in S$. $\square$

It is easy to see that $HA^1(S_+) \simeq k$ and from [Sl] we know that $HA^n(S_+) = 0$ for $n \geq 2$.

Thus in this case the long-exact sequence reduces to the exact sequence

$$0 \longrightarrow HA^1(S_+ - S_\lambda, S_+) \longrightarrow k \longrightarrow HA^1(S_+ - S_\lambda) \longrightarrow HA^2(S_+ - S_\lambda, S_+) \longrightarrow 0$$

Combining this with Prop. (2.1) we get

$$\dim HA^2(S_+ - S_\lambda, S_+) = \dim HA^1(S_+ - S_\lambda) + \chi_{S\cup\mathcal{M}}(-\lambda) - 1$$

where $\chi_{S\cup\mathcal{M}}$ is the characteristic function of $S \cup \mathcal{M}$, i.e.

$$\chi_{S\cup\mathcal{M}}(\lambda) = \begin{cases} 1 & \lambda \in S \cup \mathcal{M} \\ 0 & \lambda \notin S \cup \mathcal{M} \end{cases}$$

To compute the dimension of the tangent space we need some more information about $HA^1(S_+ - S_\lambda, S_+)$ for various $\lambda \in \mathbf{Z}$.

It is easily seen that for a monoid-like subset $K \subset S$ of a monoid S generated by three elements, we have

$$\dim HA^1(K) = \alpha_0(K) - \alpha_1(K)$$

where $\alpha_0(K)$ is the number of generators of S inside K and $\alpha_1(K)$ is the number of minimal relations of S inside K, upward bounded by 2. We shall compute the number of generators and minimal relations of S inside $K = S_+ - S_\lambda$ for various $\lambda \in \mathbf{Z}$. In fact what we shall compute the sum of the numbers

$$\dim HA^2(S_+ - S_\lambda, S_+) = \alpha_0(S_+ - S_\lambda) - \alpha_1(S_+ - S_\lambda) + \chi_{S\cup\mathcal{M}}(-\lambda) - 1$$

when λ ranges over all of $\mathbf{Z}$.
Observe that for $\lambda << 0$ we have $S_+ - S_\lambda = \emptyset$, $-\lambda \in S \cup \mathcal{M}$ and therefore

$$\dim HA^2(S_+ - S_\lambda, S_+) = 0 - 0 + 1 - 1 = 0$$

In the other end, for $\lambda >> 0$ we have $\alpha_0(S_+ - S_\lambda) = 3$, $\alpha_1(S_+ - S_\lambda) = 2$, $-\lambda \notin S \cup \mathcal{M}$ and consequently

$$\dim HA^2(S_+ - S_\lambda, S_+) = 3 - 2 + 0 - 1 = 0$$

The vanishing outside a bounded set also follows from the fact that $k[S]$ is an isolated singularity.
So consider $\lambda \in [-n, n]$ where n is big enough. A simple computation shows that for any element $\gamma \in S_+$ we have $\gamma \in S_+ - S_\lambda$ if and only if $-\lambda \notin S - \gamma$. Suppose we have given $\gamma \in S_+$. The number of λ such that $-\lambda \notin S - \gamma$ equals the cardinality of the complement (inside $[-n, n]$) of $(-\gamma + S) \cap [-n, n] = S \cap [-n + \gamma, n + \gamma]$. This number is easily seen to be $(2n + 1) - (n + \gamma + 1 - g) = n - \gamma + g$. On the other hand the number of $\lambda \in [-n, n]$ such that $\chi_{S\cup\mathcal{M}}(-\lambda) \neq 0$ is given by $n + 1 - g + |\mathcal{M}|$.

