PRODUCTS IN THE DECOMPOSITION OF HOCHSCHILD COHOMOLOGY

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Abstract. We prove that the Cup product and the Lie bracket of Hochschild cohomology are graded products with respect to the decomposition.

Let k be a commutative ring with unit and A any commutative k-algebra. In [G] Gerstenhaber studied the properties of the cup product and the Lie bracket in the Hochschild cohomology of A with values in itself. He showed that the cup product turns Hochschild cohomology into a graded commutative ring and that the bracket product *is* a graded Lie product. He also proved that the adjoint representation $\alpha \mapsto [\alpha, \gamma]$ is a graded derivation of Hochschild cohomology considered as a ring under the cup product.

Restricting to the zero characteristic case, Quillen gave a decomposition of Hochschild cohomology

$$H^n(A, A) = \bigoplus_{i=1}^{n} H^n_{(i)}(A, A)$$

using exterior powers of the cotangent complex. The decomposition was studied further by Gerstenhaber-Schack [G-S], but by using quite different methods. In this paper we show that the two binary operations are both graded with respect to the decomposition.

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Let k be a commutative ring containing the rational numbers $\mathbf{Q}$ and let V be any k-module. Let

$$TV = k \oplus V \oplus V^{\otimes 2} \oplus \ldots$$

be the graded k-bimodule where we write $(v_1, \ldots, v_n)$ for the homogenous element $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$. The unit of TV is denoted $1 \in k = V^{\otimes 0}$. All tensor products are over k.
The symmetric group S_n acts on $V^{\otimes n}$ by permutation of the factors;

$$\sigma(v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$$

We extend the action by linearity to $\mathbf{Q}[S_n]$, making $V^{\otimes n}$ into a left $\mathbf{Q}[S_n]$-module.

Let $P; I = I_1 \cup \ldots \cup I_k$ be a segmented partition of length k of the totally ordered set $I = [\mathbf{n}]$, i.e. each I_j is a segment of I and the sets I_j are pairwise disjoint. The number n is called the total weight of the partition. Define $Mor_P(I,I)$ to be the set of bijective maps $\sigma : I \to I$ such that σ is order-preserving on each I_j. A map $\sigma \in Mor_P(I,I)$ is called a multishuffle (if the length of P equals 2 this is an ordinary shuffle). Put

$$s_P = \sum_{\sigma \in Mor_P(I,I)} sgn(\sigma)\, \sigma \in \mathbf{Q}[S_n]$$

The partition P induces a natural tensor product decomposition of $V^{\otimes n}$, given by $V^{\otimes n} = V_1 \otimes \ldots \otimes V_k$ where $V_j = V^{\otimes n_j}$ and n_j is the number of elements in I_j. The action of s_P for various P of common length k and arbitrary total weight n, defines a multilinear homogenous map $s^{(k)} : (TV)^{\otimes k} \to TV$ where we use the notation

$$s^{(k)}(\underline{v}_1, \ldots, \underline{v}_k) = \underline{v}_1 \star \ldots \star \underline{v}_k$$

LEMMA **(1)**. *Let I be a totally ordered finite set and let $P; I = I_1 \cup I_2 \cup I_3$ be a partition. Let $J = I_1 \cup I_2$. Let $Q; J = I_1 \cup I_2$ be the subpartition and let $R; I = J \cup I_3$ be the "recoarsened" partition. Then we have*

$$Mor_P(I,I) = Mor_R(I,I) \times Mor_Q(J,J)$$

Proof. There is obviously a map $\phi : Mor_R(I,I) \times Mor_Q(J,J) \to Mor_P(I,I)$ given by

$$\phi(\sigma,\lambda)(j) = \begin{cases} \sigma \circ \lambda(j) & \text{if } j \in J \\ \sigma(j) & \text{if } j \notin J \end{cases}$$

If $\phi(\sigma_1, \lambda_1) = \phi(\sigma_2, \lambda_2)$ we have by definition $\sigma_1 = \sigma_2$ outside J. If $j \in J$ we have $\sigma_1 \circ \lambda_1(j) = \sigma_2 \circ \lambda_2(j)$. Now suppose $\lambda_1 \neq \lambda_2$. Then there are $i, j \in J$ such that $\lambda_1(i) < \lambda_1(j)$ and $\lambda_2(i) > \lambda_2(j)$. But σ_1 and σ_2 are order-preserving on J and we get a contradiction since $\sigma_1 \circ \lambda_1 = \sigma_2 \circ \lambda_2$. Thus $\lambda_1 = \lambda_2$ and ϕ is injective.
On the other hand let $\sigma \in Mor_P(I, I)$. Since σ is a bijection there is an order preserving bijective map $\alpha : \sigma(J) \to J$. The composition $\alpha \circ \sigma|_J \in Mor_Q(J, J)$ and the map $\beta : I \to I$ defined by $\beta = \alpha^{-1}$ on J and $\beta = \sigma$ outside J is an order preserving map on both J and I_3. Thus $\beta \in Mor_R(I, I)$. Finally, $\phi(\beta, \alpha \circ \sigma|_J) = \sigma$ and the lemma follows. $\square$

PROPOSITION **(2)**. *The bilinear map* $TV \otimes TV \xrightarrow{\star} TV$ *defines a graded commutative associative product on* TV.

