Time-space and Space-times

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Abstract. In this paper we use some of the basic concepts and philosophy of classical physics in the study of the geometry of some moduli spaces of representations (modules) of associative algebras. Based on two very simple special cases, essentially the Hilbert scheme of length 2 subschemes of the affine 3-space, we show that it make sense to think of time as a metric on the moduli space. The *laws of dynamics* of the objects parametrized by the moduli space is then formulated in much the same way as in modern physics.

Contents:

Natura non facit saltum? [**Le**]

... De là m'est venue cette pensée que le temps est une sorte d'étendue; mais de quoi est-ce une étendue? [**A**]

§0. Introduction

This paper is not primarily about physics. It may, hopefully, be of interest as an essay on the notion of time, in physics and philosophy, based on the mathematical concept of moduli space.

The problem of time and movement has a long history, and is, of course, still wide open. Seen from a classical philosophical point of view, the different mathematical models of contemporary physics are all lacking credibility. Einstein's relativity theory, although a marvel of a scientific theory, does not *explain* neither the notion of time, nor the existence of a maximal velocity, on which it is based, and the quantum theory of Bohr and Heisenberg treats time as an ad hoc parameter.

To most physicists, I believe, time is exactly that, a measurable parameter introduced to make motion look simple, see [**MTW**], p.23.

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The main purpose of this paper is to apply some basic concepts of classical physics to the study of the geometry of some algebraic geometric moduli spaces, in particular to the moduli spaces of representations (modules) of associative k-algebras, k a field. It turns out that this study leads to problems closely related to *mathematical quantum theory.* The notion of time then comes up in a natural way.

In fact, any moduli space should be considered as the *time-space* (not space-time!) of the family of objects we consider.

Measurable time, in this mathematical model, turns out to be a metric on the time-space, measuring all possible infinitesimal changes of *the state* of the objects in the family (in line with, and extending the views of Saint Augustin?).

The dynamical laws, or the *physics* of the objects we study, should then be given by a system of differential equations, related to the metric, and maybe to some natural *Lagrangian* function defined on a *phase space* associated to the moduli space. This Lagrangian should, preferably, contain all information about the *dynamics* of the objects we are studying, and the metric should, of course, be chosen such that motion looks simple.

To make this approach understandable, we shall first sketch two very elementary models for special and general relativity and for cosmology, based on these ideas and on the notion of moduli space in (non commutative) algebraic geometry, see [**La4**]. We obtain a rather coherent picture, providing us with an interesting but, most likely, just another "toy" mathematical model for the physical reality. If one needs an excuse for presenting such *physical toy models*, one may point to the possible purely mathematical interest of such a model, related to the study of the geometric structure of moduli spaces in general, but also to the possible interest it may have related to the philosophical problem of *explaining* physical theories.

The content of this paper is then: In §1 we shall consider the simplest possible model for a time-space parametrizing the family of two objects, the *observer*, and the *observed*, both situated in the classical Euclidean three dimensional space $\mathbf{E}^3$.

We shall show that this time-space furnishes an attractive basis for the study of notions like time, velocity, energy/momentum, and spin. Moreover the space-time of general relativity should be considered as a fiber space on local histories (i.e. curves) in this time-space or, equivalently, as 4-dimensional slices of the time-space.

Then, in §2 we are being slightly bolder. Assuming that the *big bang* is mathematically modelled by the versal deformation space of a double point, we obtain a rather amusing interpretation of the Hubble constant and of the Einstein-de Sitter cosmological model.

After this preparation, we shall in §3, for a given configuration space $\underline{A} := Spec(A)$, (i.e. in principle, for any commutative or noncommutative scheme, [**La4**]), first prove that the moduli space, $\underline{U}(A)$ of all A-modules of k-dimension 2 is non-commutative and contains the classical $Hilb^2_{\underline{A}}$. Then we shall show that the phase space, $\underline{P}h(A)$, parametrizing all pairs $(\underline{a}, \xi)$ of a point $\underline{a} \in \underline{A}$ and a tangent vector ξ at $\underline{a}$ is also non-commutative, and we compute it for $\underline{A} = \underline{H} = Spec(\mathbf{R}[t_1, ..., t_6])$, the time-space of §1. Generalizing and iterating the Ph-construction we construct a k-algebra of *higher differentials* $Ph^\infty(A)$, see also [**LT**], where the notion of higher differentials is treated in greater generality.

$Ph^\infty(A)$ is a graded k-algebra, with degree-0 component A, and with a universal *Dirac-derivation* $\delta \in Der_k(Ph^\infty(A))$.

Now, applying this to the case where A is the affine (commutative or non-commutative) k-algebra of an open subset of a moduli space, we may generalize the models of §1 and §2 in many directions. Any point of $\underline{Ph}^\infty(A)$ is a point $\underline{a} \in Spec(A)$ together with a momentum, i.e. a tangent vector at $\underline{a}$, an acceleration, and any number of higher order tangents of $Spec(A)$ at $\underline{a}$. Then the *law of motion* is expressed in terms of the action of the Dirac-derivation.

This may, of course, be generalized to the study of the moduli space of representations of associative k-algebras A. If, for example, $V \in Simp_n(A)$, see [**La5**], i.e. if V is a simple A-module of k-dimension n, then a preparation of V is an extension of the operators on V from A to $Ph^\infty(A)$, implying that $V \in Simp_n(Ph^\infty(A))$. Then the Dirac derivation determines a vector field in $Simp_n(Ph^\infty(A))$, and an integral curve, or a *local history*, in $Simp_n(A)$ containing V. Even though the point of departure is very different, it is not difficult to see the analogy between the above and the spectral triples of Connes, see [**Co**], or [**Schu**].

In §4 we shall end by an apology, and by recalling some of the classical enigmas of time. Is there an arrow of time? If so, may we expect this arrow to be explainable by the properties of a natural moduli space in non-commutative geometry?

This paper is a result of a long struggle trying to understand the philosophy of physics applied to mathematics (not the other way around). In fact, the origin can be found in the author's work on moduli for plane curve-singularities. During the years 1991-92, the idea came up that there should be a kind of Yang-Mills theory definable on the moduli space for singularities. This turned out to be partially true, see §4, and then the line of thought presented here became quite natural. A first version of this note was written while visiting the University of Catania in February 1992. My sincere thanks to Professor Strano and Professor Regusa for inviting me to Sicily and finding the beautiful house near Aqui Reale, at the foot of an erupting Etna.

§1. A very elementary world – the moduli space of an observer and an observed

(i): Le temps n'est ni mouvement, ni sans le mouvement.... Le temps est le nombre du mouvement selon l'avant et l'après; et il est continu car il appartient à un continu. (Aristotle, see [**C-S**].)

When I want to describe something occurring in Nature, inside or outside of me, I tend to focus on where the occurrence took place, relative to where I am. In this picture there is therefore a pair (observer, observed)=:$(\underline{o}, \underline{x})$ of locations, with reference to some space. This space is for most practical purposes modeled by the Euclidean 3-space, $\mathbf{E}^3$. This is reasonable, since the Lie algebra $so(3)$ of $SO(3)$, seems to be part of the software that comes with any human being. How else would we be able to identify spacial objects and give them names invariant under the Euclidean group G? We shall therefore assume that $so(3)$ together with the Killing form, isomorphic to $\mathbf{E}^3$, and the symmetry group G is part of our structure.

One may therefore be tempted to model physics of occurrences and the dynamics of such, on the moduli space $\underline{H} := Spec(H)$ of pairs of points of the Euclidean 3-space. This is going to be the subject of this §1.

However, we know that describing natural phenomena, like sound, light, and elementary particles, the situation is different. The objects responsible for light and elementary particles seem not to be of a local nature. Like waves they cannot be localized in space. They seem to be smeared out on space. The mathematical objects best fitted to represent such occurrences are functions ψ (real or complex) defined on portions of $\mathbf{E}^3$, or better, sections Ψ of some fibre bundle defined on $\mathbf{E}^3$.

In line with the general philosophy of this note, one would therefore be tempted to base physics of such phenomena on some moduli space of pairs of such sections (Ψ_0, Ψ_1). This turns out to be basically the same as the space of sections of bundles on the space $\underline{H} := Spec(H)$. We shall treat it in a much wider context, that of the geometry of moduli spaces of representations of associative algebras. This is the main subject of this paper, see §3.

Let, from now on, k be the field we need. In §1 and §2 we shall work with $k = \mathbf{R}$, the real numbers. Consider first the moduli space $\underline{H} := Spec(H)$ of an observer and an observed, i.e. the *moduli scheme* parametrizing all ordered pairs $(\underline{o}, \underline{x})$ of points in the Euclidean 3-space $\mathbf{E}^3$. Clearly, $\underline{H}$ is of dimension 6, containing a canonical subscheme $\underline{\Delta}$, the diagonal. If we choose some orthonormal coordinate system in $\mathbf{E}^3$, we may consider the point $(\underline{o}, \underline{x}) \in \underline{H}$ with coordinates $(o_1, o_2, o_3; x_1, x_2, x_3)$. Put

$$
\begin{aligned}
t_i &:= \tfrac{1}{2}\,(o_i - x_i), \quad i = 1, 2, 3, \\
t_j &:= \tfrac{1}{2}\,(o_j + x_j), \quad j = 4, 5, 6,
\end{aligned}
$$

then $H = k[t_1, ..., t_6]$. The Euclidean group G acts on $\underline{H} := Spec(H)$ in an obvious way, leaving $\underline{\Delta} := \{t_1 = t_2 = t_3 = 0\}$ invariant. Let $Aff(3)$ be the commutative normal subgroup of G. Given a point $\underline{t} \in \underline{H}$, corresponding to the pair $(\underline{o}, \underline{x})$ of $\mathbf{E}^3$, we have an embedding

$$SO(3) \subset G,$$

corresponding to rotations about the point $(t_4, t_5, t_6) \in \mathbf{E}^3$. The action,

$$G \times \underline{H} \to \underline{H},$$

induces a map of Lie-algebras,

$$\mathfrak{g} \to \Theta_H = Der_k(H),$$

mapping the Lie algebra $aff(3)$ of $Aff(3)$ into the commutative sub-k-Lie algebra of Θ_H generated by $< \frac{\partial}{\partial t_4}, \frac{\partial}{\partial t_5}, \frac{\partial}{\partial t_6} >$ and the Lie algebra $so(3)$ of $SO(3)$ into the sub-k-Lie algebra generated by $< t_2\frac{\partial}{\partial t_1} - t_1\frac{\partial}{\partial t_2}, t_1\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_1}, t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2} >$. Since $aff(3)$ is a normal sub Lie-algebra of $\mathfrak{g}$, we find that it defines a canonical distribution $\tilde{\Delta}$, also called *the diagonal*, such that for any point $\underline{t} \in \underline{H}$, $\tilde{\Delta}(\underline{t}) \subset T_{\underline{H},\underline{t}}$. Moreover, for any point $\underline{t} \in \underline{H}$, we have a canonical sub-Lie algebra $so(3)_{\underline{t}}$ of $\mathfrak{g}$, which together with its Killing form is isomorphic to $\mathbf{E}^3$. If $\underline{t} \in \underline{H}$, $\underline{t} \notin \underline{\Delta}$, the corresponding pair $(\underline{o}, \underline{x})$ defines an axis $\underline{ox}$ in $\mathbf{E}^3$, and a corresponding rotation about this axis, which translate into a unit vector $j \in so(3)$ such that $J = ad(j) \in End_k(so(3))$ defines a complex structure on the plane of $\mathbf{E}^3$ normal to $\underline{ox}$, i.e. $J^2 = -id$. In this way we obtain a complex line bundle $\mathcal{H}$ on $\underline{H}$, a complexification of $O_{\underline{H}}$. The obvious notion of *phase angle* corresponds to a rotation of the space $\mathbf{E}^3$ about the axis $\underline{ox}$. Physically this phase angle is obviously "unobservable".

Notice that the condition $\underline{t} \in \underline{H}$, $\underline{t} \notin \underline{\Delta}$, will be unnecessary once we have generalized to the mathematically correct moduli space, $Hilb^2(\underline{H})$, parametrizing the subschemes of length 2 of $\mathbf{E}^3$, in which the diagonal above is replaced by a blow up, the elements of which are modeled by a point of $\mathbf{E}^3$ and a tangent line.

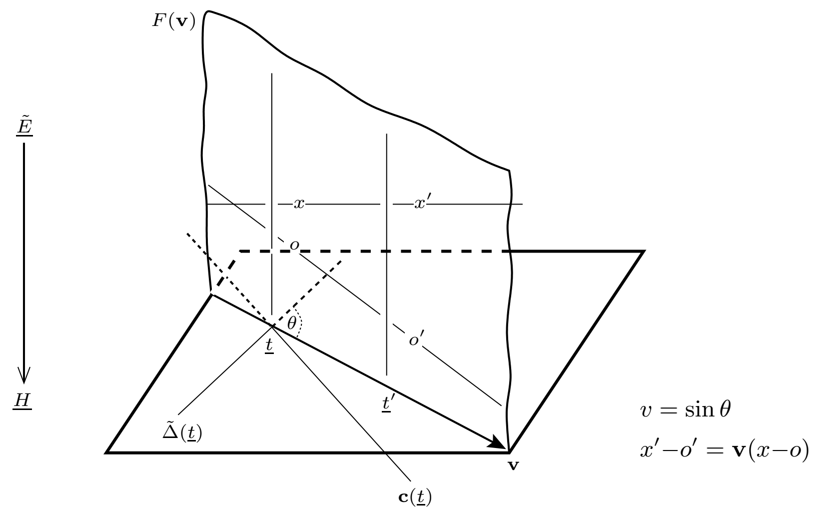


FIGURE 1

Notice also that for any $\underline{t} \in \underline{H}$ the subspaces of $T_{\underline{H},\underline{t}}$ generated by $\tilde{\Delta}(\underline{t})$ and $so(3)_{\underline{t}}$ are normal to each other in the trivial metric deduced from $\mathbf{E}^3$, by identifying an element of $T_{\underline{H},\underline{t}}$ with a pair of tangents, $(\xi, \zeta) \in T_{\mathbf{E}^3 \underline{o}} \times T_{\mathbf{E}^3 \underline{x}}$.

