# GEOMETRY OF TIME-SPACES.

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Abstract. In this paper we study the geometry of moduli spaces of representations of associative algebras $A$ over a not necessarily algebraically closed, field $k$. Using the basic concepts and the philosophy of classical physics we introduce the non-commutative *phase space* $A \to PhA$, general Lagrangians, corresponding Dirac derivations, and obtain, for every $n$ general equations of motions on the moduli space $Simp_n(A)$ of n-dimensional simple modules over A, together with a general evolution operator. The *laws of dynamics* of $Simp_n(A)$ may be given the same form as the standard physical laws. We show that a Planck's constant, a kind of Feynman's integral, a Dyson series, as well as a Fock representation is part of the picture.

Generalizing, just a little, we introduce the notion of *g-string*, i.e. an algebra $R$ together with a pair of $Ph$-points of $R$, i.e. a pair of morphisms $\epsilon_i : PhR \to k(p_i)$, $i = 1, 2$. The non-commutative tangent space between the pair of points of $PhR$ furnishes a natural candidate for the notion of *tension* of the string. Consider the moduli space, $\mathbf{M}$, of morphisms $A \to R$, then we may in a very natural way introduce the von-Neumann and Dirichlet conditions, and obtain a subscheme $String_R(A) \subseteq \mathbf{M}$, the *scheme of R-strings in A.* The dynamics of this situation is then copied from the above.

We include some simple examples, like the classical problems of an object in a central force field, the harmonic oscillator, the 0-dimensional and the 1-dimensional string in an affine space.

## §0 Introduction.

In a first paper on this subject, see[La 6], we sketched a physical "toy model", where the space-time of classical physics became a section of a universal fiber space $\tilde{E}$, defined on the moduli space $\underline{H}$, of the physical systems we chose to consider, in this case the systems composed of an observer and an observed, both sitting in Euclidean 3-space. This moduli space was called the *time-space.*

Measurable time, in this mathematical model, turned out to be a metric $\rho$ on the time-space, measuring all possible infinitesimal changes of *the state* of the objects in the family we are studying.

This lead to a "physics" where there are no infinite velocities, and where the principle of relativity comes for free. In particular, we observe that the three fundamental "gauge" groups of current quantum theory $U(1)$, $SU(2)$ and $SU(3)$ are part of the structure of the fiber space,

$$\tilde{E} \longrightarrow \underline{H}.$$

With this model in mind we embarked on the study of moduli spaces of modules over non-commutative algebras in general. The basic notions of the affine non-commutative algebraic geometry related to a (non-commutative) associative k-algebra have been treated in several texts, see [La 2,3,4,5]. Given a geometric algebra $A$, see §2, we associate a non-commutative scheme-structure on the set of isomorphism classes of simple finite dimensional modules, or representations, $Simp_{<\infty}(A)$. We show in [La 4], see §2 below, that $A$ may be recovered from the (non-commutative) structure of $Simp_{<\infty}(A)$, and that there is an underlying structure of (commutative) scheme on each component $Simp_n(A) \subset Simp_{<\infty}(A)$, parametrizing the simple representations of dimension n. In fact, we show that there is a commutative algebra $C(n)$ such that an open subvariety $U(n)$, of $Simp_1(C(n))$ is an étale covering of $Simp_n(A)$. Moreover, there exists a "universal representation" $\tilde{V} \simeq C(n) \otimes_k V$, a vector bundle of rank $n$ defined on $Simp(C(n))$, and a universal family, i.e. a morphism of algebras,

$$\tilde{\rho} : A \longrightarrow End_{Simp_n(A)}(\tilde{V}) \subseteq End_{C(n)}(\tilde{V}).$$

Obviously, $End_{C(n)}(\tilde{V}) \simeq M_n(C(n))$, and we shall use this isomorphism without further warning.

For any algebra $A$ we have also defined a *phase space* $PhA$, i.e. a universal pair of a morphism $\iota : A \to PhA$, and a derivation, $d : A \to PhA$, such that for any morphism of algebras, $A \to R$, any derivation of $A$ into $R$ decomposes into $d$ followed by an $A$- morphism $PhA \to R$. Iterating this construction we obtained a limit morphism $\iota^n : Ph^n A \to Ph^\infty A$ with image $Ph^{(n)}A$, and a universal derivation $\delta \in Der_k(Ph^\infty A, Ph^\infty A)$, the *Dirac*-derivation. For details, see §1.

A *dynamical structure of order* $n$ is now a projection $\sigma : Ph^\infty \to Ph^{(n-1)}$ or a cosection of the canonical homomorphism $\iota : Ph^{(n-1)}A \to Ph^\infty A$. If $A$ is generated by the *coordinate functions*, $\{t_i,\ i = 1, 2, ..., d\}$ any section of order $n$ is defined by a system of equations,

$$\delta^n t_p = \Gamma^p(\underline{t}_i, \underline{d}t_j, \underline{d}^2 t_k, .., \underline{d}^{n-1} t_l),\ p = 1, 2, ..., d.$$

Let,

$$\mathbf{A}(\sigma) := Ph^\infty(A)/(\delta^n t_p - \Gamma^p)$$

where $\sigma := (\delta^n t_p - \Gamma^p)$ is the two-sided $\delta$-ideal generated by the defining equations of $\sigma$. Obviously $\delta$ induces a derivation $\delta_\sigma \in Der_k(\mathbf{A}(\sigma))$, also called the Dirac derivation, and usually just denoted $\delta$.