Proof. The associativity follows from the lemma; we could as well have chosen $J = I_2 \cup I_3$, since

$$\begin{aligned}(\underline{a}_1 \star \underline{a}_2) \star \underline{a}_3 &= s_R(s_Q(\underline{a}_1, \underline{a}_2), \underline{a}_3) \\ &= s_P(\underline{a}_1, \underline{a}_2, \underline{a}_3)\end{aligned}$$

Let $I = I_1 \cup I_2$ be a partition and let $\underline{v}_1 \otimes \underline{v}_2 = \underline{v}$ be a similar splitting of $\underline{v} \in V^{\otimes n}$. Let $\rho : I \to I$ be the permutation changing I_1 and I_2, i.e. order preserving on each I_j and such that $\rho(j) < \rho(i)$ if $i \in I_1$ and $j \in I_2$. The sign of ρ is given by $sgn\rho = (-1)^{n_1 n_2}$ where n_j is the number of elements of I_j. Let $(P \circ \rho)$ denote the partition given by $I = I_2 \cup I_1$, $I_2 < I_1$. Then, using the definition of s_P, it is easily seen that $s_{(P\circ\rho)} \circ \rho = sgn(\rho)\, s_P$, and thus

$$\begin{aligned}\underline{v}_1 \star \underline{v}_2 &= s_P(\underline{v}_1 \otimes \underline{v}_2) \\ &= (sgn\rho)\, s_{P\circ\rho}(\underline{v}_2 \otimes \underline{v}_1)) \\ &= (sgn\rho)\, \underline{v}_2 \star \underline{v}_1\end{aligned}$$

Put $1 \star \underline{v} = \underline{v}$ and the proposition follows. $\square$

Define a k-linear map $\Delta : TV \to TV \otimes TV$ by

$$\begin{aligned}\Delta(v_1, \ldots, v_n) =& 1 \otimes (v_1, \ldots, v_n) + \sum_{i=1}^{n-1} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n) \\ &+ (v_1, \ldots, v_n) \otimes 1\end{aligned}$$

It is well known that Δ is a comultiplication on TV and thus induces a coalgebra structure on TV. We denote by $\Delta^{(k)}$ the iterated comultiplication.

THEOREM **(3)**. *TV with multiplication $\star$ and comultiplication Δ is a bialgebra.*

Proof. (cf.[S]) $\square$

Now suppose $V = A$ is a commutative k-algebra with units $k \hookrightarrow A$. Let A_+ be the cokernel of the unit map, and let $C_\bullet(A) = A \otimes TA_+$ be the graded commutative associative algebra with multiplication $(a \otimes \underline{a}') \star (b \otimes \underline{b}') = ab \otimes (\underline{a}' \star \underline{b}')$. Define the A-linear map $\partial : C_\bullet(A) \to C_\bullet(A)$ of degree -1 by

$$\begin{aligned}\partial(a_1, \dots, a_r) =& a_1(a_2, \dots, a_r) + \sum_{i=1}^{r-1}(-1)^i(a_1, \dots, a_i a_{i+1}, \dots, a_r)\\ &+ (-1)^r a_r(a_1, \dots, a_{r-1})\end{aligned}$$

An easy computation shows that $\partial^2 = 0$ and in addition Barr proved (cf.[B]) that

$$\begin{aligned}\partial((a_1, \dots, a_i) \star (a_{i+1}, \dots, a_n)) =& \partial(a_1, \dots, a_i) \star (a_{i+1}, \dots, a_n)\\ &+ (-1)^i(a_1, \dots, a_i) \star \partial(a_{i+1}, \dots, a_n)\end{aligned}$$

Thus $C_\bullet(A)$ is a differential graded commutative algebra, called the normalized Hochschild complex.