Now let *time* be the only sensible *measure of change* of such a system, i.e. a metric in the moduli space we are working with $\underline{H}$. We shall pick any Riemannian metric ρ, locally equivalent to the ordinary Euclidean distance corresponding to the standard choice of metric on $\mathbf{E}^3$, as explained above.

The universal family

$$\pi : \quad \underline{\tilde{E}} \longrightarrow \underline{H}$$

is, locally, the second projection of the product of $\mathbf{E}^3$ with $\underline{H}$. At a point $\underline{t} \in \underline{H}$, the fiber of π is $\mathbf{E}^3 \simeq so(3)_{\underline{t}}$ together with the ordered pair $(\underline{o}, \underline{x})$ of points corresponding to $\underline{t}$.

The tangent space $T_{\underline{H},\underline{t}}$ of $\underline{H}$ at $\underline{t}$ is a linear space of dimension 6. An element is a *tangent vector*, i.e. a direction with a length attached.

Now, the crucial point is that, with this definition of time, a *relative velocity* of the observed w.r.t. the observer, is a direction, i.e. an oriented one dimensional linear subspace $\mathbf{v}$ in $T_{H,\underline{t}}$. Thus a *relative velocity* is not a tangent vector, but the ray on which this vector sits.

A frame, or a coordinate system, of the corresponding space-time, relative to the velocity $\mathbf{v}$, centered at an event $\underline{t} \in \underline{H}$, is the subspace $F(\mathbf{v}) \subset T_{\overline{\underline{\tilde{E}},\underline{o}}}$ of the total tangent space, mapping onto the line $\mathbf{v}$ in $T_{\underline{H},t}$, parametrized by the length function along $\mathbf{v}$ and some orthogonal coordinate system in $\mathbf{E}^3$. It is a space of dimension 4 with time as one axis. See figure 1. Since the space of relative velocities, $\mathcal{V}(\underline{t})$, at $\underline{t} \in \underline{H}$ is the space of rays, therefore a 5-sphere, thus compact, and a 2-covering of the projective space $Proj(T_{H,t})$, we have minimal and maximal relative velocities at every event $\underline{t} \in \underline{H}$. The 0-velocities correspond to tangent rays in $\tilde{\Delta}(\underline{t})$. The space of 0-velocities is therefore naturally isomorphic to the 2-sphere S^2, and so isomorphic to the complex projective line $\mathbf{P}^1_{\mathbf{C}}$, via the stereographic projection. (The classical spin structure will be dealt with in a forthcoming paper).

Given a tangent vector $\underline{v} \in T_{\underline{H},\underline{t}}$ we shall denote by θ the angle between $\underline{v}$ and $\tilde{\Delta}(\underline{t})$. It is easily checked that the classical space-time velocity corresponding to the ray $\mathbf{v}$ defined by $\underline{v}$ is $v = \sin\theta$.

The maximal velocities $\mathbf{c}(\underline{t})$ correspond to lines transversal to $\tilde{\Delta}(\underline{t})$ in $T_{\underline{H},\underline{t}}$, therefore to $so(3)_{\underline{t}}$ and so to $\theta = \pi/2$.

Now, at $\underline{t} \in \underline{H}$, the subgroup $G(t) \subset G$, fixing the (observer,observed)-point pair parametrized by $\underline{t}$, acts linearly on the tangent space $T_{\underline{H},\underline{t}}$. If $t \notin \underline{\Delta}$, we have seen that $G(t) = T^1$ is the circle group. The quotient $\mathcal{E}(\underline{t}) := \mathcal{V}(\underline{t})/G(\underline{t})$ parametrizes the essential velocities. If $t \notin \underline{\Delta}$ then $\mathcal{E}(\underline{t})$ is of dimension 4. If $\underline{t} \in \underline{\Delta}$ then $G(t) = O(3)$ and $\mathcal{E}(\underline{t})$ is a sphere of dimension 2, see §3, where this is explained.

Notice also that, as explained above, the subgroup $G(t)$ defines a T^1-bundle $\mathcal{T}$ on $\underline{H} - \underline{\Delta}$.

A vector $\underline{p}$ in $T_{\underline{H},\underline{t}}$ is called a momentum, or an energy vector. The length of it, $m = \|\underline{p}\|$, is called the mass, or energy E, of $\underline{p}$. The projection, $\underline{p}_s$ of $\underline{p}$ onto the diagonal $\tilde{\Delta}(\underline{t})$, is called the *0-spin component*, and the projection $\underline{p}_c$ of $\underline{p}$ onto $\mathbf{c}(\underline{t})$ is called the *classical momentum.* The length $m_0 = m\cos\theta$ of $\underline{p}_s$ is called the rest mass, of $\underline{p}$. Notice that the length p_c of $\underline{p}_c$ is $m\sin\theta$, the product of the mass and the classical speed, i.e. the classical momentum. Therefore we have, by definition, Einstein's energy equation,

$$E^2 = p_c^2 + m_0^2.$$

Given two frames $F(\mathbf{v})$ and $F(\mathbf{v}')$ centered at the event $\underline{t}$, we shall show that the only natural linear isomorphisms connecting these two spaces, respecting the maximality of velocities and being associative w.r.t. the velocities, are the elements $L(\mathbf{v}, \mathbf{v}')$ of the Lorenz group. To see this, consider a line l in E^3, and consider the subspace $\rho^{-1}(l)$ of $\underline{H}$ consisting of the points $\underline{t}$ for which $\pi^{-1}(\underline{t})$ is E^3 with $o, p \in l$. Clearly $\underline{H}$ is the union of the $\rho^{-1}(l)$ for l running through the 4-dimensional space of all lines of E^3. Moreover

$$\rho^{-1}(l) \cap \rho^{-1}(l') = \begin{cases} \emptyset & \text{if } l \cap l' = \emptyset \\ l \cap l' & \text{if } l \cap l' = \text{point} \end{cases}$$

In fact, there is an obvious fibration,

$$\rho : \quad \underline{H} - \Delta_0 \to \{\text{Lines in } E^3\},$$

with fibers $\rho^{-1}(l) - \underline{\Delta}$. Moreover, π restricted to $\rho^{-1}(l)$ is nothing but the 1-dimensional situation, i.e. where we study the moduli of ordered pairs of points on the line $\underline{E} = l$, see figure 1.

We shall now see that we have gotten a nice model for special as well as general relativity. Notice, however, that reducing the dimension of our configuration space $\underline{E}$ to 1, we ruin the spin structure of the real world since there is no spin in dimension 1. But otherwise we find a reasonable analogy of the real thing. So let in figure 1 the vertical axis be the real space. Consider the frame F($\mathbf{v}$) above. It is the vertical plane cutting the tangent space $T_{\underline{H},t}$ of the base-plane $\underline{H}$ at $\underline{t}$ in the line $\mathbf{v}$. Let θ be the angle in $T_{\underline{H},t}$ between the diagonal $\tilde{\Delta}$ and $\mathbf{v}$. Contracting the length of $\underline{E}$ (i.e. changing the unit of length in $\underline{E}$ by a factor $\sqrt{2}$), the corresponding relative speed of our two points o, and x, along the velocity $\mathbf{v}$ is $v = \sin(\theta)$. There is an obvious map,

$$\sigma: \quad \Theta_{\underline{H}} \longrightarrow \Theta_{\tilde{E}}$$

given by

$$\sigma(\frac{\partial}{\partial t_v}) = \frac{\partial}{\partial t_v} + v\frac{\partial}{\partial x}$$

or by,

$$\sigma(\frac{\partial}{\partial t_v})(f(t,x)) = \frac{\partial f}{\partial t_v} + v\frac{\partial f}{\partial x}.$$

Consider two momenta, $p_1 = m_1 \frac{\partial}{\partial t_{v_1}}$, $p_2 = m_2 \frac{\partial}{\partial t_{v_2}}$ with the same rest mass, $m_0 = m_1 \cos\theta_1 = m_2 \cos\theta_2$. Then

$$\sigma(p_1) = m_1 \frac{\partial}{\partial t_{v_1}} + m_1 v_1 \frac{\partial}{\partial x}$$
$$\sigma(p_2) = m_2 \frac{\partial}{\partial t_{v_2}} + m_2 v_2 \frac{\partial}{\partial x}$$

and the Lorentz length of these vectors coincide with the rest mass, i.e.

$$m_1^2 - m_1^2 v_1^2 = (1 - v_1^2)(m_0^2/\cos^2(\theta_1)) = m_0^2$$
$$m_2^2 - m_2^2 v_2^2 = (1 - v_2^2)(m_0^2/\cos^2(\theta_2)) = m_0^2$$

Define the angle ϕ by $\phi = \phi_2 - \phi_1$, where $\sin\theta_1 = \tanh\phi_1$ and $\sin\theta_2 = \tanh\phi_2$. Then it is clear that there exists one and only one natural linear map $L(\mathbf{v}_1, \mathbf{v}_2)$, between the frames $F(\mathbf{v}_1)$ and $F(\mathbf{v}_2)$, given by the Lorentz transformation $L(\phi)$, i.e.

$$L(\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{pmatrix} = (1 - v_1^2)^{-1/2}(1 - v_2^2)^{-1/2} \begin{pmatrix} 1 - v_1 v_2 & v_2 - v_1 \\ v_2 - v_1 & 1 - v_1 v_2 \end{pmatrix}$$

such that

$$L(\mathbf{v}_1, \mathbf{v}_2) L(\mathbf{v}_2, \mathbf{v}_3) = L(\mathbf{v}_1, \mathbf{v}_3)$$

corresponding to the Lorentz transformation of the tangent space of the Minkowski space $F \simeq F(\mathbf{v}_i)$, between two different coordinate systems with relative velocity given by the relativistic difference of velocities,

$$\tanh(\phi_2 - \phi_1) = \frac{(v_2 - v_1)}{(1 - v_1 v_2)}.$$

Notice the consequence of this, that the sum of two momenta, one corresponding to the speed of light, and another corresponding to some speed v, has the speed of light.

Definition 1.1 A *local history* (of an observer observing an observed) is a curve γ in $\underline{H}$. A (world) *history* (of any observable observing all of space) is a vector field ξ on $\underline{H}$.

Clearly an integral curve of a history is a local history of an observer observing an observed. We may always parametrize curves of $\underline{H}$ by their length, i.e. by their eigentime, usually called t. But, be careful, eigentime for an object with observer is different from the classical *proper time*, which is measured in the corresponding space-time $S(\gamma) := \pi^{-1}(\gamma)$, see figure 2.