This $\delta$ defines a unique vector field

$$\xi \in \Theta_{Simp_n(\mathbf{A}(\sigma))},$$

and a (non-unique) derivation,

$$[\delta] \in Der_k(\mathbf{A}(\sigma), End_{C(n)}(\tilde{V})),$$

lifting $\xi$. By definition of $[\delta]$, there is now a *Hamiltonian* operator

$$Q \in M_n(C(n)),$$

satisfying the following fundamental equation,

$$\delta = [\delta] + [Q, -].$$

This equation means that for an element (an observable) $a \in \mathbf{A}(\sigma)$ the element $\delta(a)$ acts on $\tilde{V} \simeq C(n)^n$ as $[\delta](a)$ plus the Lie-bracket, $[Q, \tilde{\rho}_V(a)]$. The dynamics of the system is given in terms of the Dirac vector-field $[\delta]$, which generates a vector field $\xi$ on $Simp_n(\mathbf{A}(\sigma))$. An integral curve $\gamma$ of $\xi$ is a *solution* of *the equations of motion*. Given a state, $\psi(v_0) \in \tilde{V}(v_0) \simeq V$, and let $\gamma$ start at $v_0 \in Simp_n(\mathbf{A}(\sigma))$ and end at $v_1 \in Simp_n(\mathbf{A}(\sigma))$, with length $\tau_1 - \tau_0$, then we prove that the evolution map, $U(\tau_0, \tau_1)$ transporting $\psi(v_0)$ from time $\tau_0$, i.e. from the point representing some $V_0$, to time $\tau_1$, i.e. corresponding to some point representing $V_1$, along $\gamma$, is given as,

$$U(\tau_0, \tau_1)(\psi(v_0)) = exp(\int_\gamma Q d\tau)(\psi),$$

where $exp(\int_\gamma)$ is the non-commutative version of the classical action, to be defined later, see the proof of (1.23). Whenever $A$ is commutative and smooth, we consider the classical Lagrangians $L$, expressed in some regular coordinate system $\{t_i\}$, and produce, via the Euler-Lagrange equations,

$$\delta(\frac{\partial L}{\partial dt_i}) - \frac{\partial L}{\partial t_i} = 0$$

an order 2 dynamical system, with

$$\mathbf{A}(\sigma) \simeq PhA/(\sigma)$$

as $k$-algebras. For different Lagrangians, we may obtain different Dirac derivations on the same $k$-algebra, and therefore different dynamics of the "particles", i.e. on the universal families of the different components of $Simp_n(PhA)$, $n \geq 1$.

This machinery allows us to define Planck's *constant(s)*, $\hbar_l$, as the generator(s) of the "generalized monoid",

$$\begin{aligned}\Lambda(\sigma) :=& \{\lambda \in C(n) | \exists f_\lambda \in \mathbf{A}(\sigma), f_\lambda \neq 0, \\ & [Q, \tilde{\rho}(\delta(f_\lambda))] = \tilde{\rho}(\delta(f_\lambda)) - [\delta](\tilde{\rho}(f_\lambda)) = \lambda \tilde{\rho}(f_\lambda)\}\end{aligned}$$

which has the property that $\lambda, \lambda' \in \Lambda(\sigma), f_\lambda f_{\lambda'} \neq 0$ implies $\lambda + \lambda' \in \Lambda(\sigma)$. From this definition we may construct a general notion of *Fock space*, and a representation $\mathcal{F}$ on this space. $\mathcal{F}$ is the sub-$k$-algebra of $End_{C(n)}(\tilde{V}))$ generated by $\{a^l_+ := f_{\hbar_l}, a^l_- := f_{-\hbar_l}\}_l$, and its Lie algebra of derivations, contains a generalized Virasoro algebra, see (3.7) and (3.9) for a rather complete discussion of the rank 2, and the rank $\infty$ representation of the one-dimensional harmonic oscillator. form §3 of this paper.

Perfectly parallel with this theory of simple finite dimensional representations, we introduce the notion of *g-string* as any algebra $R$ together with a pair of PH-points, i.e. a pair of homomorphisms $\epsilon_i : PhR \rightarrow k(p_i)$, corresponding to two points $k(p_i)$ of $Simp(R)$ each outfitted with a tangent $\xi_i$. We may, clearly, consider any two points $k(p_i) \in Simp_n(PhR)$, but we shall in this paper just consider the case

$n = 1$. For any g-string, consider the *non-commutative tangent space* of the the pair of points,

$$T(R, p_1, p_2) := Ext^1_{PhR}(p_1, p_2).$$

We shall call it the *space of tensions* between the two points of the string. The von Neumann condition on the string is now simply,

$$\epsilon_i \circ d = \xi_i = 0, \ i = 1 \vee i = 2,$$

which, if $x_j, j = 1, .., n$ and $\sigma_l, l = 1, .., p$ are parameters of $A$ respectively $R$, is equivalent to the condition,

$$\frac{\partial x_j}{\partial \sigma_l}(p_i) = 0, j = 1, ..., n, \ l = 1, .., p, \ i = 1 \vee 2.$$

Any homomorphism of algebras, $\kappa : A \to R$ induces a unique commutative diagram of algebras,

$$\begin{array}{ccc} A & \xrightarrow{\kappa} & R \\ \downarrow & & \downarrow \\ PhA & \xrightarrow{Ph\kappa} & PhR. \end{array}$$

Moreover, since any derivation $\xi \in Der_k(A, R)$ has a natural lifting to a derivation $\overline{\xi} \in Der_k(PhA, PhR)$ we find, using the general machinery of deformations of diagrams, see [La 0], that any family of morphisms $\kappa$ induces a family of the above diagram. If $\tau_k, \ k = 1, ..., d$ are parameters of such a family, $\mathbf{M} = Spec(M)$, then $d\tau_i \in PhM$ corresponds to a derivation, $\tau_i \in Der_k(A, R)$, and therefore to tangents $\eta_i, \ i = 1, 2$, of $Simp_1(A)$ at the two points $k(p_i)$. The Dirichlet condition on the string is now,

$$\eta_i = 0, \ i = 1 \vee 2,$$

which is equivalent to the condition,

$$\frac{\partial x_j}{\partial \tau_l}(p_i) = 0, j = 1, ..., n, \ l = 1, .., p, \ i = 1 \vee 2.$$

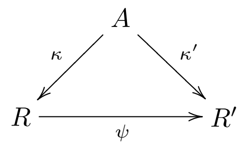
These conditions will define a new moduli space which we shall call $String_R(A)$. In the affine case the structure of this "space" is a problem, however we may of course do everything above for $A$ and $R$ replaced by projective schemes, and then all the moduli spaces exist as classical schemes. The volume form of the space the string is fanning out above a curve-element in $String_R(A)$, wil give us a Lagrangian, and then we go about defining the dynamics of the system, just like above.