DEFINITION **(4)**. **Hochschild (co-)homology** *of A with coefficients (resp. values) in the A-bimodule M is defined as (co-)homology of the complex $C_\bullet(A) \otimes_A M$ (resp. $Hom_A(C_\bullet(A), M)$).*

Let $M = A$ with the obvious bimodule structure and put

$$\begin{aligned}C^\bullet(A) &= Hom_A(C_\bullet(A), A)\\ &= Hom_k(TA_+, A)\end{aligned}$$

The induced differential δ acts on $f : TA_+ \to A$ as follows

$$\begin{aligned}\delta f(a_1, \dots, a_r) =& a_1 f(a_2, \dots, a_r) + \sum_{i=1}^{r-1}(-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_r)\\ &+ (-1)^r a_r f(a_1, \dots, a_{r-1})\end{aligned}$$

There are projection maps $p_n : TA_+ \to (A_+)^{\otimes n}$ and we say that a cochain $f \in C^\bullet(A)$ is homogenous of degree n if $f \cdot p_n = f$. Every homogenous map $g : TA_+ \to A$ of degree m may be uniquely extended to a coderivation $D_g : TA_+ \to TA_+$ defined via

$$p_n \cdot D_g = \sum_{i=1}^{n}(-1)^{(i-1)(m-1)}(p_{i-1} \otimes g \otimes p_{n-i})\Delta^{(3)}$$

The **composition product** $f \circ g$ is defined as the composition $f \cdot D_g$. Using the same terminology we write the **cup product** as

$$f \smile g = m \cdot (f \otimes g) \cdot \Delta$$

These products were originally defined by Gerstenhaber in [G]. He also defined the graded Lie product;

DEFINITION **(5)**. *The* **graded Lie product** *of the cochain complex* $C^{\bullet}(A)$ *is defined by*

$$[f, g] = f \circ g - (-1)^{mn} g \circ f$$

where f *and* g *are homogenous cochains of degree* n *and* m *respectively.*

Observe that the multiplication of A, denoted $m(a,b) = ab$ is a cocycle. Moreover, it is the coboundary of the identity map; $m = \delta\, id$. Also observe that the differential δ may be defined as the adjoint representation of m; $\delta f = -[f, m]$.

The composition product of two cocycles is not neccessarily another cocycle, but the cup product and the Lie bracket are.

PROPOSITION **(6)**. *Let* f, g *be homogenous cochains of degree* n, *respectively* m. *Then we have*

i) $\delta(f \circ f) = f \circ \delta f + (-1)^n \delta f \circ f = [f, \delta f]$ *if* n *is odd.*
ii) $\delta[f, g] = [f, \delta g] + (-1)^n [\delta f, g]$.
(iii) $\delta(f \smile g) = \delta f \smile g + (-1)^n f \smile \delta g$

Proof. (cf. [G]) □

Let I be the augmentation ideal of TA, i.e. $I = \oplus_{n \geq 1} A^{\otimes n}$, and denote by I^k the k-th shuffle power of this ideal. Let I_n^k be the image of $A^{\otimes n}$ under the left action of $s_n^{(k)} = \sum s_P$ where the sum is taken over all partitions P of total weight n and of length k.

PROPOSITION **(7)**. *For all* $1 \leq k \leq n$ *we have the equality* $p_n(I^k) = I_n^k$.

Proof. Since $p_n(I^k)$ is the image in $A^{\otimes n}$ under the action of s_P for various partitions P of total weight n and of length k, it is enough to show that the left ideal $\underline{sh}_k = (s_{P_1}, s_{P_2}, \ldots, s_{P_r}) \subset \mathbf{Q}[S_n]$, generated by all multishuffles of length k, equals the principal ideal $(s_n^{(k)})$. We need a lemma.

LEMMA **(8)**. *Given $s_n^{(k)}$ as above, there exists another element in the ring $\mathbf{Q}[S_n]$, denoted $e_n^{(k)}$, with the following properties;*

i) $e_n^{(k)}$ *is a polynomial in* $s_n^{(k)}$ *without constant term.*

ii) $sgn(e_n^{(k)}) = 1$, *where sgn is extended to all* $\mathbf{Q}[S_n]$ *by linearity.*

iii) $\partial e_n^{(k)} = e_{n-1}^{(k)}\partial$

iv) $(e_n^{(k)})^2 = e_n^{(k)}$

v) $e_n^{(k)} \cdot s_P = s_P$ *for all k-multishuffleproducts* s_P *where* P *is a partition of total weight* n *and of length* k.