The local history γ, parametrized by τ, has an *energy*, or *momentum* $\frac{\partial\gamma}{\partial\tau}(\underline{t})$ at the point $\underline{t} \in \underline{H}$. Notice that if, as above, τ is time t, then the length of $\dot{\gamma}(\underline{t}) := \frac{\partial\gamma}{\partial t}$ is 1, and $\dot{\gamma}(\underline{t})$ is just a velocity. In general therefore, we have

$$momentum =: p = \frac{\partial\gamma}{\partial\tau}(\underline{t}) = \dot{\gamma}(\underline{t}) \times \frac{\partial t}{\partial\tau}(\underline{t}).$$

So momentum is equal to velocity times "density of mass" $= \|\frac{\partial\gamma}{\partial\tau}(\underline{t})\|$. Moreover, if θ is the angle between the distribution $\tilde{\Delta}$ and the tangentspace T_γ of the curve γ in $\underline{H}$, then $\frac{\partial\theta}{\partial t}$ is the curvature of the curve. Since $\theta = \arcsin v(t)$ we obtain $\frac{\partial\theta}{\partial t} = \dot{v}(t)(1-v^2)^{-1/2}$, where we have put $\frac{\partial v}{\partial t} = \dot{v}$. That is, the curvature of the curve γ is the ordinary acceleration of our system, multiplied with the factor $(1-v^2)^{-1/2}$, as it should. From this follows

$$\frac{\partial p}{\partial t}(\underline{t}) = \frac{\partial\dot{\gamma}}{\partial t}(\underline{t}) \times \frac{\partial t}{\partial\tau}(\underline{t}) = \dot{v}(t)(1-v^2)^{-1/2}((1-v^2)^{1/2}\frac{\partial t}{\partial\tau}(\underline{t})),$$

assuming that $\frac{\partial t}{\partial\tau}(\underline{t})$ is constant, equal to the mass m. Then

$$\frac{\partial t}{\partial\tau}(\underline{t}))(1-v^2)^{1/2}$$

is the rest mass m_0, and the equation above says that,

$$\frac{\partial p}{\partial t}(\underline{t}) = curvature \times rest\ mass$$

The dynamical law, obeyed by our particle should, as usual, be given in terms of a principle of parsimony. In this simple context, where there is no symmetry to take care of, this parsimony must be expressed solely in terms of the geometry of the space $\underline{H}$. That is, the conservation of the momentum, which of course means that $\dot{\gamma}$ should be parallel to itself, or that γ should be a geodesic w.r.t. the connection of $\underline{H}$. In the "trivial" connection ∇_0 given by the Euclidean metric on the real line, the dynamical law simply says that our particle should move in a straight line. If

$$\nabla : \quad \Theta_H \longrightarrow End_{\mathbf{R}}(\Theta_H, \Theta_H)$$

is another connection on $\underline{H}$, it must be given by the trivial one plus a potential

$$A \in Hom_H(\Theta_H, End_H(\Theta_H)) = Hom_H(\Theta_H \otimes_H \Theta_H, \Theta_H).$$

A curve is a geodesic with respect to this new connection, $\nabla = \nabla_0 + A$ if and only if $\nabla_\nu(\nu) = 0$, where $\nu = \dot{\gamma}$ is the unit tangent vector to γ, i.e. if ν is parallel to itself. But this means that

$$\nabla_{0\nu}\nu = -A(\nu)(\nu) = A(\nu \otimes \nu).$$

In general, for any metric of the type considered above ρ, we shall let D be the Levi-Civita connection, see §3. For any *history* ξ we shall have the same form of equation as above. If we want ξ to be be geodesic with respect to $D + A$ then we must have

$$D_\nu\nu = -A(\nu)(\nu) = A(\nu \otimes \nu).$$

Since, as we have seen, $D_\nu\nu$ is the relative acceleration, this equation is a kind of a Lorentz world force law. Recall also (see [**SW**], p.114) that in classical relativity $-\frac{1}{3}Ric(\xi, \xi)$ is the mean relative acceleration of (all neighbors of) a geodesic reference frame ξ. Therefore, if we consider the potential $A(\nu \otimes \nu)$ as an analogue

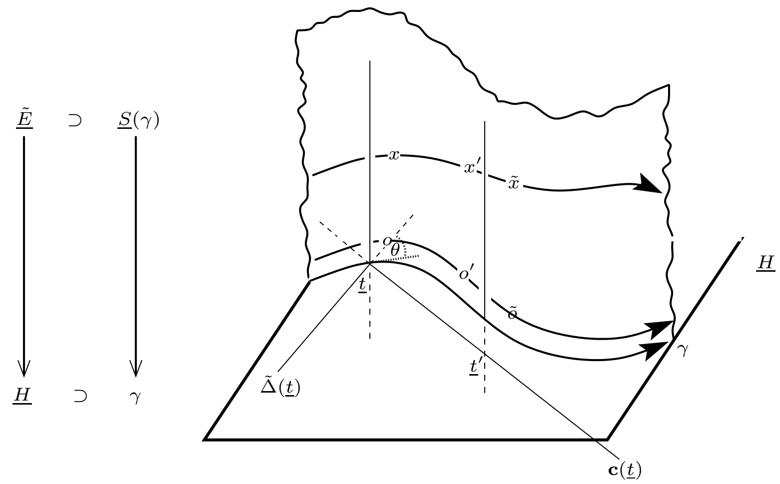


FIGURE 2

of the energy-mass-stress tensor, the above Lorentz world law looks just like the Einstein field equation. In particular,

$$R_\gamma := \text{the curvature of the curve} = \|D_\nu \nu\| = \|A(\nu \otimes \nu)\|.$$

Now, we are presented with two pictures of a history of two points, see figure 2. One in terms of a local history (of an observer observing an observed) γ in time-space $\underline{H}$, the other in terms of the lifted situation, i.e. in the corresponding picture in the 4-dimensional space-time $\underline{S}(\gamma) := \pi^{-1}(\gamma)$.

Corresponding to the local history γ there exist in $\underline{S}(\gamma)$ an observer, $\tilde{o}$ in the sense of [**SW**], and a world line $\tilde{\gamma} := \tilde{x}$.

Corresponding to a world history, ξ there exist a family of *parallel* space-times, one for each integral curve of the history. As a curiosity, one may wonder what would happen if the history has singularities. Then the otherwise parallel universes could conceivably be connected through *warp holes*, or whatever one might call these singularities, where time would come to a standstill and then go on again!

The coordinates of $\underline{S}(\dot{\gamma})$ are $\underline{x} := \underline{ox}$ in the fiber, and t along $\dot{\gamma}$. The metric $d\tau$ of the space-time $\underline{S}(\dot{\gamma})$ is given by

$$d\underline{\tau}^2 = dt^2 - d\underline{x}^2.$$

In fact, let $\dot{\gamma}_0$ be the projection of $\dot{\gamma}$ onto $\tilde{\Delta}$, and compare the spaces $S(\dot{\gamma}_0)$ and $S(\dot{\gamma})$. Clearly the point $(t_0, 0) \in S(\dot{\gamma}_0)$ corresponds naturally to to the point $(t, t\underline{v}) \in S(\dot{\gamma})$, where $\underline{v}$ is the relative velocity defined by $\dot{\gamma}$. Put $v = |\underline{v}|$, then we find

$$t_0^2 = (1 - v^2)t^2,$$

which proves that $d\underline{\tau}^2 = dt^2 - d\underline{x}^2$ is invariant.

From this follows:

$$\begin{aligned} d\underline{x}^2\sigma &= sin^2\theta d\underline{t}^2 = d\underline{u}^2, \\ d\underline{\tau}^2\sigma &= cos^2\theta d\underline{t}^2 = d\underline{\kappa}^2 \end{aligned}$$

where $d\underline{t}^2$ is the metric in $\underline{H}$, $d\underline{u}^2$ is the orthogonal projection onto $\mathbf{c}$, and $d\underline{\kappa}^2$ is the metric of the orthogonal projection onto $\tilde{\Delta}$.

From this we see that we may identify the fiber $\pi^{-1}(\underline{t})$ with the *visible universe*, through $\underline{t}$, i.e. with the integral manifold $\underline{U}(\underline{t})$ of the distribution **c** through $\underline{t}$. Moreover $\underline{U}(\underline{t})$ is easily seen to be naturally isomorphic to the past light cone of $\tilde{o}$ in $S(\gamma)$ at $\underline{t}$. In fact, there is a unique *light-ray* connecting the point $\underline{t} \in \underline{H}$ with a point sitting on $\underline{\mathbf{\Delta}}$, together with the direction towards x on the *nightsphere*, see §3. Therefore, if we at the point $\underline{t} \in \underline{U}(\underline{t}) \subset \underline{H}$, consider the orthogonal decomposition of the tangent space $T_{\underline{H},\underline{t}}$ in the sum of $\tilde{\Delta}$ and **c**, we obtain a decomposition of the metric in $\underline{H}$,

$$d\underline{t}^2 = d\underline{u}^2 + d\underline{\kappa}^2,$$

corresponding to the above decomposition of the metric in the induced space-time,

$$d\underline{\kappa}^2 = d\underline{t}^2 - d\underline{u}^2.$$

In fact, the tangent space of $\underline{S}(\gamma)$ at the corresponding point is identified with the subspace of $T_{\underline{H},\underline{t}}$ generated by T_γ and **c**, and the metric in $\underline{S}(\gamma)$ is obtained via the projection onto $\tilde{\Delta}$, i.e.,

$$d\underline{\tau}^2 = cos^2\theta d\underline{t}^2,$$

properly interpreted.

Conversely, if **M** is a space-time, and $\tilde{o}$ is an observer in a reference frame, we obtain for each world line $\tilde{x}$ a local history γ in the time-space $\underline{H}$. Consider for every point $\tilde{o}(\tau)$ the past light cone, P_τ of $\tilde{o}(\tau)$ and the point $x(\tau) = P_\tau \cap \tilde{x}$, and let γ consist of all the pairs $(o(\tau), x(\tau))$, identifying each past lightcone P_τ with $\mathbf{E}^3$, using the reference frame.

From this we also deduce that geodesic curves in time-space, with constant rest-mass, correspond to geodesic curves in the corresponding space-time, see §3.

Example 1.2 Reduce to the 1-dimensional space situation above, $\rho^{-1}(l)$, and consider a history γ and the corresponding curvature $R_\gamma = \dot{v}(t)(1-v^2)^{-1/2}$. In the corresponding space-time $S(\gamma)$ we have a 2-velocity given by

$$u = ((1-v^2)^{-1/2}, v(1-v^2)^{-1/2})$$

and a 2-acceleration,

$$a = \dot{v}(1-v^2)^{-3/2}.$$

(i) Assume the history is such that the 2-acceleration is constant and equal to g. An easy calculation shows that in the space-time $S(\gamma)$ this gives the curves

$$\tilde{x}: \ (x-x_0)^2 - (t-t_0)^2 = 1/g^2,$$

while in the time-space $\underline{H}$ a corresponding history looks like,

$$t_1 - t_2 = cosh(t_1 + t_2).$$

Put $x = t_1 - t_2$ and $t = t_1 + t_2$, then the speed of the history is $v = sin\,\theta = tanh(t)$, and the acceleration, i.e. the curvature of the hyperbola, is $cosh^{-2}(t)$. Since $(1-v^2)^{-1} = cosh^2(t)$, it follows from the usual Lorentz-transformation that the acceleration of the "mass-point" in its rest system is the product $cosh^{-2}(t)\,cosh^2(t) = 1$, that is, constant.

(ii) Assume now that the curvature R_γ is constant $= 1/r$, then

$$\dot{v}(t)(1-v^2)^{-1/2} = 1/r.$$

This gives us circular history γ in $\underline{H}$, and a sinus curve, an occilator, in the corresponding $S(\gamma)$. This is reasonable, since the constant curvature now relates to the relative situation.

(iii) In §2 and §3 we shall see that there are natural metrics on similar moduli spaces, which are closely related to the Einstein-de Sitter and to the Schwarzschild metrics.

§2. The corresponding cosmological model

Le physicien doit démontrer ce qu'il a l'intention de considérer, ou bien il doit se taire. Le commencement du Monde par création n'est pas physique et ne peut etre prouvé au niveau de la physique (Albert le Grand, 1206-1280, see J.P. Luminet in [**KS**]).

Albert le Grand may be right, but we shall nevertheless try to find some canonical connection on $\underline{H}$, stemming from deformation theory, with an initial object, mimicking a cosmological "big bang".

Notice that we have no hope of finding a model for the universe, with the big bang scenario, using the elementary set up of §1. In fact the model we have considered presupposes a Eucledean ambient space, and has no room for an origin. This is in fact also true for the most popular cosmological model in classical relativity, the Einstein-de Sitter model, even though physicists use it with, seemingly, very positive results.

We must look at another model, for which there is a special point. The easiest one can cook up is the versal space of a double point a on a line, i.e. the algebraic object defined by the k-algebra,

$$O := k \longrightarrow k[x]/(x^2).$$

Since the algebra cohomology, see [**LaPf**],

$$H^1(k, k[x]/(x^2); k[x]/(x^2)) = Hom_{k[x]}((x^2), k[x]/(x^2)) \simeq k^2,$$

we obtain a monomial basis $\{1, x\}$ of $H^1(k, k[x]/(x^2); k[x]/(x^2))$, a versal space $\underline{H}_0 = Spec(H_0)$, $H_0 = k[u_0, u_1]$ and a versal family given in terms of $F = x^2 + u_1 x + u_0 \in H_0[x]$, i.e. defining a morphism,

$$H_0 \longrightarrow H_0[x]/(F) =: \tilde{O}.$$

Corresponding to this family we have a Kodaira-Spencer map,

$$g: \quad Der(H_0) \longrightarrow H^1(H_0, H_0[x]/(x^2 + u_1 x + u_0); H_0[x]/(x^2 + u_1 x + u_0))$$

defined by

$$g(\frac{\partial}{\partial u_0}) = \frac{\partial F}{\partial u_0} = 1, \quad g(\frac{\partial}{\partial u_1}) = \frac{\partial F}{\partial u_1} = x$$

The kernel is readily seen to be,

$$V = < \delta_0, \delta_1 >,$$

where we have put

$$\delta_0 = \frac{1}{2} u_1 \frac{\partial}{\partial u_0} + \frac{\partial}{\partial u_1}, \quad \delta_1 = u_0 \frac{\partial}{\partial u_0} + \frac{1}{2} u_1 \frac{\partial}{\partial u_1}.$$

The discriminant, $\Delta = (1/4)u_1^2 - u_0$, describes the locus $\underline{\Delta}$ of the double points in the family. On $\underline{H}_0 = Spec(H_0)$ we have, in line with our general principles, an obvious logarithmic connection, given by the webb $\{a\delta_0 + b\delta_1 | a, b \in R\}$. Since $[\delta_0, \delta_1] = (1/2)\delta_0$, the structure coeficient being constant, this connection is integrable. The picture of this situation is drawn in figure 3. One checks that the affine

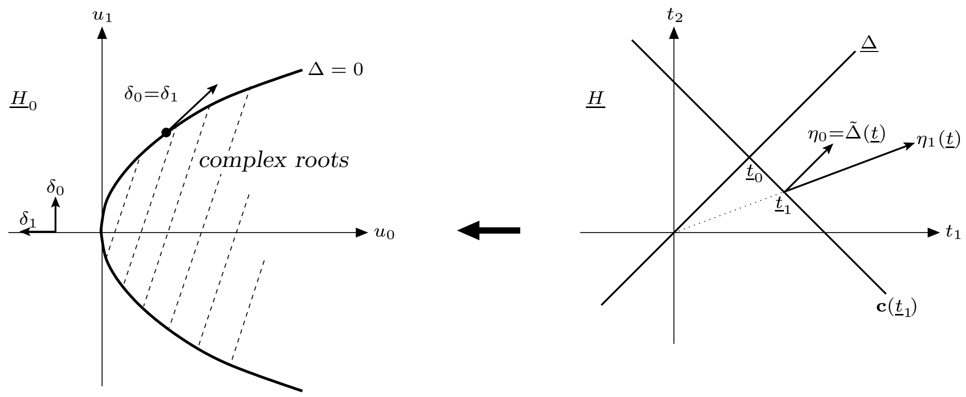


FIGURE 3

group, in one dimension, operates on the lattice $<\delta_0, \delta_1>$ as

$$b^* = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

where $b \in R$.