LEMMA **(2.2)**. *For a symmetric monoid S we have $\mathcal{M} = \{m\}$, the maximal gap.*

Proof. For a symmetric monoid the set of gaps $\mathcal{H} = (m - S) \cap \mathbf{Z}_+ = \hat{m}$, thus for all $h \in \mathcal{H} - \{m\}$ there is a $\lambda \in S_+$ such that $h + \lambda = m \notin S$. $\square$

Recall that a symmetric monoid generated by three elements has to minimal relations. The following theorem is due to Buchweitz ([B]) and the formula $c = r_1+r_2-g_1-g_2-g_3$ is proved by Herzog in [H]. We give an alternative proof for the dimension formula, involving cohomology of monoid-like sets as described above.

THEOREM **(2.3 (Buchweitz))**. *For a symmetric monoid $S =< g_1, g_2, g_3 >$ with minimal relations R_1, R_2 of degree r_1, r_2 the dimension of the moduli space is given by*

$$\dim_k T^1(k[S]) = r_1 + r_2 - g_1 - g_2 - g_3 + 1 = c$$

Proof. From the discussion above we get

$$\dim_k T^1(k[S]) = \sum_{\lambda \in \mathbf{Z}} HA^2(S_+ - S_\lambda, S_+)$$

and thus

$$\begin{aligned} \dim_k T^1(k[S]) &= (n - g_1 + g) + (n - g_2 + g) + (n - g_3 + g) \\ &\quad - (n - r_1 + g) - (n - r_2 + g) + (n + 1 - g + |M|) - (2n + 1) \\ &= r_1 + r_2 - g_1 - g_2 - g_3 + 1 = c \end{aligned}$$

□

For the non-symmetric case the situation is somewhat more complicated.
We shall compute the cardinality of $\mathcal{M}$ when S is non-symmetric, generated by three elements. Let $S =< g_1, g_2, g_3 >$ be a numerical semigroup generated by three elements and let $s \neq 0$ be an element of S. Let $B(s) = \{b_0, \ldots, b_{s-1}\}$ be the Apéry set with respect to s , i.e. $b_0 = s$ and for $i \geq 0$ b_i is the least integer in S having s-residue distinct from those of $b_0, \ldots, b_{i-1}$. For the set $B(s)$ we define

$$\tilde{B}(s) = \{b \in B(s) \,|\, b \neq b_0, b + b_i \notin B(s) \quad \text{for every } i = 0, \ldots, s-1\}$$

PROPOSITION **(2.4 (Cavahiere-Niesi))**. *Let S be a numerical monoid generated by three elements. Then the cardinality $|\tilde{B}(s)|$ of $\tilde{B}(s)$ is independent of choice of s and we have*

$$|\tilde{B}(s)| = \begin{cases} 1 & \textit{if } S \textit{ is symmetric} \\ 2 & \textit{if } S \textit{ is non-symmetric} \end{cases}$$

Proof. see [CN] (2.6 and 5.1) □

Let $x \in B(s)$ and suppose $x - s \in S$. Then $x, x - s$ have the same s-residue, contrary to the definition of B. Thus $(\tilde{B}(s) - s) \cap S = \emptyset$.

We can sharpen the result of the Cavahiere-Niesi-proposition, using the set $\tilde{B}(s) - s$.

PROPOSITION **(2.5)**. *The set* $\tilde{B}(s) - s$ *is independent of choice of* s *and we have*

$$\tilde{B}(s) - s = \mathcal{M}$$

Proof. Let $x \in \tilde{B}(s) \subset B(s)$ and suppose $x - s \in S$. Then $x, x - s$ have the same s-residue, contrary to the definition of $B(s)$. Thus

$$(\tilde{B}(s) - s) \cap S = \emptyset$$

Let $s = g_1 + g_2 + g_3$. Then $g_1, g_2, g_3 \in B(s)$. For $x \in \tilde{B}(s)$ we have $x + g_1, x + g_2, x + g_3 \notin B(s)$, by definition. Thus, for each $i = 1, 2, 3$, there is a $t_i > 0$ such that