At the end of §3, we then add some simple examples. Finally in §4, we shall introduce interactions, lifetime, decay and creation of particles.

When $A$ is the coordinate $k$-algebra of a moduli space, we should also consider the family of Lie algebras of *essential* automorphisms of the objects classified by $Spec(A)$, and obtain a general form for Yang-Mills theory, see [Bj-La] and [La-Pf], for the case of plane curve singularities. This offers us a model for the notion of *gauge particles and gauge fields*, coupling with ordinary particles via representations onto the corresponding simple modules. In a forthcoming paper we shall go back to our "toy" model, where the standard Gauge groups, $U(1)$, $SU(2)$, and $SU(3)$ pop up canonically and show that the results above can be used to construct a general geometric theory closely related to general relativity and to quantum theory, generalizing both.

**§1 Phase spaces and the Dirac derivation.**

Given a $k$-algebra $A$, denote by $A/k - \underline{alg}$ the category where the objects are homomorphisms of $k$-algebras $\kappa : A \to R$, and the morphisms, $\psi : \kappa \to \kappa'$ are commutative diagrams,




and consider the functor,

$$Der_k(A, -) : A/k - \underline{alg} \longrightarrow \underline{Sets}.$$

It is representable by a k-algebra-morphism,

$$\iota : A \longrightarrow Ph(A),$$

with a *universal family* given by a universal derivation,

$$\tilde{\delta} : A \longrightarrow Ph(A).$$

Ph (A) is relatively easy to compute. It can be constructed as the non-commutative versal base of the deformation functor of the morphisme $\rho : A \to k[\epsilon]$, see [La 6].

Clearly we have the identities,

$$\tilde{\delta}_* : Der_k(A, A) = Mor_A(Ph(A), A),$$

and,

$$\tilde{\delta}^* : Der_k(A, Ph(A)) = End_A(Ph(A)),$$

the last one associating $\tilde{\delta}$ to the identity endomorphisme of $Ph$. Let now $V$ be a right $A$-module, with structure morphism $\rho : A \to End_k(V)$. We obtain a universal derivation,

$$c : A \longrightarrow Hom_k(V, V \otimes_A Ph(A)),$$

defined by, $c(a)(v) = v \otimes \tilde{\delta}(a)$. Using the long exact sequence,

$$0 \to Hom_A(V, V \otimes_A Ph(A)) \to Hom_k(V, V \otimes_A Ph(A)) \to^{\iota}$$
$$Der_k(A, Hom_A(V, V \otimes_A Ph(A))) \to^{\kappa} Ext^1_A(V, V \otimes_A Ph(A)) \to 0,$$

we obtain the non-commutative Kodaira-Spencer class,

$$c(V) := \kappa(c) \in Ext^1_A(V, V \otimes_A Ph(A)),$$

inducing the Kodaira-Spencer morphism,

$$g : \Theta_A := Der_k(A, A) \longrightarrow Ext^1_A(V, V),$$

via the identity, $\tilde{\delta}_*$. If $c(V) = 0$, then the exact sequence above proves that there exist a $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ such that $\tilde{\delta} = \iota(\nabla)$. This is just another way of proving that $\tilde{\delta}$ is given by a connection,

$$\nabla : Der_k(A, A) \longrightarrow Hom_k(V, V).$$

As is well known, in the commutative case, the Kodaira-Spencer class gives rise to a Chern character by putting,

$$ch^i(V) := 1/i!\ c^i(V) \in Ext^i_A(V, V \otimes_A Ph(A)),$$

and if $c(V) = 0$, the curvature $R(V)$ induces a curvature class,

$$R_\nabla \in H^2(k, A; \Theta_A, End_A(V)).$$

Any $Ph(A)$-module $W$, given by its structure map,

$$\rho_W : Ph(A) \longrightarrow End_k(W)$$

corresponds bijectively to an induced $A$-module structure on $W$, and a derivation $\delta_\rho \in Der_k(A, End_k(W))$, defining an element,

$$[\delta_\rho] \in Ext^1_A(W, W),$$

Fixing this element we find that the set of $Ph(A)$-module structures on the $A$-module $W$ is in one to one correspondence with,

$$End_k(W)/End_A(W).$$

Conversely, starting with an $A$-module $V$ and an element $\delta \in Der_k(A, End_k(V))$, we obtain a $Ph(A)$-module $V_\delta$. It is then easy to see that the kernel of the natural map,

$$Ext^1_{Ph(A)}(V_\delta, V_\delta) \to Ext^1_A(V, V),$$

induced by the linear map,

$$Der_k(Ph(A), End_k(V_\delta)) \to Der_k(A, End_k(V))$$

is the quotient,

$$Der_A(Ph(A), End_k(V_\delta))/End_k(V),$$

and the image is a subspace $[\delta_\rho]^\perp \subseteq Ext^1_A(V, V))$, which is rather easy to compute, see examples below.

*Remark.* Since $Ext^1_A(V, V)$ is the tangent space of the miniversal deformation space of $V$ as an $A$-module, we see that the non-commutative space $PhA$ also parametrizes the set of *generalized momenta*, i.e. the set of pairs of a simple module $V \in Simp(A)$, and a tangent vector of $Simp(A)$ at that point.