Proof. We have $sgn\, s_n^{(k)} \neq 0$, in fact Loday (see [L]) gives the formula $sgn\, s_n^{(k)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} i^n$. Put

$$e_k^{(k)} = \frac{1}{k!} s_k^{(k)} = \frac{1}{k!} \sum_{\sigma \in S_n} sgn(\sigma)\sigma = \epsilon_k$$

Suppose we have found $e_k^{(k)}, e_{k+1}^{(k)}, \ldots, e_{n-1}^{(k)}$ satisfying the given conditions. Let $e_{n-1}^{(k)} = p(s_{n-1}^{(k)})$ where p is the polynomial of i), and define

$$e_n^{(k)} = p(s_n^{(k)}) + (1 - p(s_n^{(k)})) \cdot \frac{s_n^{(k)}}{sgn\, s_n^{(k)}}$$

We start by proving the lemma for $e_k^{(k)}$. By construction it satisfies i) and ii). Furthermore $\partial\epsilon_k = 0 = e_{k-1}^{k}\partial$. $\epsilon_k^2 = \epsilon_k$ and the only k-shuffling in s_k is multiplication by ϵ_k. Hence ϵ_k satisfies i)-v).
Consider $e_n^{(k)}$. Once more; by construction it satisfies i) and ii). In [L] Loday proves that $\partial s_n^{(k)} = s_{n-1}^{(k)}\partial$ and therefore

$$\begin{aligned}
\partial e_n^{(k)} &= \partial p(s_n^{(k)}) + \partial(1 - p(s_{n-1}^{(k)})) \cdot \frac{s_n^{(k)}}{sgn\, s_n^{(k)}} \\
&= p(s_{n-1}^{(k)})\partial + \frac{1}{sgn\, s_n^{(k)}}(1 - p(s_{n-1}^{(k)})) \cdot s_{n-1}^{(k)}\partial \\
&= e_{n-1}^{(k)}\partial + + \frac{1}{sgn\, s_n^{(k)}}(1 - e_{n-1}^{(k)}) \cdot s_{n-1}^{(k)}\partial \\
&= e_{n-1}^{(k)}\partial
\end{aligned}$$

since $s_{n-1}^{(k)} = \sum_P s_P$ and $s_{n-1}^{(k)} - e_{n-1}^{(k)} s_{n-1}^{(k)} = 0$. Furthermore, $\partial(e_n^{(k)})^2 = (e_{n-1}^{(k)})^2\partial = e_{n-1}^{(k)}\partial = \partial e_n^{(k)}$. Hence $\partial((e_n^{(k)})^2 - e_n^{(k)}) = 0$ and therefore

$(e_n^{(k)})^2 = e_n^{(k)}$ (this is a consequence of Prop 2.1 in [B]). The equalities

$$\begin{aligned}
&\partial e_n^{(k)} s_P[r_1,\dots,r_n] \\
&\quad = e_{n-1}^{(k)} \partial s_P[r_1,\dots,r_n] \\
&\quad = \sum_{j=0}^{n-1} (-1)^{\alpha_j} e_{n-1}^{(k)} s_{P_j}[r_1,\dots,r_{\alpha_j}] \otimes \partial[r_{\alpha_j+1},\dots,r_{\alpha_{j+1}}] \otimes [r_{\alpha_{j+1}+1},\dots,r_n] \\
&\quad = \sum_{j=0}^{n-1} (-1)^{\alpha_j} s_{P_j}[r_1,\dots,r_{\alpha_j}] \otimes \partial[r_{\alpha_j+1},\dots,r_{\alpha_{j+1}}] \otimes [r_{\alpha_{j+1}+1},\dots,r_n] \\
&\quad = \partial s_P[r_1,\dots,r_n]
\end{aligned}$$

where P_j is the induced partition on $I-\{j\}$, implies that $\partial(e_n^{(k)} s_P - s_P) = 0$ hence, as another consequence of Prop. 2.1 in [B],

$$e_n^{(k)} s_P - s_P = sgn(e_n^{(k)} s_P - s_P)\epsilon_n = 0$$

Thus we have also proved v), which completes the proof (This proof is an immediate generalization of Barr‘s proof (cf.[B]) in the case $k = 2$). $\square$

Going back to the proof of Proposition 7, we obviously have inclusions

$$(e_n^{(k)}) \subset (s_n^{(k)}) \subset \underline{sh}_k$$

Lemma 8 says that $s_P = e_n^{(k)} \cdot s_P \in (e_n^{(k)})$ for all partitions P and the inclusions must be equalities. $\square$

REMARK **(9)**. The ideal $(s_n^{(k)}) = (e_n^{(k)}) \neq (1)$ because $e_n^{(k)}$ is an idempotent different from 1 and therefore a zero-divisor. Thus $e_n^{(k)}$ cannot be a unit, consequently TA is infinitly generated as k-algebra with generators in all degrees.