Notice that we may identify $\underline{H}_0 - \underline{\Delta}$ with the Lie group $\exp\{\delta_0, \delta_1\}$, and notice also that a geodesic through the point $(u_0(0), u_1(0))$ with momentum $m_0\delta_0 + m_1\delta_1$ is given by the differential equations

$$\frac{\partial u_0}{\partial t} = \tfrac{1}{2}\, au_1 + bu_0$$
$$\frac{\partial u_1}{\partial t} = a + \tfrac{1}{2}\, bu_1$$

where $\gamma = (u_0, u_1)$, and $(m_0, m_1) = (a(1/2)u_1(0) + bu_0(0), a + b1/2u_1(0))$. From this one finds,

$$a = \frac{1}{(1/4)u_1(0)^2 - u_0(0)}\,(\tfrac{1}{2}\, u_1(0)m_0 - u_0(0)m_1)$$
$$b = \frac{1}{(1/4)u_1(0)^2 - u_0(0)}\,(-m_0 + \tfrac{1}{2}\, u_1(0)m_1)$$

And the solution,

$$u_1(t) = -(2/b)a + e^{(b/2)t}$$

together with the solution of the equation

$$\frac{\partial u_0}{\partial t} = bu_0 + (a/2)(e^{(b/2)t} - 2a/b).$$

Now consider the map

$$k[u_0, u_1] \longrightarrow k[t_0, t_1],$$

given by sending

$$u_0 \longmapsto t_0t_1$$
$$u_1 \longmapsto (-t_0 - t_1),$$

corresponding to the map of the roots of $x^2 + u_1 x + u_0$ to the symmetric functions in these roots. On the scheme level, this looks like

$$\underline{H} \longrightarrow \underline{H}_0$$

mapping (t_0, t_1) to (u_0, u_1), identifying $\underline{H}_0$ with $\underline{H}/Z_2$. It is easy to see that this map induces the following map on the vector fields,

$$\eta_0 = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1} \longmapsto -2\delta_0$$
$$\eta_1 = t_0 \frac{\partial}{\partial t_0} + t_1 \frac{\partial}{\partial t_1} \longmapsto 2\delta_1$$

We observe that $[\eta_0, \eta_1] = \eta_0$. Therefore $\{\eta_0, \eta_1\}$ defines an integral connection on $\underline{H}$. The picture of $\underline{H}$ is given in figure 3. The distribution $\tilde{\Delta}$, induced by the action of the translation group of $\mathbf{R} = E^1$, is given by η_0.

Notice also that the only geodesics that emanate from the origin, the big bang, are the radial half lines, i.e. the integrals of the distribution η_1. We have the following relations,

$$m_0^2 = \text{rest mass}^2 = \tfrac{1}{2}(t_1 + t_2)^2$$
$$x = m_0 \tan\theta = \tfrac{1}{2}(t_1 - t_2)$$
$$v = \sin\theta = \tfrac{x/m_0}{\sqrt{1+(x/m_0)^2}}$$
$$r^2 = \tfrac{1}{2}(t_1^2 + t_2^2)$$

and therefore,

$$\sin\theta = \tfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$
$$\tan\theta = x/m_0$$
$$x = r\sin\theta = rv.$$

Remark 2.1 Suppose we are at the point $\underline{t}_1 \in \underline{H}$, i.e. suppose $\underline{t}_1$ corresponds to a pair (o, x), i.e. me o, watching a point x, and measure the relative velocity $\mathbf{v}$ of (o, x). Since the assumption is that we are on a geodesic, emanating from the big bang, we calibrate our position, using the translation group of E^1, and we therefore know, exactly *where we are.*

Our *visible universe* is then given by the normal section of $\tilde{\Delta}(\underline{t}_1)$, through $\underline{t}_1$, see figure 3. Now, a photon leaving sometime after the big bang travelling via the star x, arriving at the observer, i.e. me, at the point $\underline{t}_0$ on the intersection of the *visible universe* with $\underline{\Delta}$, must have used some time T equal to the sum of the time T_1 of reaching $\underline{t}_1$ and the time $T_{1,0}$ to pass from $\underline{t}_1$ to $\underline{t}_0$. To make sense of this we have to identify, or fuse, the visible universe and the space $\mathbf{c}(\underline{t}_1)$, see figure 3, i.e. the space of light-rays emanating from the big bang. This we can do by considering a 1-dimensional Friedmann equation. Define the energy-density $e^2(\underline{t}_1)$ of the universe at $\underline{t}_1$ to be the inverse of the length of η_1, the *momentum of the universe.* For the universe, i.e. the vacuum, to conserve its energy, we must have a Friedmann equation,

$$R(\underline{t})e^2(\underline{t}) = C$$

where $R(\underline{t})$ is the "unit length" of the visible universe at $\underline{t}$, and C a constant along $\mathbf{c}$. This amounts to changing the metric of $\underline{H}$ dividing ρ by $(1+\tan^2(\theta))$, or multiplying

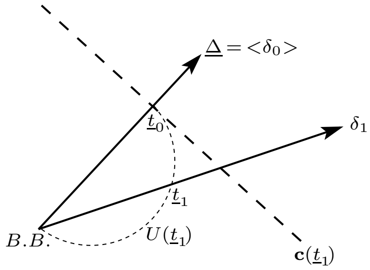


Figure 4

with $\cos^2(\theta)$. This does not change the connection. However, the distance in each visible universe, see figure 4, is now

$$d(o,x) = m_0 \int_0^x d(x/m_0)/(1+(x/m_0)^2) = m_0 \arctan(x/m_0) = m_0\theta$$

which means that the visible universe is closed and, in fact, a circle with diameter given by m_0, see figure 4. Moreover the proper time spent since the big bang is now

$$T_* = \int_{r_0}^{r} \cos^2(\theta)\, dr$$

along the $\mathbf{v}$. Since θ is constant along $\mathbf{v}$ and $r\cos(\theta) = m_0$, we find

$$T_* = \cos(\theta) m_0 = m_0(1-v^2)^{1/2},$$

which is consistent with the fact that viewed from the space-time frame of $\underline{t}_0$, the time would be m_0. With this, the observer finds that in the visible universe the receding speed of the observed x is $\sin(\theta)$, while the distance to x is $m_0\theta$. Therefore the Hubble "constant" is

$$H(m_0) = \sin(\theta)/(m_0\theta)$$

which is almost as the expected $1/m_0$. In fact the difference is the factor $\theta/\sin(\theta)$ with $\pi/2 \geq \|\theta/\sin(\theta)\| \geq 1$, such that we find,

$$2/\pi \times 1/m_0 \leq H(m_0) \leq 1/m_0.$$

From this it follows that the youngest part of our universe, i.e. the part in the observer's neighbourhood, is m_0 units old. Moreover, the Hubble "constant" is not constant, it is decreasing with increasing distance, reaching $2/\pi \sim 2/3$ times the maximum value at the big bang. Recall that 2/3 is exactly the value given by the Einstein-de Sitter universe in dimension 3.

To see that the Einstein-de Sitter universe should enter this picture, we extend our 1-dimensional model to 3-dimensions. This is easy. The squared energy-density $e^2(\underline{t}_1)$ of the universe at $\underline{t}_1$ is as above the inverse of the squared length of η_1, the momentum of the universe, and for the universe to conserve its energy, we must have the Friedmann equation in dimension 3,

$$R(\underline{t})^3 e^2(\underline{t}) = C$$

where $R(\underline{t})$ as above is the "unit length" of the visible universe at $\underline{t}$, and C a constant along $\mathbf{c}(\underline{t})$.

Since

$$e^2(\underline{t}) = m_0^{-2}\cos^2(\theta)$$

we find

$$R(\underline{t}) = C^{1/3}r^{-2/3}$$

The corresponding metric in the 4-space-time with time axis $\mathbf{v}$ must be

$$-dt^2 + C^{-2/3}(r^{2/3})^2 d\underline{x}^2$$

i.e. essentially the Einstein-de Sitter metric.

Since we have assumed that our model space is the Euclidean 3-dimensional space, we may never see the moment of the big bang. If we had assumed the model space compact, our visible space would have been compact and therefore contain the point of the big bang, which we therefore should be able to see!

This is coherent with a general philosophy, accepting that vision, or perception communicated by light, constitutes the space of the universe. Each reality, i.e. each fiber of π, representing a pair (o, x) in the model space E^3, induces an (algebraic or differentiable) isomorphism between the model space and the visible universe at that event. But these spaces are certainly not equal!

§3. Moduli spaces of representations of associative algebras

We must confess in all humility that, while number is a product of our mind alone, space has a reality beyond the mind whose rules we cannot completely prescribe. (Carl-Friedrich Gauss (1830).)

The space $\underline{H}$ of §1 is the space of all ordered pairs of points of $\mathbf{E}^3$. Clearly there can be no physical difference in interchanging the observer with the observed. So we really want to work in the space of unordered pairs of points, as we saw in §2, i.e. we must consider the moduli space of all unordered pairs of points in $\mathbf{E}^3$, the space $Hilb^2(\mathbf{A}^3)$ parametrizing all length 2 subschemes of $\mathbf{A}^3$, classically constructed by blowing up the diagonal of $(\mathbf{A}^3 \times \mathbf{A}^3)/Z_2$.

Now it turns out that this space is really non-commutative, see [**La4**]. Let A be any commutative k-algebra, and consider the *swarm* of 2-dimensional representations of A, see also [**La5**]. In particular, consider a double point, $V = k[\epsilon]$, of $Spec(A)$, as an A-module. Any such corresponds to a point in $Spec(A)$ together with a tangent-direction. Suppose first, in line with §1, that $A = k[x_1, x_2, x_3]$ is a polynomial k-algebra, such that $Spec(A)$ may be identified with the scheme underlying the Eucledean 3-space, and let us compute the deformation functor Def_V. The computation is basically the same for 3 as for 2 variables, so to make the formulas more amenable, we shall restrict to $A = k[x_1, x_2]$.

Lemma 3.1 The (non-commutative) formal moduli, $H(V)$ of the A-module $V = A/(x_1^2, x_2)$, is given as the completion of the k-algebra,

$$U := U(A) = k\{t_1, t_2, \omega_1, \omega_2\}/(y_1, y_2)$$

where

$$y_1 = [t_1, t_2] - t_1[\omega_1, \omega_2] \qquad y_2 = [t_1, \omega_2] - [t_2, \omega_1] - \omega_1[\omega_1, \omega_2],$$

and where the family of left U-and right H-modules,

$$U \otimes_k k^2$$

is defined by the actions of x_1 and x_2, given by,

$$x_1 = \begin{pmatrix} 0 & -t_1 \\ 1 & -\omega_1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} t_2 & -t_1\omega_2 \\ \omega_2 & t_2 - \omega_1\omega_2 \end{pmatrix}.$$

Proof: Consider the obvious free resolution of $V := A/(x_1^2, x_2)$ as an H-module,

$$V \xleftarrow{\rho} A \xleftarrow{d_0} A^2 \xleftarrow{d_1} A \xleftarrow{d_2} 0$$

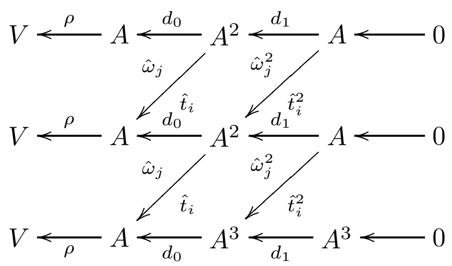
where we have

$$d_0 = (x_1^2, x_2), \quad d_1 = \begin{pmatrix} x_2 \\ -x_1^2 \end{pmatrix}.$$

Consider the Yoneda complex, and pick a basis

$$\{\hat{t}_1, \hat{t}_2; \hat{\omega}_1, \hat{\omega}_2, \}$$

of $Ext^1_H(V, V)$ represented by the morphisms of the diagram



Here,

$$\hat{t}_1 = (1, 0), \qquad \hat{t}_2 = (0, 1,);$$

$$\hat{\omega}_1 = (x_1, 0), \quad \hat{\omega}_2 = (0, x_1)$$

and

$$\hat{t}_1^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{t}_2^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and finally,

$$\hat{\omega}_1^2 = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \quad \hat{\omega}_2^2 = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix}.$$

Using this it is easy to see that

$$\hat{t}_i \cup \hat{t}_i = 0, \quad \hat{t}_1 \cup \hat{t}_2 = -\hat{t}_2 \cup \hat{t}_1 = \hat{y}_1,$$

and that

$$\hat{t}_1\hat{\omega}_2^2 = \hat{\omega}_1 \hat{t}_2^2 = -\hat{y}_2, \quad \hat{\omega}_i \hat{t}_i^2 = 0, \quad \hat{\omega}_i \hat{\omega}_j^2 = 0, \quad \hat{t}_2 \hat{\omega}_1^2 = \hat{\omega}_2 \hat{t}_1^2 = \hat{y}_2,$$

therefore

$$-y_2 = \hat{t}_1 \cup \hat{\omega}_2 = \hat{\omega}_1 \cup \hat{t}_2 = -\hat{t}_2 \cup \hat{\omega}_1 = -\hat{\omega}_2 \cup \hat{t}_1, \quad \hat{\omega}_i \cup \hat{\omega}_j = \hat{t}_i \cup \hat{t}_j = 0.$$

Now, consider the dual basis $\{t_1, t_2; \omega_1, \omega_2\}$ generating the hull of the deformation functor $Def_{k[\epsilon]}$, we find after a simple computation of the third order Massey products the formulas we want.