**Example 1.1.** (i) Let $A = k[t]$, then obviously, $Ph(A) = k < t, dt >$ and $\tilde{\delta}$ is given by $\tilde{\delta}(t) = dt$, such that for $f \in k[t]$, we find $\tilde{\delta}(f) = J_t(f)$ with the notations of [La 4], i.e. the non-commutative derivation of $f$ with respect to $t$. One should also compare this with the non-commutative Taylor formula of loc.cit. If $V \simeq k^2$ is an $A$-module, defined by the matrix $X \in M_2(k)$, and $\delta \in Der_k(A, End_k(V))$, is defined in terms of the matrix $Y \in M_2(k)$, then the $Ph(A)$-module $V_\delta$ is the $k < t, dt >$-module defined by the action of the two matrices $X, Y \in M_2(k)$, and we find

$$e^1_V := dim_k Ext^1_A(V, V) = dim_k End_A(V) = dim_k\{Z \in M_2(k) |\ [X, Z] = 0\}$$
$$e^1_{V_\delta} := dim_k Ext^1_{Ph(A)}(V_\delta, V_\delta) = 8 - 4 + dim\{Z \in M_2(k) |\ [X, Z] = [Y, Z] = 0\}.$$

We have the following inequalities,

$$2 \leq e^1_V \leq 4 \leq e^1_{V_\delta} \leq 8.$$

(ii) Let $A = k^2 \simeq k[x]/(x^2 - r^2),\ r \in k,\ r \neq 0$, then,

$$PhA = k < x, dx > /((x^2 - r^2), x \cdot dx + dx \cdot x).$$

Notice that $PhA$ just has 2 points, i.e. simple representations, given by,

$$k(r) : x = r, dx = 0,\ k(-r) : x = -r,\ dx = 0.$$

An easy computation shows that,

$$Ext^1_{PhA}(k(\alpha), k(\alpha)) = 0,\ \alpha = r, -r,\ Ext^1_{PhA}(k(\alpha), k(-\alpha)) = k \cdot \omega,$$

where $\omega$ is represented by the derivation given by $\omega(x) = 2r,\ \omega(dx) = t \in k$ where $t$ is the *tension of this string of dimension* $-1$*, see end of* §*2, and end of* §*3. Notice also that this is an example of the existence of tangents between different points, in non-commutative algebraic geometry.*

*(iii) Now, let* $A = k[x_1, x_2, x_3]$ *and consider,*

$$PhA = k < x_1, x_2, x_3, dx_1, dx_2, dx_3 > /([x_i, x_j], \tilde{\delta}[x_i, x_j]),$$

*and consider any 2-dimensional representation of* $PhA$*. It is an easy computation that any such is given by the actions,*

$$x_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},\ x_2 = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},\ x_3 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix},$$

*and,*

$$dx_1 = \begin{pmatrix} \alpha_{1,1} & (a_1 - a_2) \\ (a_2 - a_1) & \alpha_{2,2} \end{pmatrix},$$
$$dx_2 = \begin{pmatrix} \beta_{1,1} & (b_1 - b_2) \\ (b_2 - b_1) & \beta_{2,2} \end{pmatrix},$$
$$dx_3 = \begin{pmatrix} \gamma_{1,1} & (c_1 - c_2) \\ (c_2 - c_1) & \gamma_{2,2} \end{pmatrix}$$

*The intrinsic kinetic moment is now given by,*

$$L_{1,2} := x_1 dx_2 - x_2 dx_1 = \begin{pmatrix} (a_1\beta_{1,1} - b_1\alpha_{1,1}) & (a_2 b_1 - a_1 b_2) \\ (a_1 b_2 - a_2 b_1) & (a_2\beta_{2,2} - b_2\alpha_{2,2}) \end{pmatrix},$$

*etc. And the isospin has the form,*

$$I_1 := [x_1, dx_1] = \begin{pmatrix} 0 & (a_1 - a_2)^2 \\ (a_2 - a_1)^2 & 0 \end{pmatrix},$$

*etc.*

*The phase-space construction may, of course, be iterated. Given the k-algebra A we may form the sequence,* $\{Ph^nA\}_{1\leq n}$*, defined inductively by*

$$Ph^0A = A,\ Ph^1A = PhA, ..., Ph^{n+1}A := PhPh^nA.$$

*Let* $i_0^n : Ph^nA \rightarrow Ph^{n+1}A$ *be the canonical imbedding, and let* $d_n : Ph^nA \rightarrow Ph^{n+1}A$ *be the corresponding derivation. Since the composition of* $i_0^n$ *and the derivation* $d_{n+1}$ *is a derivation* $Ph^nA \rightarrow Ph^{n+2}$*, there exist by functoriality a homomorphism* $i_1^{n+1} : Ph^{n+1}A \rightarrow Ph^{n+2}$*, such that,*

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$

*Notice that we compose functions and functors from left to right. Clearly we may continue this process constructing new homomorphisms,*

$$\{i_j^n : Ph^nA \rightarrow Ph^{n+1}A\}_{0\leq j\leq n},$$

*with the property,*

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}.$$

*Notice also that we have the "bi-gone",* $i_0^0 i_0^1 = i_0^0 i_1^1$ *and the "hexagone",*

$$\begin{aligned} i_0^1 i_0^2 &= i_0^1 i_1^2 \\ i_1^1 i_0^2 &= i_0^1 i_2^2 \\ i_1^1 i_1^2 &= i_1^1 i_2^2, \end{aligned}$$

*and, in general,*

$$\begin{aligned} i_p^n i_q^{n+1} &= i_{q-1}^n i_p^{n+1},\ p<q \\ i_p^n i_p^{n+1} &= i_p^n i_{p+1}^{n+1} \\ i_p^n i_q^{n+1} &= i_q^n i_{p+1}^{n+1},\ q<p, \end{aligned}$$

*which is all easily proved by composing with* $i_0^{n-1}$ *and* $d_{n-1}$*. Thus, the* $Ph^*A$ *is a semi-simplicial algebra. The corresponding descent and cohomology properties will be postponed.*