Now consider the augmentation ideal I of TA. The indecomposables with respect to the multiplication $\star$ (the dual notion of primitive elements in a coalgebra) are given by $Q = I/I^2$ and if Q is flat over $A/I \simeq k$ (i.e. I is quasi-regular in the sence of Quillen), we have the equality $S(Q) = \oplus_{p\geq 0} I^p/I^{p+1}$, where $S(Q)$ is the graded symmetric algebra on Q. If $char(k) = 0$, the dual version of the Poincaré-Birkhoff-Witt-theorem (cf.[Q]) now gives the decomposition

$$TA \simeq S(Q) = \oplus_{p\geq 0} I^p/I^{p+1}$$

Tensor product and direct sum commutes, consequently

$$\begin{aligned}
C_\bullet(A) &= A \otimes TA = A \otimes (\oplus_{p\geq 0} I^p/I^{p+1}) \\
&= \oplus_{p\geq 0}(A \otimes I^p/I^{p+1})
\end{aligned}$$

But $C_\bullet(A)$ is a differential graded commutative algebra and the splitting induces a splitting of cohomology. Thus we have the following theorem, essentially due to Quillen (cf.[Q])

THEOREM **(10)**. *Let k, A and I be as above. Then Hochschild cohomology decomposes into a direct sum*

$$\begin{aligned} H^\bullet(A,A) &= \bigoplus_{p\geq 1} H^\bullet(Hom_k(I^p/I^{p+1},A)) \\ &= \bigoplus_{p\geq 1} H^\bullet_{(p)}(A,A) \end{aligned}$$

where in particular $H^n_{(p)}(A,A)=0$ if $p>n$.

REMARK **(11)**. Using Proposition 7 it is easy to see that this decomposition is the same as the decomposition induced by the λ-filtration of Loday in [L]. It also coincides with the decomposition of Gerstenhaber-Schack (cf. [G-S]) which extends the definition of commutative algebra cohomology made by Harrison in [H].

Now suppose we have $Ext^1_k(TA/I^p,A)=0$ for all $p\geq 0$. Then the map

$$Hom_k(TA/I^{p+1},A)\to Hom_k(I^p/I^{p+1},A)$$

is surjective for all $p\geq 0$. We lift the cochains of $Hom_k(I^p/I^{p+1},A)$ to cochains of $Hom_k(TA/I^{p+1},A)$ and study their behaviour with respect to the various products.

Let f and g be cochains of degree n and m and let $\underline{a}=\underline{a}_1\star\ldots\star\underline{a}_k$ where $\underline{a}_i\neq 1$. Using the coalgebra structure of TA and sigma notation we write

$$\begin{aligned} \Delta\underline{a} &= \Delta(\underline{a}_1\star\ldots\star\underline{a}_k) \\ &= \Delta\underline{a}_1\star\ldots\star\Delta\underline{a}_k \\ &= (\sum \underline{a}_{1(1)}\otimes\underline{a}_{1(2)})\star\ldots\star(\sum \underline{a}_{k(1)}\otimes\underline{a}_{k(2)}) \\ &= \sum \pm(\underline{a}_{1(1)}\star\ldots\star\underline{a}_{k(1)})\otimes(\underline{a}_{1(2)}\star\ldots\star\underline{a}_{k(2)}) \end{aligned}$$

where the sign is the sign of the appropriate permutation of the coordinates of the $n+m-1$-tuple $(a_1,\ldots,a_{n+m-1})$. Notice that we may have $\underline{a}_{i\,(j)}=1$. Using the definition of the cup product we get

$$\begin{aligned} f\smile g\,(\underline{a}) &= m\cdot(f\otimes g)\cdot\Delta(\underline{a}) \\ &= \sum \pm f(\underline{a}_{1(1)}\star\ldots\star\underline{a}_{k(1)})\cdot g(\underline{a}_{1(2)}\star\ldots\star\underline{a}_{k(2)}) \end{aligned}$$

Now put $J_i = \{j \in \{1,\ldots,k\} \,|\, \underline{a}_{j(i)} \neq 1\}$ for $i = 1, 2$ and let s be the number of elements in J_1 and t the number of J_2. We obviously have $J_1 \cup J_2 = \{1,\ldots,k\}$ but the two sets are not neccessarily disjoint.