$$x - t_i s + g_i \in S$$

Now suppose $t_i > 2$. Then $t_i - 1 > 0$ and we get

$$x - s = (x - t_i s + g_i) + ((t_i - 1)s - g_i) \in S$$

since $s - g_i \in S$. But this contradicts the fact that $x \in B(s)$. Thus $t_i = 1$ and $x - s + g_i \in S$ $\forall i = 1, 2, 3$ and $x - s + S_+ \subset S$.
On the other hand, if $h \notin S$ and $h + S_+ \subset S$ we have $h + s \in S$ and consequently $h + s \in B(s)$. Obviously $h + s \neq s$. Since $h + b \in S$ for all $b \in B(s)$ we have $h + b + s \in S$ and the minimality property ensures that $h + b + s \notin B(s)$. Thus $h + s \in \tilde{B}(s)$ and $h \in \tilde{B}(s) - s$, proving that the two sets are equal. □

COROLLARY **(2.6)**. *If* S *is a numerical semigroup generated by three elements, and* $\mathcal{M}$ *is as defined above, then we have*

$$|\mathcal{M}| = \begin{cases} 1 & S \textit{ symmetric} \\ 2 & S \textit{ is not symmetric} \end{cases}$$

Proof. Combine the two last propositions. □

In the non-symmetric case there are three minimal relations R_1, R_2, R_3 of degree r_1, r_2, r_3. Using the same argument as above we obtain the formula

$$\begin{aligned} &\dim_k(\oplus_{\lambda \in \mathbf{Z}} HA^2(S_+ - S_\lambda, S_+)) \\ &\qquad = r_1 + r_2 - g_1 - g_2 - g_3 + |\mathcal{M}| - (n - r_3 + g) + |K'| \end{aligned}$$

where $K' = \{\lambda \in \mathbf{Z} \,|\, r_1, r_2, r_3 \in S_+ - S_\lambda\}$.
Now $\gamma \in S_+ - S_\lambda$ if and only if $\lambda \in \gamma - \tilde{\mathcal{H}}$, where $\mathcal{H}$ is the extended set of gaps, i.e. $\tilde{\mathcal{H}} = \mathbf{Z} - S$. Thus we can write

$$K' = (r_1 - \tilde{\mathcal{H}}) \cap (r_2 - \tilde{\mathcal{H}}) \cap (r_3 - \tilde{\mathcal{H}})$$

or more convenient as a disjoint union

$$K' = K \cup [r_3 + 1, n]$$

where $K = (r_1 - \tilde{\mathcal{H}}) \cap (r_2 - \tilde{\mathcal{H}}) \cap (r_3 - \mathcal{H})$ and $|K'| = |K| + n - r_3$. There is a one-to-one correspondance between the set K and

$$L = \mathcal{H} \cap (r_3 - r_1 + \tilde{\mathcal{H}}) \cap (r_3 - r_2 + \tilde{\mathcal{H}})$$

given by $a \mapsto r_3 - a$. Let

$$N = \mathcal{H} \cap [(r_3 - r_1 + S) \cup (r_3 - r_2 + S)]$$

Then $|N| + |K| = g$. The minimality of the relation R_3 implies $r_3 - r_1, r_3 - r_2 \in \mathcal{H}$. With this notation we have

$$\dim_k T^1(k[S]) = r_1 + r_2 - g_1 - g_2 - g_3 + |\mathcal{M}| - |N|$$

The next lemma gives a generalization of the fact that for a symmetric monoid $\tilde{\mathcal{H}} = m - S$, where m is the maximal gap.