*The system of k-algebras and homomorphisms of k-algebras* $\{Ph^nA, i_j^n\}_{n,0\leq j\leq n}$ *has an inductive (direct) limit,* $Ph^\infty A$*, together with homomorphisms,*

$$i_n : Ph^nA \longrightarrow Ph^\infty A$$

*satisfying,*

$$i_j^n \circ i_{n+1} = i_n,\ j = 0, 1, .., n.$$

*Moreover, the family of derivations,* $\{d_n\}_{0\leq n}$ *define a unique derivation,*

$$\delta : Ph^\infty A \longrightarrow Ph^\infty A,$$

*such that,*

$$i_n \circ \delta = d_n \circ i_{n+1}.$$

*Put*

$$Ph^{(n)}(A) := im\ i_n \subseteq Ph^\infty A$$

*The k-algebra* $Ph^\infty A$ *has a descending filtration of two-sided ideals,* $\{F^n\}_{0\leq n}$ *given by:*

$$F^n = Ph^\infty A \cdot im(\delta^n) \cdot Ph^\infty A$$

*such that the derivation* $\delta$ *induces derivations,*

$$\delta_n : F^n \longrightarrow F^{n+1}.$$

**Definition 1.2.** *For a given $k$-algebra $A$, the $k$-algebra $Ph^{\infty}(A)$ will be called the $k$-algebra of higher differentials, and the completion of $Ph^{\infty}(A)$ in the topology given by the filtration $\{F^n\}_{0\leq n}$, denoted by $\mathcal{D}(A)$, will be called the formalized $k$-algebra of higher differentials.*

Clearly $\delta$ defines a derivation on $\mathcal{D}(A)$, and an isomorphism of $k$-algebras,

$$\epsilon(t) := exp(t\delta) : \mathcal{D}(A) \to \mathcal{D}(A).$$

**Proposition 1.3.** *Let $A$ be a finitely generated $k$-algebra, generated by $\{t_i\}_{i=1,..,n}$ and let $k(0)$ be a one-dimensional representation, i.e. a point of $Simp_1(A)$, corresponding to a two-sided maximal ideal $\mathfrak{m} \subset A$. We may assume $\mathfrak{m} = (t_1, ..., t_n)$. Then the elements $\tau_i := \delta t_i + 1/2\delta^2 t_i + ... + 1/n!\delta^n t_i + ... \in \mathcal{D}$ generates a sub algebra of $\mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D})$, isomorphic to $A_{\mathfrak{m}}$, the image of $A$ in $H(k(0)$.*

*Proof.* Let $f \in k < t_1, ..., t_n >$ be a relation for the $k$-algebra $A$. Then, computing $\epsilon(f)$, there is a Taylor series,

$$\begin{aligned} &f(t_1 + \delta t_1 + 1/2\delta^2 t_1 + ... + 1/n!\delta^n t_1 + ..., .., t_n + \delta t_n + 1/2\delta^2 t_n)1/n!\delta^n t_n + ...) \\ =&f(t_1, ..., t_n) + \delta f(t_1, ..., t_n) + 1/2\delta^2 f(t_1, ..., t_n) + ... + 1/n!\delta^n f(t_1, ..., t_n) + ... \end{aligned}$$

Since $f(t_1, ..., t_n)$, and therefore also the right hand side, must be 0 in $\mathcal{D}$, it is clear that the left hand side must vanish. Modulo $\mathfrak{m}$ this is, however, simply $f(\tau_1, ..., \tau_n)$. It follows that the $k$-algebra $A_0$ generated in $\mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D})$ is a quotient of $A$. By the versality of $H(k(0))$, the homomorphism $A \to \mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D})$ factors via $A \to H(k(0))$, and the morphism $H(k(0)) \to \mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D})$ induces a surjective map on tangent spaces, and is therefore injective.

□

**Definition 1.4.** *Let $A$ be a finitely generated $k$-algebra. Then*

$$\iota_0 : A \longrightarrow \mathcal{D}$$

*will be called the versal family of $A$, as moduli of its points.*

This is just another way of expressing the content of (1.3).

As above, the kernel of the natural map,

$$Ext^1_{Ph^{\infty}(A)}(V_\delta, V_\delta) \to Ext^1_A(V, V),$$

induced by the linear map,

$$Der_k(Ph^{\infty}(A), End_k(V_\delta)) \to Der_k(A, End_k(V))$$

is the quotient,

$$Der_A(Ph^{\infty}(A), End_k(V_\delta))/End_A(V),$$

and the image is a subspace $[\delta_\rho]^{\perp} \subseteq Ext^1_A(V, V))$, which is computable, see examples below.

**Remark 1.5.** *Once we have determined the moduli spaces $Simp(A) \subset Ind(A)$ of all isomorphism classes of the simple, resp. indecomposable representations of $A$, i.e. the time-spaces, we have a series of geometries to consider, $Simp(Ph^n(A) \subset Ind(Ph^n(A))$, parametrizing simple and indecomposable representations of $A$, together with their first $n$ higher order momenta. In dimension 1, these amounts to points with momenta, and their first $n$ higher order analogies. Moreover $Ph^\infty(A)$ parametrizes representations of $A$ with fixed velocity, acceleration, and any number of higher order changes of velocity, called from now on higher order momenta. A fundamental problem of (our model of) physics can now be stated as follows: If we prepare an object so that we know its momentum, and its higher order momenta up to a certain order, what can we infer on its behavior in the future?*

Let us first consider the following situation: Let $A$ be a finitely generated $k$-algebra. As we shall see, in §2, the set $Simp_n(A)$ of isomorphism classes of $n$-dimensional $A$-modules has a canonical scheme structure, see also [Procesi] or [La 4]. Since, in general, $Ph^\infty(A)$ is far from finitely generated, we do not have a natural scheme structure on $Simp_n(Ph^\infty(A))$ Therefore the map induced by the obvious forgetful functor,

$$\omega : Simp(Ph^\infty(A)) \longrightarrow Rep(A),$$

is just a set-theoretical map with a well defined tangent map,

$$T_\omega : Ext^1_{Ph^\infty(A)}(V,V) \longrightarrow Ext^1_A(V,V).$$

The preparations made on the $A$-module (the object) $V$, by fixing its structure as a $Ph^\infty(A)$-module, forces it to change in the following way: The derivation $\delta \in Der_k(Ph^\infty(A))$ maps via the structure homomorphism of the module $V$,

$$\rho_V : Ph^\infty(A) \longrightarrow End_k(V)$$

to an element $\delta_V \in Der_k(Ph^\infty(A), End_k(V))$ and via the first of the canonical linear maps,

$$Der_k(Ph^\infty(A), End_k(V)) \longrightarrow Ext^1_{Ph^\infty(A)}(V,V) \longrightarrow Ext^1_A(V,V)$$

to a distribution $\tilde{\delta}$ on $Simp_n(A)$. Denote by $\delta(V)$ its value in the tangent space $Ext^1_{Ph^\infty(A)}(V,V)$ of $Simp_n(Ph^\infty(A))$, at the point represented by the module $V$, see [La 4].