Notice that to see that our result holds, we just have to compute the tangent situation and check that our formulas give us a lifting of the quadratic relations and of the corresponding H-action. □

By a simple computation one checks that the k-points of $\underline{U}(A)$ form an open dense subset of $Hilb^2\mathbf{A}^2$ containing V. It glues together with the corresponding versal family containing $W = k[x_1,x_2]/(x_1,x_2^2)$, and the union, which we still denote $\underline{U}(A)$, is a non-commutative version of $Hilb^2\mathbf{A}^2$. In fact we find the eigenvalues of x_1 are the solutions of the equation, $\lambda^2 + \omega_1\lambda + t_1 = 0$. The corresponding eigenvalues of x_2 are $t_2 + \omega_2\lambda$ (refer to §2). When the discriminant $\Delta \neq 0$, any k-point of $\underline{U}$ corresponds to an unordered pair of different points in $\underline{A}$. The set of representations that are double points is given by $\Delta = \omega_1^2 - 4t_1 = 0$. In this case we may pick a basis for k^2, given by $\psi_1 = (1,0)$, $\psi_2 = (\omega_1/2, 1)$, such that, expressed in terms of these basis elements the action of the x_1 and x_2 are given by,

$$x_1 = \begin{pmatrix} -\omega_1/2 & 0 \\ 1 & -\omega_1/2 \end{pmatrix}, \quad x_2 = \begin{pmatrix} t_2 & 0 \\ 0 & t_2 \end{pmatrix} + x_1\omega_2.$$

This corresponds to a double point of $\mathbf{A}^2$, or to the pair of a point,

$$(-\omega_1/2, -\omega_1\omega_2/2 + t_2) \in \mathbf{A}^2$$

and a tangent-direction,

$$(1,\omega_2): \quad (\begin{pmatrix} -\omega_1/2 & 0 \\ 1 & -\omega_1/2 \end{pmatrix}, \begin{pmatrix} -\omega_1\omega_2/2 + t_2 & 0 \\ \omega_2 & -\omega_1\omega_2/2 + t_2 \end{pmatrix})$$

at that point. Notice that the trivial module structure on k^2 is not part of $\underline{U} = Simp_1 U$. Thus, $Simp_1(\underline{U})$, i.e. the k-points of $\underline{U}$, is identified with $Hilb^2\mathbf{A}^2$, which is the blow up of $(\mathbf{A}^2 \times \mathbf{A}^2)/\mathbf{Z}_2$ along the diagonal.

Notice that there are also other simple representations of U. The Weyl representation,

$$wey: \quad U \longrightarrow Diff_k(A) = k[t_1, t_2, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}] \subset End_k(A),$$

defined by the canonical representation of U on A, i.e. mapping the ω_i to the $\frac{\partial}{\partial t_i}$, shows that $k|t_1, t_2]$ is a simple representation of U. And there is a surjective homomorphism,

$$ham: \quad U \longrightarrow k[t_1, t_2, \xi_1, \xi_2],$$

the "Hamilton representation", mapping the ω_i to the commuting ξ_i.

Now go back to §1. There we considered the moduli space $\underline{H}$ of ordered pairs of two points (observer, observed) in $\mathbf{E}^3$, as the base space for our "geometry". In §2 it turned out that the reasonable thing to do, was to consider the Hilbert scheme $Hilb^2_{\mathbf{E}^3}$, and the map $(\underline{H} - \underline{\Delta}) \longrightarrow Hilb^2_{\mathbf{E}^3}$, and deduce from the latter the distribution (δ_0, δ_1). Since $Hilb^2_{\mathbf{E}^3} \subset \underline{U}(\mathbf{E}^3)$, as k-point sets, it is reasonable to think that the "physical" significance sits in the space $\underline{U} := \underline{U}(\mathbf{E}^3)$, which is non-commutative. We observe that our $\underline{H}$ is the obvious double covering on the Hamiltonian representation of U, separating the observer from the observed. Moreover, let $\underline{\tilde{H}}$ be the blow-up of $\underline{H}$ in $\underline{\Delta}$, replacing any point of $\underline{\Delta}$ with the projective 2-space, i.e. a 2-sphere with diametrically opposed points identified, then we see that there is a surjective morphism,

$$\underline{\tilde{H}} \longrightarrow Simp_1(U).$$

Observe now that parity inversion P, in $\mathbf{E}^3$, together with time-inversion T, in $\underline{H}$ reduces to the identity in $\underline{U} = Hilb^2(\mathbf{E}^3)$. We have PT invariance as we should!

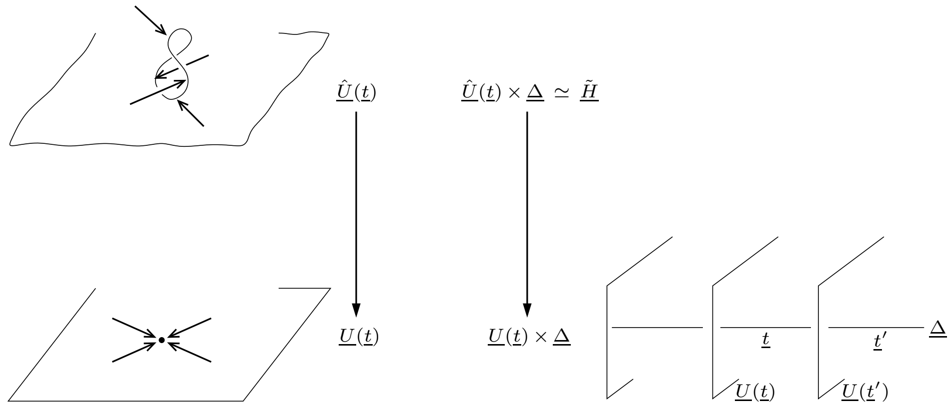


FIGURE 5

Let us, as above, pick the following coordinates for H, $t_i = \frac{1}{2}(x_i + o_i),\ i = 4,5,6$, in the direction of $\underline{\Delta}$ and $t_i = \frac{1}{2}(x_i - o_i),\ i = 1,2,3$, directed along $\mathbf{c}$. Let $\underline{c}$ be the maximal integral submanifold of the distribution $\mathbf{c}$ defined by $\{t_i = 0,\ i = 4,5,6\}$. Since

$$\underline{\tilde{H}} \subset \underline{H} \times \mathbf{P}^2, \quad \underline{\tilde{H}} = \underline{\tilde{c}} \times, \underline{\Delta},$$

where $\underline{\tilde{c}}$ is the blow up of $\underline{c}$ in the origin $\{t_i = 0,\ i = 1,2,3.\}$, it is easy to see that the Fubini-Studi metric on $\mathbf{P}^2$, with the projective coordinates $(\lambda_1 : \lambda_2 : \lambda_3)$, is given, in the affine piece defined by $\lambda_1 = 1$, by

$$(1 + \lambda_2^2 + \lambda_3^2)^{-1}(d\lambda_2^2 + d\lambda_3^2) - (1 + \lambda_2^2 + \lambda_3^2)^{-2}(\lambda_2 dt_2 + \lambda_3 dt_3)^2.$$

In the affine piece of $\underline{\tilde{c}}$ defined by $t_1 \neq 0$, we have

$$\lambda_2 t_1 = t_2, \quad \lambda_3 t_1 = t_3,$$

and the metric is given by

$$ds^2 = dt_1^2 + (1 + r^{-2})dt_2^2 + (1 + r^{-2})dt_3^2 - r^{-4}(t_2 dt_2 + t_3 dt_3)^2,$$

where we have put $r^2 = t_1^2 + t_2^2 + t_3^2$. The corresponding metric in $\underline{\tilde{H}}$ is then given as

$$dt^2 = ds^2 + d\lambda^2,$$

where $d\lambda^2 = dt_4^2 + dt_5^2 + dt_6^2$ is the metric along $\tilde{\Delta}$.

As we have seen above, the real moduli space for the two-point situation, is not $\underline{H}$, nor its blow-up along $\underline{\Delta}$, but actually, a quotient space of this blow up, the non-commutative Hilbert-scheme $\underline{U} = Hilb^2_{\mathbf{E}^3}$. It is difficult to draw a picture of this space, but if we reduce the configuration space $\mathbf{E}^3$ to a plane, with polar coordinates (r, ϕ), and the diagonal $\underline{\Delta}$ to a line, with coordinate κ, it looks like figure 5. In this picture the visible universe, or the light-section $\underline{U}(\underline{t})$ of $\mathbf{c}(\underline{t})$, looks like a plane with the origin blown up. A point of the exceptional curve, which in the complete model is a projective plane, and in figure 5 a circle, corresponds to a double point, with a given tangent-line normal to $\tilde{\Delta}$. Therefore $\underline{U}(\underline{t})$ is no more outfitted with a centre for the polar coordinates, there is no origin in the $\underline{U}(\underline{t})$. The subset $r = 0$ corresponds to a sphere, not a point. We have computed the metric of $\underline{U}$ above, and in our simplified picture of figure 5, the metric looks like

$$dt^2 = dr^2 + (r^2 + 1)d\phi^2 + d\lambda^2.$$

Now, put $\rho = r + h$ such that $\rho^2 = r^2 + 1$, then $2\rho d\rho = 2rdr$. Obviously, h is not a constant, but rather a kind of diameter of the "black hole" at $r = 0$. We find

$$dr^2 = (\rho/r)^2 d\rho^2.$$

With this, our metric becomes

$$dt^2 = (1 - h/\rho)^{-2} d\rho^2 + \rho^2 d\phi^2 + d\lambda^2.$$

Multiply with $r/\rho = (1 - h/\rho)$, and get

$$d\sigma^2 = (1 - h/\rho)^{-1} d\rho^2 + r\rho d\phi^2 + (1 - h/\rho) d\lambda^2.$$

The metric for the space-time corresponding to 0 velocity is then

$$g' = (1 - h/\rho)^{-1} d\rho^2 + r\rho d\phi^2 - (1 - h/\rho) dt^2.$$

If we may assume r and ρ are very close to each other, this is amazingly (?) like the Schwarzschild metric, see [**SW**], p.30,

$$g = (1 - (2\mu/r))^{-1} dr \otimes dr + r^2 p^* h - (1 - (2\mu/r)) dt \otimes dt,$$

r corresponding to our ρ and μ, the radius of the black hole, to $1/2h$. Notice that along the direction of $\tilde{\Delta}$, i.e. the $\frac{\partial}{\partial t}$ the term $(1 - (2\mu/r))$ is constant.

The non-commutative phase space. Now, consider any moduli scheme $\underline{M}$, say the 6-dimensional *space-time* $\underline{H}$ considered above. Clearly there cannot, in general, be any objective origin in $\underline{M}$. The *physically interesting* functions measuring change of *the interesting observables* parametrizing the moduli space, must be "functions" of pairs of points and tangents in $\underline{M}$.

The classical phase space of Hamiltonian mechanics is considered to be the space of all pairs (point, momentum) of the configuration space, and the regular functions on this phase space are the basic tools of Hamiltonian mechanics.