Let $I \subset TA$ be the augmentation ideal and let $I^s = I \star \ldots \star I$ be the s-fold shuffle product. Assume $f \in Hom_k(TA/I^p, A)$ and $g \in Hom_k(TA/I^q, A)$. Then $f(I^s) = 0$ if $s \geq p$ and $g(I^t) = 0$ if $t \geq q$. Consider the product $f(I^s)g(I^t)$ where $s+t \geq p+q-1$. Either we have $s \geq p$, implying that $f(I^s) = 0$ and the product vanishes, or $p \geq s+1$. In that case $s+t \geq p+q-1 \geq s+1+q-1 = s+q$, therefore $t \geq q$ and consequently $f(I^s)g(I^t) = f(I^s)0 = 0$. Thus $f(I^s)g(I^t) = 0$ if $s+t \geq p+q-1$. Our assumptions were that $f(I^p) = g(I^q) = 0$ and $s+t \geq k$, and we have shown that the cup product $(f \smile g)(I^k)$ vanishes whenever $k \geq p+q-1$, obtaining the following lemma;

LEMMA **(12)**. *Let $f \in Hom_k(TA/I^p, A)$, $g \in Hom_k(TA/I^q, A)$. Then the product $f \smile g \in Hom_k(TA/I^{p+q-1}, A)$.*

□

THEOREM **(13)**. *Let k be a commutative ring, containing the rational numbers. Let A be a commutative k-algebra and let I be the augmentation ideal of TA. Suppose I/I^2 is flat over $A/I \simeq k$ and that $Ext^1_k(TA/I^p, A) = 0$ for all $p \geq 0$. Then the cup-product of Hochschild cohomology :*

$$\smile : H^n_{(i)}(A,A) \times H^m_{(j)}(A,A) \longrightarrow H^{n+m}_{(i+j)}(A,A)$$

is graded with respect to decomposition degree.

Proof. Let $\overline{f} \in H^n_{(i)}(A,A)$ and $\overline{g} \in H^m_{(j)}(A,A)$ be represented by cochains $f \in Hom_k(TA/I^{i+1}, A)$ and $g \in Hom_k(TA/I^{j+1}, A)$. Then by Lemma 12 we have $f \smile g \in Hom_k(TA/I^{i+j+1}, A)$. The image of this element in $Hom_k(I^{i+j}/I^{i+j+1}, A)$ is a cocycle, representing the product $\overline{f} \smile \overline{g}$. □

In the notation of Gerstenhaber and Schack [G-S] the graded cup product takes the form

$$\smile : H^{p,i}(A,A) \times H^{q,j}(A,A) \longrightarrow H^{p+q,i+j}(A,A)$$

LEMMA **(14)**. *Let $f \in Hom_k(TA/I^{p+1}, A)$, $g \in Hom_k(TA/I^{q+1}, A)$. Then the composition $f \circ g \in Hom_k(TA/I^{p+q}, A)$.*

Proof. Let $f = f \cdot p_n$ and $g = g \cdot p_m$. The composition product defined by

$$f \circ g = f \cdot D_g = f \cdot p_n \cdot D_g = f \cdot \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} (p_{i-1} \otimes g \otimes p_{n-i}) \cdot \Delta^{(3)}$$

satisfies $f \circ g = (f \circ g) \cdot p_{n+m-1}$. Suppose $f(I^{p+1}) = g(I^{q+1}) = 0$ and let P be a partition of total weight $n+m-1$ and of length $p+q$. As before we write $s_P(a_1, \ldots, a_{n+m-1}) = \underline{a}_1 \star \ldots \star \underline{a}_{p+q} \in I^{p+q}$. We must show that $(f \circ g)(\underline{a}_1 \star \ldots \star \underline{a}_{p+q}) = 0$.

We shall study the element

$$\begin{aligned} Y &= \Delta^{(3)}(\underline{a}_1 \star \ldots \star \underline{a}_{p+q}) \\ &= (\Delta^{(3)}\underline{a}_1) \star \ldots \star (\Delta^{(3)}\underline{a}_{p+q}) \end{aligned}$$

where the equality holds because Δ is an algebra map respecting $\star$. Using the sigma notation for comultiplication we can write

$$\Delta^{(3)}\underline{a}_i = \sum \underline{a}_{i\,(1)} \otimes \underline{a}_{i\,(2)} \otimes \underline{a}_{i\,(3)}$$

Multiplication in the graded algebra $TA \otimes TA \otimes TA$ is defined componentwise and we have

$$\begin{aligned} Y = \sum \pm (\underline{a}_{1\,(1)} \star \ldots \star \underline{a}_{p+q\,(1)}) \otimes (\underline{a}_{1\,(2)} \star \ldots \star \underline{a}_{p+q\,(2)}) \\ \otimes (\underline{a}_{1\,(3)} \star \ldots \star \underline{a}_{p+q\,(3)}) \end{aligned}$$

which we simply write

$$Y = \sum \pm y_{(1)} \otimes y_{(2)} \otimes y_{(3)}$$

and where the sign is the sign of the appropriate permutation of the coordinates of the $n+m-1$-tuple $(a_1, \ldots, a_{n+m-1})$. Notice that we may have $\underline{a}_{i\,(j)} = 1$.
Grouping the terms in the sum by fixing $y_{(2)}$, we may write