Suppose first that, for an *infinitely prepared* $A$-module $V$ (see (1.5) above), $\delta(V)$ is zero, see [La 6]. This means that the image of $\delta$ in $Der_k(Ph^\infty(A), End_k(V))$ is an inner derivation given by an endomorphism $Q \in End_k(V)$, such that for every $f \in Ph^\infty(A)$, we find $\delta(f)(v) = (Qf - fQ)(v)$. This $Q$ is the corresponding *Hamiltonian*, (or Dirac operator in the terminology of Connes, see [Schücker]), and we have a situation that is very much like classical quantum mechanics, i.e. a set-up where the objects are represented by a fixed Hilbert space $V$ and an algebra of *observables* $Ph^\infty(A)$ acting on it, with time, and therefore also energy, represented by a special Hamiltonian operator $Q$.

In quantum mechanics we also find that there is a Planck's constant, making differences of energy- values an integral multiple of a certain minimal positive value.

This can be seen to be a general fact, see also loc.cit. Let $\{v_i\}_{i\in I}$ be a basis of $V$ (no longer assumed to be finite dimensional), formed by eigenvectors of $Q$, and let the eigenvalues be given by,

$$Q(v_i) = \kappa_i v_i.$$

Consider the set $\Lambda(\delta)$ of real numbers $\lambda$ defined by,

$$\Lambda(\delta) := \{\lambda \in \mathbf{R}|\ \exists f_\lambda \in Ph^\infty(A), f_\lambda \neq 0, \rho_V(\delta(f_\lambda)) = \lambda\rho_V(f_\lambda) \in End_k(V)\}.$$

Since $\delta = [Q, -]$ is a derivation, if $f_\lambda$ and $f_\mu$ are eigenvectors for $\delta$ in $V$, then if $f_\lambda f_\mu$ is non-trivial, it is also an eigenvector, with eigenvalue $\lambda + \mu$, implying that if $\lambda, \mu \in \lambda(\delta)$, with $f_\lambda f_\mu = 0$, we must have $\lambda + \mu \in \Lambda(\delta)$. Now,

$$\lambda f_\lambda \cdot v_i = \delta(f_\lambda) \cdot v_i = (Qf_\lambda - f_\lambda Q(v_i) = Q(f_\lambda \cdot v_i) - \kappa_i f_\lambda \cdot v_i,$$

implying,

$$Q(f_\lambda \cdot v_i) = (\kappa_i + \lambda) \cdot (f_\lambda \cdot v_i).$$

If $f_\lambda \cdot v_i \neq 0$, it follows that: $\kappa_i + \lambda = \kappa_j$ for some $j \in I$. Therefore

$$f_\lambda \cdot v_i = \alpha v_j,\ \alpha \in \mathbf{R}, \text{and } \lambda = \kappa_i - \kappa_j.$$

and so,

$$\Lambda(\delta) \subset \{\kappa_i - \kappa_j|\ i, j\},$$

Planck's constant $\hbar$ should be a generator of the monoid $\Lambda(\delta)$, when this is meaningful.

We can show, see example 3.5-3.7, that $\Lambda(\delta)$ is an additive monoid in the infinite dimensional case of the classical oscillator.

See also that when $\{f_\lambda\}_\lambda$ generate $End_k(V)$ we must have $\Lambda(\delta) = \{\kappa_i - \kappa_j|\ i, j\}$, and that when $\hbar$ "tends to 0", any $f \in Ph^\infty(A)$ maps every eigenspace $V(\kappa_i)$ into itself, see §3. In the generic case when all $\kappa_i$ are different, the image of $Ph^\infty(A)$ into $End_k(V)$ becomes commutative, a ring of functions on the spectrum of $Q$.

A system characterized by a $Ph^\infty(A)$-module $V$, for which $\delta(V) = 0$, (the stationary case) is now said to be in *state* $\psi$ if we have chosen an element $\psi \in V$. The Dirac derivation $\delta$ defines a Hamiltonian operator $Q$, (a Dirac operator), and time, i.e. $\delta$, now push the state $\psi$ into the state,

$$exp(\tau Q)(\psi) \in V,$$

corresponding to the isomorphism of the module $V$ defined by the inner isomorphism of the algebra of observables, $Ph^\infty(A)$ defined by $U := exp(\tau\delta)$, whenever this is well defined. This is a well known situation i classical quantum mechanics, corresponding to the equivalence between the set-ups of Schrödinger and Heisenberg.

To treat the situation when $[\delta] \neq 0$, we first have to take a new look at noncommutative algebraic geometry, as developed in [La 3,4,5].

**§2. Non-commutative deformations and the structure of the moduli space of simple representations.**

In [La 2], [La 3] and [La 4,5], we introduced non-commutative deformations of families of modules of non-commutative $k$-algebras, and the notion of *swarm* of right modules (or more generally of objects in a $k$-linear abelian category). Let $\underline{a}_r$ denote the category of $r$-pointed not necessarily commutative $k$-algebras $R$. The objects are the diagrams of k-algebras,

$$k^r \xrightarrow{\iota} R \xrightarrow{\pi} k^r$$

such that the composition of $\iota$ and $\pi$ is the identity. Any such *$r$-pointed $k$-algebra* $R$ is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of $R$ is the bilateral ideal $Rad(R) := ker\pi$, such that $R/Rad(R) \simeq k^r$. The dual k-vector space of $Rad(R)/Rad(R)^2$ is called the tangent space of $R$.

For $r = 1$, there is an obvious inclusion of categories

$$\underline{l} \subseteq \underline{a}_1$$

where $\underline{l}$, as usual, denotes the category of commutative local Artinian $k$-algebras with residue field $k$.