In our case, see §1, a momentum is uniquely represented as a tangent vector of $\underline{H}$, so we are interested in the space of all pairs (point, tangent vector at the point) of $\underline{H}$. Clearly a nonzero momentum determines a mass, i.e. the length of the tangent vector, and a velocity, the direction of the tangent vector. The proper phase space of $H = k[t_1, t_2, \ldots, t_6]$ or of $\underline{H}$ is therefore given as the space of surjective algebra homomorphisms,

$$H \longrightarrow k[\epsilon].$$

The coordinate ring of this space is given as a quotient, Ph, of the algebra of non-commuting polynomials in the variables $\tau_i,\ \omega_j,\ i,j = 1, \ldots, 6$ together with a surjection of Ph-algebras,

$$Ph \otimes H \longrightarrow Ph \otimes k[\epsilon],$$

determined by the action T_i of t_i on $Ph \otimes k[\epsilon]$. Clearly,

$$T_i = \begin{pmatrix} \tau_i & 0 \\ \omega_i & \tau_i \end{pmatrix},$$

where τ_i corresponds to t_i and ω_i to $\frac{\partial}{\partial t_i}$ and the only relations among the T_i's are that they commute, i.e. that $[T_i, T_j] = 0$, which imposes the relations $[\tau_i, \tau_j] = 0,\ \omega_i\tau_j + \tau_i\omega_j = \omega_j\tau_i + \tau_j\omega_i$, on Ph. Therefore we have

Definition 3.2 The non-commutative phase space $Ph(H) =: Ph$ for H, the moduli space for all pairs of a k-point and a non-zero tangent vector in that point, is given as the k-algebra,

$$Ph = k < \tau_1, \dots, \tau_6, \omega_1, \dots \omega_6 > /([\tau_i, \tau_j], [\tau_i, \omega_j] - [\tau_j, \omega_i] |\ i, j = 1, \dots, 6)$$

together with the family of left Ph-and right H-algebras

$$Ph \otimes_k k(\underline{t})[\epsilon],$$

with H-acting on the right as,

$$(1 \otimes 1) * t_i = (\tau_i \otimes 1) + \omega_i \otimes \delta_i(t_i)\epsilon$$

where $\delta_i = \frac{\partial}{\partial t_i}$ is the vector field dual to ω_i.

By functoriality there is an embedding $i: \ H \subset Ph$, and a universal derivation

$$\tilde{\delta}: \quad H \longrightarrow Ph,\ \tilde{\delta}(t_i) = \omega_i.$$

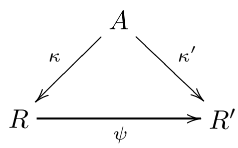
Moreover, given a vector field of $\underline{H}$, i.e. a derivation $\xi \in \theta_H := Der_k(H, H)$ there is an obvious homomorphism of k-algebras,

$$e_\xi: \quad Ph \longrightarrow H.$$

In fact, for every point $\underline{t} \in \underline{H}$, there is a corresponding point $(\underline{t}, \xi(\underline{t})) \in \underline{Ph}$. Therefore any $f \in Ph$ has a value $f(\underline{t}, \xi(\underline{t}))$ and e_θ is this evaluation map. Any quadratic $g \in Ph$, of the form, $g = \sum_{i,j=1,\dots 6} g_{i,j}\omega_i\omega_j$ therefore is a candidate for a metric on $\underline{H}$. However, there is an essential problem of invariance, which we shall discuss later, and which implies that there is essentially a unique choice, the usual $\Delta = \sum \omega_i^2$. First let us generalize the Ph-construction to any k-algebra A.

The general case. The non-commutative phase-space, Ph constructed above is a special case of a more general construction.

Given a k-algebra A, denote by A/k-$\underline{alg}$ the category where the objects are homomorphisms of k-algebras $\kappa: \ A \to R$, and the morphisms $\psi: \ \kappa \to \kappa'$ are commutative diagrams,



and consider the functor

$$Der_k(A, -): \quad A/k\text{-}\underline{alg} \longrightarrow \underline{Sets}.$$

It is representable by a k-algebra morphism

$$\iota: \quad A \longrightarrow Ph(A),$$

with a *universal family* given by a universal derivation

$$\tilde{\delta}: \quad A \longrightarrow Ph(A).$$

In fact this can be constructed as the non-commutative versal base of the deformation functor of the morphisme $\rho: \ A \to k[\epsilon]$.

Clearly we have the identities

$$\tilde{\delta}_*: \quad Der_k(A, A) = Mor_A(Ph(A), A),$$

and

$$\tilde{\delta}^* : \quad Der_k(A, Ph(A)) = End_A(Ph(A)),$$

the last one associating $\tilde{\delta}$ to the identity endomorphism of Ph. Let now V be a right A-module, with structure morphism $\rho : A \to End_k(V)$. We obtain a universal derivation,

$$c : \quad A \longrightarrow Hom_k(V, V \otimes_A Ph(A)),$$

defined by $c(a)(v) = v \otimes \tilde{\delta}(a)$. Using the long exact sequence

$$0 \to Hom_A(V, V \otimes_A Ph(A)) \to Hom_k(V, V \otimes_A Ph(A)) \to^\iota$$
$$Der_k(A, Hom_A(V, V \otimes_A Ph(A))) \to^\kappa Ext^1_A(V, V \otimes_A Ph(A)) \to 0,$$

we obtain the non-commutative Kodaira-Spencer class,

$$c(V) := \kappa(c) \in Ext^1_A(V, V \otimes_A Ph(A)),$$

inducing the Kodaira-Spencer morphism,

$$g : \quad \Theta_A := Der_k(A, A) \longrightarrow Ext_A(V, V),$$

via the identity, $\tilde{\delta}_*$. If $c(V) = 0$, then the exact sequence above proves that there exist a $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ such that $\tilde{\delta} = \iota(\nabla)$. This is just another way of proving that $\tilde{\delta}$ is given by a connection,

$$\nabla : \quad Der_k(A, A) \longrightarrow Hom_k(V, V).$$

The Kodaira-Spencer class gives rise to a Chern character by putting,

$$ch^i(V) := \frac{1}{i!}\, c^i(V) \in Ext^i_A(V, V \otimes_A Ph(A)),$$

and if $c(V) = 0$, the curvature $R(V)$ induces a curvature class,

$$R_\nabla \in H^2(k, A; \Theta_A, End_A(V)).$$

Any $Ph(A)$-module W, given by its structure map,

$$\rho_W : \quad Ph(A) \longrightarrow End_k(W),$$

corresponds bijectively to an induced A-module structure on W, and a derivation $\delta_\rho \in Der_k(A, End_k(W))$, defining an element

$$[\delta_\rho] \in Ext^1_A(W, W).$$

Fixing this element we find that the set of $Ph(A)$-module structures on the A-module W is in one to one correspondence with

$$End_k(W)/End_A(W).$$

Conversely, starting with an A-module V and an element $\delta \in Der_k(A, End_k(V))$, we obtain a $Ph(A)$-module V_δ. It is then easy to see that the kernel of the natural map

$$Ext^1_{Ph(A)}(V_\delta, V_\delta) \to Ext^1_A(V, V),$$

induced by the linear map

$$Der_k(Ph(A), End_k(V_\delta)) \to Der_k(A, End_k(V)),$$

is the quotient

$$Der_A(Ph(A), End_k(V_\delta))/End_k(V),$$

and the image is a subspace $[\delta_\rho]^\perp \subseteq Ext^1_A(V, V))$, which is rather easy to compute, see examples below.

Remark. Since $Ext^1_A(V,V)$ is the tangent space of the miniversal deformation space of V as an A-module, we see that the non-commutative space PhA also parametrizes the set of *generalized momenta*, i.e. the set of pairs of a simple module $V \in Simp(A)$, and a tangent vector of $Simp(A)$ at that point.

Example 3.3. Let $A = k[t]$, then obviously, $Ph(A) = k < t, \omega >$ and $\tilde{\delta}$ is given by $\tilde{\delta}(t) = \omega$, such that for $f \in k[t]$, we find $\tilde{\delta}(f) = J_t(f)$ with the notations of [**La5**], i.e. the non-commutative derivation of f with respect to t. One should also compare this with the non-commutative Taylor formula of [**La5**]. If $V \simeq k^2$ is an A-module, defined by the matrix $X \in M_2(k)$, and $\delta \in Der_k(A, End_k(V))$ is defined in terms of the matrix $Y \in M_2(k)$, then the $Ph(A)$-module V_δ is the $k < t, \omega >$-module defined by the action of the two matrices $X, Y \in M_2(k)$, and we find

$$e^1_V : \ = dim_k Ext^1_A(V,V) = dim_k End_A(V) = dim_k \{Z \in M_2(k) | \ [X,Z] = 0\}$$

$$e^1_{V_\delta} : \ = dim_k Ext^1_{Ph(A)}(V_\delta, V_\delta) = 8 - 4 + dim\{Z \in M_2(k) | \ [X,Z] = [Y,Z] = 0\}.$$

We have the following inequalities:

$$2 \leq e^1_V \leq 4 \leq e^1_{V_\delta} \leq 8.$$

The phase-space construction may, of course, be iterated. Given the k-algebra A we may form the sequence $\{Ph^n A\}_{1 \leq n}$, defined inductively by

$$Ph^0 A = A, \quad Ph^1 A = PhA, ..., Ph^{n+1} A = PhPh^n A.$$

Let $i^n_0 : \ Ph^n A \to Ph^{n+1} A$ be the canonical embedding, and let $d_n : \ Ph^n A \to Ph^{n+1} A$ be the corresponding derivation. Since the composition of i^n_0 and the derivation d_{n+1} is a derivation $Ph^n A \to Ph^{n+2}$, there exists by functoriality a homomorphism $i^{n+1}_1 : \ Ph^{n+1} A \to Ph^{n+2}$, such that

$$d_n \circ i^{n+1}_1 = i^n_0 \circ d_{n+1}.$$

Notice that we compose functions and functors from left to right. Clearly we may continue this process constructing new homomorphisms,

$$\{i^n_j : \ Ph^n A \to Ph^{n+1} A\}_{0 \leq j \leq n},$$

with the property

$$d_n \circ i^{n+1}_{j+1} = i^n_j \circ d_{n+1}.$$

The system of k-algebras and homomorphismas of k-algebras $\{Ph^n A, i^n_j\}_{n, 0 \leq j \leq n}$ has an inductive (direct) limit, $Ph^\infty A$, together with homomorphisms

$$i_n : \quad Ph^n A \longrightarrow Ph^\infty A,$$

satisfying

$$i^n_j \circ i_{n+1} = i_n, \quad j = 0, 1, ..., n.$$

Moreover, the family of derivations, $\{d_n\}_{0 \leq n}$ defines a unique derivation

$$\delta : \quad Ph^\infty A \longrightarrow Ph^\infty A,$$

such that

$$i_n \circ \delta = d_n \circ i_{n+1}.$$

The k-algebra $Ph^\infty A$ has a descending filtration of two-sided ideals, $\{F^n\}_{0 \leq n}$ given by

$$F^n = Ph^\infty A \cdot im(\delta^n) \cdot Ph^\infty A$$

such that the derivation δ induces derivations,

$$\delta_n : F^n \longrightarrow F^{n+1}.$$

Definition 3.4 For a given k-algebra A, the k-algebra $Ph^\infty(A)$ will be called the k-algebra of higher differentials, and the completion of $Ph^\infty(A)$ in the topology given by the filtration $\{F^n\}_{0\leq n}$, denoted by $\mathcal{D}(A)$, will be called the formalized k-algebra of higher differentials. Clearly δ defines a derivation on $\mathcal{D}(A)$, and an isomorphism of k-algebras,

$$\epsilon := exp(\delta) : \quad \mathcal{D}(A) \to \mathcal{D}(A).$$

Proposition 3.5 Let A be a finitely generated k-algebra, which is generated by $\{t_i\}_{i=1,...,n}$, and let $k(0)$ be a one-dimensional representation, i.e. a point of $Simp_1(A)$, corresponding to a two-sided maximal ideal $\mathfrak{m} \subset A$. We may assume $\mathfrak{m} = (t_1, ..., t_n)$. Then the elements $\tau_i := \delta t_i + 1/2\delta^2 t_i + \, ... \, + 1/n!\delta^n t_i + \, ... \in \mathcal{D}$ generate a subalgebra of $\mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D})$, isomorphic to A.

Proof: Let $f \in k < t_1, \ldots, t_n >$ be a relation for the k-algebra A. Then, computing $\epsilon(f)$, there is a Taylor series,

$$f(t_1 + \delta t_1 + \tfrac{1}{2}\,\delta^2 t_1 + \ldots + \tfrac{1}{n!}\,\delta^n t_1 + \ldots, \ldots, t_n + \delta t_n + \tfrac{1}{2}\,\delta^2 t_n 1 + \ldots + \tfrac{1}{n!}\,\delta^n t_n + \ldots)$$
$$= f(t_1, \ldots, t_n) + \tfrac{1}{2}\,\delta^2 f(t_1, \ldots, t_n) + \ldots + \tfrac{1}{n!}\,\delta^n f(t_1, \ldots, t_n) + \ldots$$

Since $f(t_1, \ldots, t_n)$, and therefore also the left hand side must be 0 in $\mathcal{D}$, it is clear that the right hand side must vanish. Modulo $\mathfrak{m}$ this is, however, simply $f(\tau_1, \ldots, \tau_n)$. It follows that the k-algebra A_0 generated in $\mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D})$ is a quotient of A. By the universality of the Ph-construction, the only relations between the $\tau_i' s$ are those induced from the relations of A by applying δ^p. Therefore $A_0 \simeq A$. □

Definition 3.6 Let A be a finitely generated k-algebra. Then

$$\iota_0 : \quad A \longrightarrow \mathcal{D}$$

will be called the versal family of A, as moduli of its points. This is just another way of expressing the content of (3.5).