LEMMA **(2.7)**.

$$\tilde{\mathcal{H}} = M - S = \bigcup_{m \in M} (m - S)$$

Proof. Since $M \subset \mathcal{H}$ it is obvious that $m - S \subset \tilde{\mathcal{H}}$.
On the other hand, for each $h \in \tilde{\mathcal{H}}$ we have $h + S_+ \subset S$, or $h + S_+ \not\subset S$. In the first case $h \in M$ and $h = m - 0 \in M - S$. In the other case there exists $\gamma \in S_+$ such that $h + \gamma \in \tilde{\mathcal{H}}$: Repeat the argument until $h + s + S_+ \subset S$, where $s = \lambda_1 + \ldots + \lambda_k$. Then $h + s \in M$ and we get $\tilde{\mathcal{H}} \subset M - S$. $\square$

Using the S-order relation on $\mathbf{Z}$, i.e. $n_1 \leq n_2$ if $n_2 - n_1 \in S$ we recognize the set N as the elements lying in between $\{m_1, m_2\}$ and $\{r_3 - r_1, r_3 - r_2\}$.
Thus we have the following theorem.

THEOREM **(2.8)**. *For a non-symmetric monoid $S = < g_1, g_2, g_3 >$ with minimal relations R_1, R_2, R_3 of degree $r_1 < r_2 < r_3$ the dimension of the tangent space of the deformation functor equals the conductor c of the monoid, i.e.*

$$\dim_k T^1(k[S]) = r_1 + r_2 - g_1 - g_2 - g_3 + |\mathcal{M}| - |N|$$

where N is the set

$$N = [(m_1 - S) \cup (m_2 - S)] \cap [(r_3 - r_1 + S) \cup (r_3 - r_2 + S)]$$

Proof. Follows from the above discussion. $\square$

We can give another formula for this dimension, using the result of [B], stating that the dimension equals $2g+1$.
Let $\mathcal{M} = \{m_1 > m_2\}$. It is easily seen that the set $[0, m_1]$ is contained in the disjoint union

$$[0, m_1) \subset S \cup \hat{m_1} \cup (\hat{m_2} - \hat{m_1})$$

Counting the elements inside $[0, m_1]$ we get the formula $2g + |(\hat{m_2} - \hat{m_1})|$. Because of the maximality property of m_2 we have $m_1 - m_2 \in \mathcal{H}$. Furthermore we have $m_1 - (m_1 - m_2) = m_2 \notin S$ and Lemma 2.7 gives $m_1 - m_2 \in m_2 - S$, and thus $2m_2 - m_1 \in S$.

LEMMA **(2.9)**. *There is a one-to-one-correspondance*

$$(2m_2 - m_1)\hat{} \to \hat{m_2} - \hat{m_1}$$

given by $s \mapsto s + (m_1 - m_2)$.

Proof. Let $u \in S$ such that $m_2 - u \in \hat{m_2}$. Suppose $m_2 - u \notin \hat{m_1}$, i.e. $m_1 - m_2 + u \notin S$. Since $m_2 \notin S$ we have $m_1 - m_2 + u \notin m_1 - S$ and we must have $m_1 - m_2 + u \in m_2 - S$, say $m_1 - m_2 + u = m_2 - t$, $t \in S$. But the we have $(m_2 - u) - (m_1 - m_2) = t \in S$, and we have proved that $m_1 - m_2 \leq m_2 - u \leq m_2$ or $0 \leq (m_2 - u) - (m_1 - m_2) = 2m_2 - m_1$. For $0 \leq \gamma \leq 2m_2 - m_1$, $\gamma \in S$ we have $m_1 - m_2 \leq \gamma + m_1 - m_2 \leq m_2$. Suppose $\gamma + m_1 - m_2 \leq m_1$, i.e. $m_1 - (\gamma + m_1 - m_2) = m_2 - \gamma \in S$. Then $m_2 \in S$ which is obvious a contradiction. It follows that $\gamma + m_1 - m_2 \in \hat{m_2} - \hat{m_1}$. $\square$

Using this lemma we get a formula relating the maximal elements (including the Frobenius number m_1)

$$m_1 + (2m_2 - m_1)\hat{} + |N| = r_1 + r_2 - g_1 - g_2 - g_3$$

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