$$y = \sum_{y_{(2)}} \sum \pm y_{(1)} \otimes y_{(2)} \otimes y_{(3)}$$

Put $y = (id \otimes g \otimes id)(Y)$. Choose one "in the middle"-term

$$y_{(2)} = \underline{a}_{1\,(2)} \star \ldots \star \underline{a}_{p+q\,(2)} = \underline{a}_{i_1\,(2)} \star \ldots \star \underline{a}_{i_s\,(2)}$$

where the last equality holds since we assume that $\underline{a}_{i_j\,(2)} \neq 1$ for all $j = 1, 2, \ldots, s$ and $\underline{a}_{j\,(2)} = 1$ for all $j \neq i_1, i_2, \ldots, i_s$.
Let y' be the part of y with this $y_{(2)}$ fixed;

$$y' = \sum \pm y_{(1)} \otimes (\underline{a}_{i_1\,(2)} \star \ldots \star \underline{a}_{i_s\,(2)}) \otimes y_{(3)}$$

We shall use the notation

$$\{j_1, \ldots, j_k\} = \{1, 2, \ldots, p+q\} - \{i_1, \ldots, i_s\}$$

for the indices where $\underline{a}_{j\,(2)} = 1$. Thus we can write

$$y_{(1)} = \underline{a}_{j_1(1)} \star \ldots \star \underline{a}_{j_k(1)} \star \underline{a}_{i_1(1)} \star \ldots \star \underline{a}_{i_s(1)}$$

and

$$y_{(3)} = \underline{a}_{i_1(3)} \star \ldots \star \underline{a}_{i_s(3)} \star \underline{a}_{j_1(3)} \star \ldots \star \underline{a}_{j_k(3)}$$

Let further

$$z = (\underline{a}_{i_1\,(1)} \star \ldots \star \underline{a}_{i_s\,(1)}) \otimes (\underline{a}_{i_1\,(2)} \star \ldots \star \underline{a}_{i_s\,(2)}) \otimes (\underline{a}_{i_1\,(3)} \star \ldots \star \underline{a}_{i_s\,(3)})$$

and put $\underline{b} = \underline{a}_{j_1} \star \ldots \star \underline{a}_{j_k}$. Obviously $z, \underline{b} \neq 1$. Let $|z|$ be the tensor-degree of z. Consider

$$\sum_{i=1}^{n} (-1)^{(i-1)(m-1)} (p_{i-1} \otimes g \otimes p_{n-i})(y')$$

It is easy to see that this sum has the same terms as the ones we obtain when forming the product $(1 \otimes g \otimes 1)(z) \star \underline{b}$. To prove that the sums are equal we must show that the signs of each term coincides in the two cases.
Let α be some term of $(1 \otimes g \otimes 1)(z) \star \underline{b}$; put

$$\alpha = (-1)^{|\alpha|} a_{t_1} \otimes \ldots \otimes a_{t_i} \otimes (1 \otimes g \otimes 1)(z) \otimes a_{t_{i+1}} \otimes \ldots \otimes a_{t_r}$$

If α' is another term in the shuffle product $(1 \otimes g \otimes 1)(z) \star \underline{b}$, "obtained " from α by "moving" a_{t_l} to the "other" side of $(1 \otimes g \otimes 1)(z)$;

$$\begin{aligned}\alpha' = (-1)^{|\alpha'|} a_{t_1} \otimes \ldots \otimes a_{t_{l-1}} \otimes a_{t_{l+1}} \otimes \ldots \otimes a_{t_i} \otimes (1 \otimes g \otimes 1)(z) \\ \otimes a_{t_{i+1}} \otimes \ldots \otimes a_{t_k} \otimes a_{t_l} \otimes a_{t_{k+1}} \otimes \ldots \otimes a_{t_r}\end{aligned}$$

then we have

$$\begin{aligned}|\alpha'| &= |\alpha| + (i-l) + |(1 \otimes g \otimes 1)(z)| + (k-i) \\ &= |\alpha| + |(1 \otimes g \otimes 1)(z)| + (k-l)\end{aligned}$$