For any right A-module, with structure morphism $\rho : \; A \to End_k(V)$, we can now copy what we did above, and obtain a universal derivation

$$c : \quad A \longrightarrow Hom_k(V, V \otimes_A Ph^\infty(A)),$$

defined by, $c(a)(v) = v \otimes \delta(a)$. Using the long exact sequence

$$0 \to Hom_A(V, V \otimes_A Ph^\infty(A)) \to Hom_k(V, V \otimes_A Ph^\infty(A)) \to_\iota$$
$$Der_k(A, Hom_A(V, V \otimes_A Ph^\infty(A))) \to_\kappa Ext^1_A(V, V \otimes_A Ph^\infty(A)) \to 0,$$

we obtain the non-commutative Kodaira-Spencer class

$$c(V) := \kappa(c) \in Ext^1_A(V, V \otimes_A Ph^\infty(A)),$$

inducing the Kodaira-Spencer morphism

$$g : \quad \Theta_A := Der_k(A, A) \longrightarrow Ext_A(V, V),$$

via the identity, $\tilde{\delta}_*$. If $c(V) = 0$, then the exact sequence above proves that there exists a $\nabla \in Hom_k(V, V \otimes_A Ph^\infty(A))$ such that $\tilde{\delta} = \iota(\nabla)$. This is just another way of proving that $\tilde{\delta}$ is given by a connection

$$\nabla : \quad Der_k(A, A) \longrightarrow Hom_k(V, V).$$

As above, the Kodaira-Spencer class gives rise to a Chern character by putting

$$ch^i(V) := \frac{1}{i!}\, c^i(V) \in Ext^i_A(V, V \otimes_A Ph^\infty(A)),$$

and if $c(V) = 0$, the curvature $R(V)$ induces a curvature class,

$$R_\nabla \in H^2(k, A; \Theta_A, End_A(V)).$$

As above, the kernel of the natural map

$$Ext^1_{Ph^\infty(A)}(V_\delta, V_\delta) \to Ext^1_A(V, V),$$

induced by the linear map

$$Der_k(Ph^\infty(A), End_k(V_\delta)) \to Der_k(A, End_k(V))$$

is the quotient

$$Der_A(Ph^\infty(A), End_k(V_\delta))/End_A(V),$$

and the image is a subspace $[\delta_\rho]^\perp \subseteq Ext^1_A(V, V))$, which is computable, see examples below.

Remark 3.7 So, once we have determined the moduli space $\underline{A} := Spec(A)$ of our system, i.e. the time-space, we have a series of *geometries* to consider, $\underline{P}h^n(A) := Ind(Ph^n(A))$, parametrizing indecomposable representations of A, together with their first n higher order momenta. In dimension 1, this amounts to points with momenta, and their first n higher order analogies. Moreover $Ph^\infty(A)$ parametrizes representations of A with a velocity, an acceleration, and any number of higher order changes of velocity, called from now on *higher order momenta.* A fundamental problem of (our model of) physics can now be stated as follows: If we prepare an object so that we know its momentum, or its higher order momenta up to a certain order, what can we infer on its behavior in the future?

Let us first consider the following situation: Let A be a finitely generarated k-algebra, and consider the map

$$\omega : \quad Simp(Ph^\infty(A)) \longrightarrow Rep(A),$$

and put

$$Simp^\omega(Ph^\infty(A)) := \omega^{-1}(Simp(A).$$

For every integer $n \geq 1$ the set $Simp_n(A)$ of isomorphism classes of n-dimensional A-modules has a canonical scheme structure, see [**Pr**] or [**La5**]. Since $Ph^\infty(A)$ is far from finitely generated, the induced map

$$\omega : \quad Simp^\omega_n(Ph^\infty(A)) \longrightarrow Simp_n(A)$$

is just a set-theoretical map, but it has a well defined tangent map,

$$T_\omega : \quad Ext_{Ph^\infty(A)}(V, V) \longrightarrow Ext_A(V, V).$$

The preparations made on the object V, by fixing its structure as a $Ph^\infty(H)$-module, forces it to change in the following way: The derivation $\delta \in Der_k(Ph^\infty(H))$ maps via the structure homomorphism of the module V,

$$\rho_V : \quad Ph^\infty(H) \longrightarrow End_k(V),$$

to an element $\delta_V \in Der_k(Ph^\infty(H), End_k(V))$, and via the canonical linear maps

$$Der_k(Ph^\infty(H), End_k(V)) \longrightarrow Ext^1_{Ph^\infty}(V, V) \longrightarrow Ext^1_A(V, V)$$

to a vector field $\tilde{\delta}$ on $Simp_n(A)$). Denote by $\delta(V)$ its value in the tangent space $Ext^1_A(V,V)$ of $Simp_n(A)$, at the point V, see [**La5**]. Suppose first that $\delta(V)$ is non-zero. This means that V has a non-trivial deformation. If this deformation can be *integrated* V is *pushed by time* into non-isomorphic modules. Since the problem of measurement in this case is something classical quantum mechanics is not considering, we shall return to this in a later paper.

If, however, $\delta(V) = 0$, the representation V stays within its isomorphism class through time. But this means that the image of δ in $Der_k(A, End_k(V))$ is an inner derivation given by an endomorphism $\phi \in End_k(V)$, such that $\delta(f)(v) = (f\phi - \phi f)(v)$. This ϕ is the corresponding *Hamiltonian* (or Dirac operator in the terminology of Connes), and we have a situation that is very much like classical quantum mechanics, i.e. a set-up where the objects are represented by a Hilbert space V and an algebra of *observables* $Ph^\infty(A)$ acting on it, with time, and therefore also energy, represented by a special Hamiltonian operator ϕ.

In quantum mechanics we also find that there is a Planck's constant, making differences of energy-values an integral multiple of a certain minimal positive value. This can be seen, in this mini-model, as follows: Let $\{v_i\}_{i\in I}$ be a basis of V (no longer assumed to be finite-dimensional), formed by eigenvectors of ϕ, and let the eigenvalues be given by

$$\phi(v_i) = \kappa_i v_i.$$

Consider the set $\Lambda(\delta)$ of real numbers λ defined by

$$\Lambda(\delta) := \{\lambda \in \mathbf{R} \mid \exists f_\lambda \in Ph^\infty(A), \delta(f_\lambda) = \rho_V \lambda \rho_V(f_\lambda)\}.$$

Then,

$$\lambda f_\lambda \cdot v_i = \delta(f_\lambda)\cdot v_i = (f_\lambda\phi - \phi f_\lambda)(v_i) = \kappa_i f_\lambda \cdot v_i - \phi(f_\lambda \cdot v_i),$$

implying

$$\phi(f_\lambda \cdot v_i) = (\kappa_i - \lambda)\cdot(f_\lambda \cdot v_i).$$

It follows that $\kappa_i - \lambda = \kappa_j$ for some $j \in I$. Therefore

$$f_\lambda \cdot v_i = \alpha v_j, \quad \alpha \in \mathbf{R}, \quad \text{and } \lambda = \kappa_i - \kappa_j,$$

and

$$\Lambda(\delta) \subset \{\kappa_i - \kappa_j \mid i, j\}.$$

Moreover, if f_λ and f_μ are eigenvectors for δ in V, then $f_\lambda f_\mu$ is also an eigenvector with eigenvalue $\lambda+\mu$, implying that if $\lambda, \mu \in \lambda(\delta)$ then we must have $\lambda+\mu \in \Lambda(\delta)$. Therefore $\Lambda(\delta)$ is an additive monoid. Planck's constant $\hbar$ should be a generator of this monoid. See also that "when $\hbar$ tends to 0", any $f \in Ph^\infty(A)$ maps every eigenspace $V(\kappa_i)$ into itself. In the generic case when all κ_i are different, the image of $Ph^\infty(A)$ into $End_k(V)$ becomes commutative, a ring of functions on the spectrum of ϕ.

The system charaterized by the A-module V for which $\delta_V = 0$ is now said to be in *state* ψ if we have chosen an element $\psi \in V$. The Dirac derivation δ now defines a Hamiltonian operator ϕ (a Dirac operator), and time is assumed to push the system into the state

$$exp(t\phi)(\psi) \in V$$

whenever this is well defined.

Now go back to our physical model H or U, of §1 and §2, and notice that in $Ph(H)$ we have

$$[\tau_i, \omega_j] - [\tau_j, \omega_i] = (\tau_i\omega_j - \tau_j\omega_i) - (\omega_j\tau_i - \omega_i\tau_j),$$

such that the relations in Ph simply says that the *angular momentum*

$$L_{i,j} := \tau_i\omega_j - \tau_j\omega_i$$

is equal to

$$L_{i,j}* := \omega_j\tau_i - \omega_i\tau_j,$$

and therefore independent upon the non-commutativity in $Ph(H)$. As above we see that $Ph(H)$ is, in a natural way, a graded k-algebra with $deg\ t_i = 0,\ deg\ dt_i = 1,\ i = 1,2,\dots,6$, and $H = k[t_1,...,t_6]$ identified with the 0-th component.

As we noticed above, the geometry of $\underline{H}$ is, of course, dependent on the choice of a Riemannian metric g. At any point $\underline{t} \in \underline{H}$ we shall pick a coordinate system, as above, $\{t_1,...,t_6\}$ such that we may express the metric as

$$g = \sum_{i=1,\dots,6} dt_i^2.$$

Then, $\frac{\partial}{\partial t_i}$ and dt_i are dual with respect to g. Given this metric, it is easy to see that there is a unique symplectic structure on the spectrum of $Ph(H)$, i.e. on $Simp_1(Ph(H))$, given by $\sum_{i=1}^6 dt_i \wedge d(dt_i)$. But, do not confound $d(dt_i)$ and d^2t_i!

Since we have fixed a metric, therefore a duality $\Omega_H \simeq \theta_H$, there is a surjective homomorphism

$$ham: \quad Ph(H) \longrightarrow \mathbf{R}[t_1,..t_6,\xi_1,..,\xi_6],$$

the Hamilton representation, i.e. the representation on the classical phase space, mapping the dt_i to the commuting ξ_i. Notice that we use the same notation, dt_i, for the element in $Ph(H)$ as for the corresponding element in Ω_H. This, we hope, is not going to produce too much confusion. We also have the surjective homomorphism,

$$wey: \quad Ph(H) \longrightarrow A(6) := Diff_k(H) = \mathbf{R}[t_1,\dots,t_6,\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_6}],$$

the Weyl representation, defined by the canonical representation of $Ph(H)$ on H, i.e. mapping the dt_i to the $\frac{\partial}{\partial t_i}$. This is uniquely defined, once the metric is fixed, i.e. it does not depend upon a change of coordinates that conceves the metric. The corresponding representation,

$$hei: \quad Ph(H) \longrightarrow \mathbf{C}\{q_1,..,q_6;p_1,..,p_6\}/([q_i,q_j],[q_i,p_j], i \neq j, [p_i,q_i] = \imath I),$$

the Heisenberg representation, is then defined by mapping t_i to q_i, and dt_j to $-\imath\frac{\partial}{\partial t_j} := p_j$.

This is the restriction of a natural representation of Ph on the H-module $\mathcal{H}$. Any element $\psi \in \mathcal{H}$ may, in a unique way, be considered as a sum of complex functions of the form $f(\underline{t})e^{\imath\phi(\underline{t})}$, where f and ϕ are regular functions on $\underline{H}$, i.e. elements of H, thus real functions. Therefore, if we put,

$$\begin{aligned} e^{\imath\phi(\underline{t})} * t_i &= t_i e^{\imath\phi(\underline{t})} \\ e^{\imath\phi(\underline{t})} * dt_i &= \frac{\partial}{\partial t_i}(e^{\imath\phi(\underline{t})-\imath\pi/2}) \end{aligned}$$

we observe that dt_j corresponds to $-\imath\frac{\partial}{\partial t_j}$, and we easily check that $[t_i,dt_j] = \imath I\delta_{i,j}$ in $\mathcal{H}$. Notice that in this picture, E, the energy, is $-\imath\frac{\partial}{\partial t_j} = p_j$, therefore conjugate to time, see [**El**] p.48. Moreover we see that the metric $g = \sum_{i=1,\dots,6} dt_i^2 \in Ph(H)$ maps to the corresponding Schrödinger operator $\Delta = -\sum_{i=1,\dots,6}(\frac{\partial}{\partial t_i})^2$.

Now, go back to §1, and consider the problem of defining a geometry on the space of pairs (ψ_0, ψ_1) of functions on $\mathbf{E}^3$. The tensor product $\psi_0 \otimes \psi_1$ is a function on $\underline{H}$, and it is reasonable to assume that for physically interesting states (ψ_0, ψ_1), we have $\psi_0 \otimes \psi_1 \in \mathcal{H}$, which is a simple representation of Ph, therefore just a point of the non-commutative geometry defined by $Ph(H)$.

Moreover, as we have seen, any connection on Θ_H determines a $Ph(H)$-module structure on Θ_H.

The moduli space of pairs of localizable events in $\mathbf{E}^3$ is, as we know, parametrized by $Simp_1(H)$, and the dynamics is gouverned by the Dirac derivation of $(Ph^\infty(H))$. The moduli space of pairs of non-localizable events is parametrized by one fat point, $\mathcal{H}$ of $Simp(Ph(H))$ (resp. $Simp(Ph^\infty)(H)$), or more generally, by an $\underline{H}$-bundle with connection, $(\mathbf{B}, \nabla)$, considered as a representation of $Simp(Ph^\infty)$. The dynamics is gouverned by the Dirac operator of $End_k(\mathcal{H})$.