Fix a not necessarily commutative $k$-algebra $A$ and consider a right $A$-module $M$. The ordinary deformation functor

$$Def_M : \underline{l} \to \underline{Sets}$$

is then defined. Assuming $Ext^i_A(M, M)$ has finite $k$-dimension for $i = 1, 2$, it is well known, see [Sch], or [La 2], that $Def_M$ has a pro-representing hull $H$, *the formal moduli of* $M$. Moreover, the tangent space of $H$ is isomorphic to $Ext^1_A(M, M)$, and $H$ can be computed in terms of $Ext^i_A(M, M)$, $i = 1, 2$ and their *matric* Massey products, see [La 2].

In the general case, consider a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of right $A$-modules. Assume that,

$$dim_k Ext^1_A(V_i, V_j) < \infty.$$

Any such family of $A$-modules will be called a *swarm*. We shall define a deformation functor,

$$Def_{\mathcal{V}} : \underline{a}_r \to \underline{Sets}$$

generalizing the functor $Def_M$ above. Given an object $\pi : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the $k$-vector space and left $R$-module $(R_{i,j} \otimes_k V_j)$. It is easy to see that $End_R((R_{i,j} \otimes_k V_j)) \simeq (R_{i,j} \otimes_k Hom_k(V_i, V_j))$. Clearly $\pi$ defines a $k$-linear and left $R$-linear map,

$$\pi(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i,$$

inducing a homomorphism of $R$-endomorphism rings,

$$\tilde{\pi}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right $A$-module structure on the $V_i'$s is defined by a homomorphism of $k$-algebras, $\eta_0 : A \to \oplus_{i=1}^r End_k(V_i)$. Let

$$Def_{\mathcal{V}}(R) \in \underline{Sets}$$

be the set of isoclasses of homomorphisms of k-algebras,

$$\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

such that,

$$\tilde{\pi}(R) \circ \eta' = \eta_0,$$

where the equivalence relation is defined by inner automorphisms in the $k$-algebra $(R_{i,j} \otimes_k Hom_k(V_i, V_j))$ inducing the identity on $\oplus_{i=1}^r End_k(V_i)$. One easily proves that $Def_{\mathcal{V}}$ has the same properties as the ordinary deformation functor and we prove the following, see [La 2]:

**Theorem 2.1.** *The functor $Def_{\mathcal{V}}$ has a pro-representable hull, i.e. an object $H$ of the category of pro-objects $\hat{\underline{a}}_r$ of $\underline{a}_r$, together with a versal family,*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathcal{V}}(H/\mathfrak{m}^n),$$

*where $\mathfrak{m} = Rad(H)$, such that the corresponding morphism of functors on $\underline{a}_r$,*

$$\kappa : Mor(H, -) \to Def_{\mathcal{V}}$$

*defined for $\phi \in Mor(H, R)$ by $\kappa(\phi) = R \otimes_\phi \tilde{V}$, is smooth, and an isomorphism on the tangent level. Moreover, $H$ is uniquely determined by a set of matric Massey products defined on subspaces,*

$$D(n) \subseteq Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{n-1}}, V_k),$$

*with values in $Ext^2(V_i, V_k)$.*

The right action of $A$ on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j)),$$

and the k-algebra $O(\mathcal{V})$ acts on the family of $A$-modules $\mathcal{V} = \{V_i\}$, extending the action of $A$. If $dim_k V_i < \infty$, for all $i = 1, ..., r$, the operation of associating $(O(\mathcal{V}), \mathcal{V})$ to $(A, \mathcal{V})$ turns out to be a closure operation.

Moreover, we prove the crucial result,

**A generalized Burnside theorem 2.2.** *Let $A$ be a finite dimensional $k$-algebra, $k$ an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of all simple $A$-modules, then*

$$\eta : A \longrightarrow O(\mathcal{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

*is an isomorphism.*

We also prove that there exists, in the non-commutative deformation theory, an obvious analogy to the notion of pro-representing (modular) substratum $H_0$ of the formal moduli $H$, see [La 0] and [La-Pf]. The tangent space of $H_0$ is determined by a family of subspaces

$$Ext^1_0(V_i, V_j) \subseteq Ext^1_A(V_i, V_j), \qquad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathcal{V}$, and by a subspace,

$$T_0(\Delta) \subseteq \prod_i Ext^1_A(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of $A$-modules generated by the family $\mathcal{V} = \{V_i\}_{i=1}^r$, see [La 1]. If $\mathcal{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some Artinian $k$-algebra $A$, we show that the above notion of *almost split sequence* coincides with that of Auslander, see [R].

Using this we consider, in [La 2], the general problem of classification of iterated extensions of a family of modules $\mathcal{V} = \{V_i,\}_{i=1}^r$, and the corresponding classification of filtered modules with graded components in the family $\mathcal{V}$, and extension type given by a directed representation graph $\Gamma$, see §3. The main result is the following, see [La 4],

**Proposition 2.3.** *Let $A$ be any $k$-algebra, $\mathcal{V} = \{V_i\}_{i=1}^r$ any swarm of $A$-modules, i.e. such that,*

$$\dim_k Ext^1_A(V_i, V_j) < \infty \quad \textit{for all } i, j = 1, \dots, r.$$

*(i): Consider an iterated extension $E$ of $\mathcal{V}$, with representation graph $\Gamma$. Then there exists a morphism of $k$-algebras*

$$\phi : H(\mathcal{V}) \to k[\Gamma]$$

*such that*

$$E \simeq k[\Gamma] \otimes_\phi \tilde{V}$$

*as right $A$-algebras.*

*(ii): The set of equivalence classes of iterated extensions of $\mathcal{V}$ with representation graph $\Gamma$, is a quotient of the set of closed points of the affine algebraic variety*

$$\underline{A}[\Gamma] = Mor(H(\mathcal{V}), k[\Gamma])$$

*(iii): There is a versal family $\tilde{V}[\Gamma]$ of $A$-modules defined on $\underline{A}[\Gamma]$, containing as fibers all the isomorphism classes of iterated extensions of $\mathcal{V}$ with representation graph $\Gamma$.*