On the other hand, producing the same effect on the terms of y', i.e changing

$$\beta = (-1)^{|\beta|} a_{t_1} \otimes \ldots \otimes a_{t_i} \otimes z \otimes a_{t_{i+1}} \otimes \ldots \otimes a_{t_r}$$

to

$$\beta' = (-1)^{|\beta'|} a_{t_1} \otimes \ldots \otimes a_{t_{l-1}} \otimes a_{t_{l+1}} \otimes \ldots \otimes a_{t_i} \otimes z \otimes a_{t_{i+1}} \\ \otimes \ldots \otimes a_{t_k} \otimes a_{t_l} \otimes a_{t_{k+1}} \otimes \ldots \otimes a_{t_r}$$

the sign equation is $|\beta'| = |\beta| + |z| + (k-l)$. Thus we get

$$\begin{aligned} |\alpha'| - |\alpha| &= |(1 \otimes g \otimes 1)(z)| + (k-l) \\ &= |z| - (m-1) + (k-l) \\ &= |\beta'| - |\beta| - (m-1) \end{aligned}$$

But if $p_{j-1} \otimes g \otimes p_{n-j}(\beta) \neq 0$ then $p_{j-2} \otimes g \otimes p_{n-j+1}(\beta') \neq 0$, producing an additional change in sign given by multiplication by $(-1)^{m-1}$, and we obtain the same effect on the signs in both expressions. Consequently we have the equality

$$(1 \otimes g \otimes 1)(z) \star \underline{b} = \pm \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} (p_{i-1} \otimes g \otimes p_{n-i}(y')$$

and y is a sum of such terms.
Using the vanishing property of g we see that if $s \geq q+1$, then $(1 \otimes g \otimes 1)(z) = 0$. If $s \leq q$ we have $k = p+q-s \geq p+q-q = p$ and $(1 \otimes g \otimes 1)(z) \star \underline{b} \in I^{p+1}$. But then $f((1 \otimes g \otimes 1)(z) \star \underline{b}) = 0$ and in both cases we get $(f \circ g)(\underline{a}_1 \star \ldots \star \underline{a}_{p+q}) = 0$ as expected. □

The second main theorem of this paper now follows as a corollary.

THEOREM **(15)**. *Let k be a commutative ring, containing the rational numbers. Let A be a commutative k-algebra and let I be the augmentation ideal of TA. Suppose I/I^2 is flat over $A/I \simeq k$ and that $Ext^1_k(TA/I^p, A) = 0$ for all $p \geq 0$. There is an anti-commutative product on Hochschild cohomology :*

$$[-,-] : H^{n+1}_{(i+1)}(A,A) \times H^{m+1}_{(j+1)}(A,A) \longrightarrow H^{n+m+1}_{(i+j+1)}(A,A)$$

The product is graded up to a shift in the decomposition degree.

Proof. Let $\overline{f} \in H^{n+1}_{(i+1)}(A,A)$ and $\overline{g} \in H^{m+1}_{(j+1)}(A,A)$ be represented by cochains $f \in Hom_k(TA/I^{i+2}, A)$ and $g \in Hom_k(TA/I^{j+2}, A)$. Then by Lemma 14 we have $f \circ g \in Hom_k(TA/I^{i+j+2}, A)$ and the same for $[f,g]$. The image of this element in $Hom_k(I^{i+j+1}/I^{i+j+2}, A)$ is a cocycle, representing the product $[\overline{f}, \overline{g}]$. □

Notice that the conditions of Theorem 13 and 15 are fullfilled e.g. if k is a field of characteristic 0. Furthermore, if we put $i = j = 0$ the product of Theorem 15 is precisely the Lie-bracket in Harrison cohomology $[-,-] : Ha^{n+1}(A,A) \times Ha^{m+1}(A,A) \longrightarrow Ha^{n+m+1}(A,A)$.

REFERENCES

[B] BARR, M., "Harrison Homology, Hochschild Homology and Triples"
J. of Alg. 8 (1968) pp.314-323.

[G] GERSTENHABER, M., "The cohomology structure of an associative ring"
Ann. of math. 78 no. 2 (1963) pp.267-288.

[G-S] GERSTENHABER, M., SCHACK, S.D., "A Hodge-type decomposition for commutative algebra cohomology"
J. of Pure Appl. Alg. 48 (1987) pp.229-247.

[H] HARRISON, D.K., "Commutative Algebras and cohomology"
Trans. Amer. Math. Soc. 104 (1962) pp.191-204.

[L] LODAY, J.-L., "Opérations sur l'homologie cyclique des algèbres commutatives"
Invent. Math. 96 (1989) pp.205-230.

[Q] QUILLEN, D., "On the (co-)homology of commutative rings"
Proc. of Symp. in Pure Math. Vol XVII (1970) pp.65-87.

[R] RONCO, M., "Sur l'homologie d'André-Quillen"
Preprint IRMA, Strasbourg (1990).

[S] SWEEDLER, M., "Hopf Algebras"
Benjamin, New York (1969).