Thus it seems that we may obtain a common model for R.T. and for classical Q.T., together with a precise formulation of the second postulate of quantum mechanics, mapping the observables dt_i (momentum) to the section ξ_i of the cotangent bundle in the classical Hamiltonian case, to the vector field $p_i := -\imath \frac{\partial}{\partial t_i}$, in the Heisenberg formulation, and to the differential form $dt_i = dt_i$ on $\underline{H}$ in the relativistic model.

Notice that the kinetic moments defined above operate on $\mathcal{H}$ such that

$$[L_{i,j}, L_{k,l}] = \begin{cases} 0 & \text{if } \{i,j\} \cap \{k,l\} = \emptyset \\ \imath L_{i,l} & \text{if } i \neq j = k \neq l. \end{cases}$$

Put,

$$L_x := L_{2,3}, \quad L_y := L_{3,1}, \quad L_z := L_{1,2}, \quad L^2 = L_x^2 + L_y^2 + L_z^2,$$

then the representation $\mathcal{H}$ of $Ph(H)$ may be cut up into a sum of the irreducible representations of the spin-group, and we are in a known situation, see e.g. [**El**], p.198. Moreover there is a natural homomorphism of Lie algebras,

$$ham_* : \quad InnerDer(Ph(H)) \longrightarrow PoissDer(k[\underline{t}, \underline{\xi}]),$$

mapping $ad(h) = [h, -]$ to the Poisson derivative $\{h, -\}$.

Now, $\theta_H = Der_k(H, H)$ is an H-module, and we may therefore consider the Kodaira-Spencer class,

$$c(\theta_H) \in Ext^1_H(\theta_H, \theta_H \otimes_H Ph(H)).$$

Since θ_H is ffree over H the *ext*-group is 0, so there exists a natural connection

$$\nabla : \quad Der_k(H, H) \to End_k(\theta_H).$$

We are interested in a special one, the Levi-Civita connection, i.e. the unique one without torsion and conserving the metric g. The last condition is the same as asking the Dirac derivation, that degenerates into the Dirac operator, the Hamiltonian, to kill $g \in Ph(H)$. Fix this connection. Then, the physically interesting histories $\xi \in \theta_H = Der_k(H, H)$, should satisfy the following *equation of motion*,

$$\nabla_\xi \xi = 0,$$

i.e. they should be geodesic.

Summing up, it is clear that the non-commutative geometries defined by $Ph(H)$ and $Ph^\infty(H)$, are much richer than the geometry of $\underline{H}$. We observe that among the points of $Ph(H)$ we find, together with the closed k-points of $\underline{H}$, also H and $\mathcal{H}$ and any fibre-bundle (**B** or any principal bundle $\tilde{\mathfrak{g}}$ on $\underline{H}$, equipped with a connection. This follows by just composing the representations *wey*, or *hei* with the connection, depending on whether the bundle is real or complex. In the mini-model of §1 we may look at the tangent bundle, $\Theta_{\underline{H}}$. The subbundles or sub-quotients of $\Theta_{\underline{H}}$ that are simple representations of $Ph(H)$ must be of interest. They represent, in a sense, all phenomena that are defined only in terms of location, velocities, including spin and rotations, and their dynamics. Together with the representation *hei* on $\mathcal{H}$, representing light, i.e. the phenomenon that we call a *photon*, these subquotients of $\Theta_{\underline{H}}$ should take care of the *most elementary phenomena* for which we cannot associate other observables than those related to location and velocities.

In this picture a photon $\psi \in \mathcal{H}$ is like a clock, its spinning phase angle must measure time, and, as we have seen, actually define space-time and causality. In fact, let us look at a photon represented by the *field* $\psi \in \mathcal{H}$. The classical equations of Maxwell lead to the equation

$$\nabla^2\psi - \frac{n^2}{c^2}(\frac{\partial}{\partial t})^2\psi = 0,$$

see e.g. [**El**], chapter 14, where $n/c = v^{-1}$. Here n is the index of refraction of the medium, c and v are the speed of light, respectively in vacuum (where we put it equal to 1) and in the medium we are in, measured by an observer that sits in vacuum (?). We shall, below, show that this simply means that the Lagrangian $L \in Ph$ is the standard metric in $\underline{H}$, i.e.

$$d\underline{t}^2 = v^{-2}(d\underline{u}^2 + d\underline{\kappa}^2),$$

lifted to $Ph(H)$, i.e. the *kinetic energy*,

$$L = v^{-2}\sum_{i=1,2,\ldots,6} dt_i^2,$$

and that the "field equation" above is the eigenvalue problem

$$L\psi = (\imath\hbar\cdot\nu)^2\psi,$$

at the "point" $\mathcal{H}$. Here $\hbar$ is Planck's constant and ν is the frequency of the photon, i.e. $\hbar\cdot\nu$ is the *energy*. Then one sees that the index of refraction, $n = v^{-1}$, is related to the length-magnification in the time-space corresponding to the velocity of light in the medium. Notice that in this formulation, the speed of light in the medium is considered to be isotropic, at every point equal in all directions. (We know, however, see e.g. [**LePi**], that the speed of light in a "moving medium" is not isotropic. At least it does not look like isotropic, when measured (observed) from outside the medium, from a view-point sitting in vacuum.)

Notice that the observable $dt_i \in Ph(H)$, the momentum, "in the ith-direction" will act on $\psi = exp(ik(\underline{t})) \in \mathcal{H}$ as

$$dt_i\psi = \imath\frac{\partial\psi}{\partial t_i}.$$

Clearly,

$$L\psi = -\imath L(k(\underline{t}))\cdot\psi + \sum_{i=1,2,\ldots,6} k_i^2\cdot\psi,$$

where

$$k_i = dt_i(k(\underline{t})) = \frac{\partial k}{\partial t_i}$$

and

$$L(k(\underline{t})) = \sum_{i=1,2,\ldots,6} \frac{\partial^2}{\partial t_i^2} k.$$

Notice that $\sum_{i=1,2,\ldots,6} k_i^2$ is the square of the length of the gradient $\nabla k(\underline{t})$, and that $L(k(\underline{t}))\psi = 0$ means that ψ represents a wave. Now a photon is modelled by a section $\psi = exp(\imath(k(\underline{t}))$ of $\mathcal{H}$ without rest mass i.e. such that

$$\delta\psi = 0, \ \forall \delta \in \tilde{\Delta}.$$

This implies that the gradient ∇k is normal to $\tilde{\Delta}$, therefore is in **c**. If now ψ is an eigenvector for L and if $L(k) = 0$ we find

$$L\psi = (\sum_{i=1,2,\ldots,6} k_i^2)\psi = \|\nabla k\|^2 \psi.$$

Assuming $\|\nabla k\|$ constant implies

$$(\hbar \cdot \nu)^2 = \|\nabla k\|^2,$$

and we find

$$\nabla^2 \psi = (\hbar \cdot \nu)^2 \psi.$$

This should be interpreted as: The energy of the photon in the state ψ is $\|\nabla k\|$. If ξ is a unit vector, say the tangent vector of a history γ, then the momentum of the photon, measured by ξ is just the length of the projection of ∇k along ξ, as it should! This defines an *optical direction*, determined by the direction in which we observe the photon with maximal energy. Clearly this optical direction is given by the tangent vector $\nabla k(\underline{t})$, at any point $\underline{t} \in \underline{H}$. Measuring time along the optical direction, we find

$$\nabla^2 \psi = (\frac{\partial}{\partial t})^2 \psi,$$

i.e. the classical field equation referred to above. Notice that we have admitted an optical direction depending upon time, i.e. upon $\underline{t} \in \underline{H}$. However, to maintain constant energy, i.e. constant frequency, the length of $\nabla k(\underline{t})$ is also constant. The discussion above shows that we find, just like in the classical case, that the optical ray of the photon in the visible universe $\underline{U}(\underline{t})$ through $\underline{t} \in \underline{H}$, is a history γ in $\underline{U}(\underline{t})$, parallell to ∇k, therefore given by the parsimony condition, asking

$$(\hbar \cdot \nu) \int_{\gamma} L d\gamma$$

to be optimal. Thus the photon that reaches the observer from the observed source is given by a state-vector $\psi = exp(\imath k(\underline{t}))$, where ∇k is the tangent of a geodesic history γ relating these points in $\underline{H}$. Our point of view is then that it is not light that travels faster or slower. It is just the metric of the time-space that is not uniform.

Remark 3.8 Let us try out a definition of a *graviton*. If we have given a metric ρ, the time on $\underline{H}$, and its Levi-Civita connection, ∇, then any geodesic $\xi \in \Theta_H$, with $\|\xi\| = 1$ will have the properties:

a. $\nabla_\xi \xi = 0$

b. If $\nu \in \Theta_H$ then $\nabla_\nu \xi \perp \xi$.

If we consider ξ as a history, in the sense of Definition 1.1, and if we assume that the properties of this history is a consequence only of processes of the space, e.g. a consequence of gravitation alone, it is reasonable to think that it would be invariant under the action of $aff(3) \simeq \tilde{\Delta}$, i.e. invariant under a coordinate change conserving all spatial relations between an observer and an observed, we must have

c. If $\nu \in \tilde{\Delta}$, then $\nabla_\nu \xi = 0$, i.e. ξ must be light-like.

Consider ξ as a *particle*, i.e. as a state of the system defined by the $Ph(H)$-module Θ_H (and an element of some basis of this module). Call it *a graviton* if it is an eigenvector for the Lagrangian $L \in Ph(H)$, i.e. for $L = g = \sum_{i=1,\dots,6} dt_i^2$. Then

$$L\xi = (\sum_{i=1,2,3} dt_i^2)\xi = \kappa \cdot \xi.$$

Let $\{v_1, v_2, \xi\}$ be an orthonormal basis for $\mathbf{c}(\underline{t})$, such that $\{v_1, v_2\}$ is a basis for the plane $\pi(\underline{t}) := \mathbf{c}(\underline{t}) \cap \perp \xi(\underline{t})$. Then there is a *relative velocity operator* , see [**SW**] p.55, defined on the plane $\pi(\underline{t})$,

$$\nabla_{-}\xi \in End_k(\pi(\underline{t})),$$

with the following property:

$$\nabla_{v_1}\nabla_{v_1}\xi + \nabla_{v_2}\nabla_{v_2}\xi = \kappa\xi.$$

See [**SW**], p.242.

§4. Time, what is it?

... to seek simplicity and to distrust it. (Alfred North Whithead.)

Now, the problem of what time "is" occupies quite a lot of scientists, although maybe not as many as one would have thought, given the importance of the subject. The quotation from Aristotle proves that relating time to movement has a more than 2000 years history. But the linearity of time, its one-dimensionality, has never been doubted until recently. Moreover, the existence of an arrow of time seems to be a psychological necessity. As Peter Coveney and Roger Highfield express this in [**CH**], p.260: *If we dismiss the arrow of time as an illusion, we must forfeit all the insights we have gained. This would surely be an enormous sacrifice – and all we would gain are the absurdities of a world-view in which bowls of soup could heat up of their own accord, and snooker balls mysteriously pop out of their pockets. The objective existence of the arrow of time is an idea that cannot be denied.*

The *arrow of time* is also considered to be equivalent to the notion of causality, and, of course, to the flow of time. When we think in terms of the above model, see §1-3, it seems to me that the *undeniable* notion of irreversibility of time, commonly captured in the kind of statements quoted above, or in: *we have never seen the pieces of a broken glass to fuse, then jump onto the table and touch a human hand,* is not so obvious, but maybe explainable. A physical scenario like the above would require a moduli space, i.e. a time-space, classifying the states of the system composed of a human, a glass, a table and a floor in a 3-dimensional space, with gravity. Moreover we must admit as part of our system the states representing the glass decomposed into any number of pieces. The moduli space of such a system must be highly complex and definitely non-commutative. The points corresponding to the states where the glass is broken may have no tangents pointing towards systems where the pieces have fused into a glass. No curve may lead from such a special point (state) back to the regular point. The time-space may not be reversible for all its histories. This is true for many non-commutative moduli spaces, see [**La4**]. To talk about time reversibility we also need a *memory of what has passed.* If part of the information of a system is lost along a history γ, i.e. if the space of deformations of the system $X(\gamma(1))$ does not include $X(\gamma(0))$, then our memory is not complete, and time is not reversible. This is encoded in the geometry of the non-commutative time-space. The notion of time reversability is therefore only interesting in case the history γ takes us along a smooth part of $\underline{U}$, with no catastrophes. This situation, like sitting in a room with white walls, leaves us with no memories. Nothing happens! A reasonable interpretation of this is that the observational part of the world is, in fact, highly non-smooth and non-commutative, defining an arrow of time, within the histories we observe that we are part of. At the same time, this does not imply that all histories (even geodesics) are uniquely determined by the the tangent (preparation) at any point. We may have jumps that make the local determination impossible. So causality, in the usual elementary sense, may simply be false.

Things are as they are because they were as they were. I like the simplicity of this quotation from Thomas Gould (1972). But it may well be false!

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