To any, not necessarily finite, swarm $\underline{c} \subset \underline{mod}(A)$ of right-$A$-modules, we have associated two associative $k$-algebras, see [La 3] and [La 4], $O(|\underline{c}|, \pi) = \varprojlim_{\mathcal{V} \subset |\underline{c}|} O(\mathcal{V})$, and a sub-quotient $\mathcal{O}_\pi(\underline{c})$, together with natural $k$-algebra homomorphisms,

$$\eta(|\underline{c}|) : A \longrightarrow O(|\underline{c}|, \pi)$$

and,

$$\eta(\underline{c}) : A \longrightarrow \mathcal{O}_\pi(\underline{c})$$

with the property that the $A$-module structure on $\underline{c}$ is extended to an $\mathcal{O}$-module structure in an optimal way. We then defined an *affine non-commutative scheme* of right $A$-modules to be a swarm $\underline{c}$ of right $A$-modules, such that $\eta(\underline{c})$ is an isomorphism. In particular we considered, for finitely generated $k$-algebras, the swarm $Simp^*_{<\infty}(A)$ consisting of the finite dimensional simple $A$-modules, and the *generic* point $A$, together with all morphisms between them. The fact that this is a swarm, i.e. that for all objects $V_i, V_j \in Simp_{<\infty}$ we have $dim_k Ext^1_A(V_i, V_j) < \infty$, is easily proved. We have in [La 4] proved the following result, (see (4.1), loc.cit. and Lemma above.)

**Proposition 2.4.** *Let $A$ be a geometric $k$-algebra, then the natural homomorphism,*

$$\eta(Simp^*(A)) : A \longrightarrow \mathcal{O}_\pi(Simp^*_{<\infty}(A))$$

*is an isomorphism, i.e. $Simp^*_{<\infty}(A)$ is a scheme for $A$.*

In particular, $Simp^*_{<\infty}(k < x_1, x_2, ..., x_d >)$, is a scheme for $k < x_1, x_2, ..., x_d >$. To analyze the local structure of $Simp_n(A)$, we need the following, see [La 4],(3.23),

**Lemma 2.5.** *Let $\mathcal{V} = \{V_i\}_{i=1,..,r}$ be a finite subset of $Simp_{<\infty}(A)$, then the morphism of $k$-algebras,*

$$A \to O(\mathcal{V}) = (H_{i,j} \otimes_k Hom_k(V_i, V_j))$$

*is topologically surjective.*

*Proof.* Since the simple modules $V_i$, $i = 1, .., r$ are distinct, there is an obvious surjection, $\eta_0 : A \to \prod_{i=1,..,r} End_k(V_i)$. Put $\mathfrak{r} = ker\eta_0$, and consider for $m \geq 2$ the finite-dimensional $k$-algebra, $B := A/\mathfrak{r}^m$. Clearly $Simp(B) = \mathcal{V}$, so that by the generalized Burnside theorem, see [La 2], (2.6), we find, $B \simeq O^B(\mathcal{V}) := (H^B_{i,j} \otimes_k Hom_k(V_i, V_j))$. Consider the commutative diagram,

$$\begin{array}{ccccc} A & \longrightarrow & (H^A_{i,j} \otimes_k Hom_k(V_i, V_j)) =: O^A(\mathcal{V}) & & \\ \downarrow & & \downarrow & \searrow & \\ B & \longrightarrow & (H^B_{i,j} \otimes_k Hom_k(V_i, V_j)) & \xrightarrow{\alpha} & O^A(\mathcal{V})/\mathfrak{m}^m \end{array}$$

where all morphisms are natural. In particular $\alpha$ exists since $B = A/\mathfrak{r}^m$ maps into $O^A(\mathcal{V})/rad^m$, and therefore induces the morphism $\alpha$ commuting with the rest of the morphisms. Consequently $\alpha$ has to be surjective, and we have proved the contention.

□

*Localization and topology.* Let $s \in A$, and consider the *open subset* $D(s) = \{V \in Simp(A) |\ \rho(s)$ invertible in $End_k(V)\}$. The Jacobson topology on $Simp(A)$ is the topology with basis $\{D(s) |\ s \in A\}$. It is clear that the natural morphism,

$$\eta : A \to O(D(s), \pi)$$

maps $s$ into an invertible element of $O(D(s), \pi)$. Therefore we may define the localization $A_{\{s\}}$ of $A$, as the k-algebra generated in $O(D(s), \pi)$ by $im\, \eta$ and the inverse of $\eta(s)$. This furnishes a general method of localization with all the properties one would wish. And in this way we also find a canonical (pre)sheaf, $\mathcal{O}$ defined on $Simp(A)$.

**Definition 2.6.** *When the $k$-algebra $A$ is geometric, such that $Simp^*(A)$ is a scheme for $A$, we shall refer to the presheaf $\mathcal{O}$, defined above on the Jacobson topology, as the structure presheaf of the scheme $Simp(A)$.*

We shall now see that the Jacobson topology on $Simp(A)$, restricted to each $Simp_n(A)$ is the Zariski topology for a classical scheme-structure.

Recall first that a standard n-commutator relation in a $k$-algebra $A$ is a relation of the type,

$$[a_1, a_2, ..., a_{2n}] := \sum_{\sigma \in \Sigma_{2n}} sign(\sigma) a_{\sigma(1)} a_{\sigma(2)} ... a_{\sigma(2n)} = 0$$

where $\{a_1, a_2, ..., a_{2n}\}$ is a subset of $A$. Let $I(n)$ be the two-sided ideal of $A$ generated by the subset,

$$\{[a_1, a_2, ..., a_{2n}] |\ \{a_1, a_2, ..., a_{2n}\} \subset A\}.$$

Consider the canonical homomorphism,

$$p_n : A \longrightarrow A/I(n) =: A(n).$$

It is well known that any homomorphism of $k$-algebras,

$$\rho : A \longrightarrow End_k(k^n) =: M_n(k)$$

factors through $p_n$, see e.g. [Formanek].

**Corollary 2.7.** *(i). Let $V_i, V_j \in Simp_{\leq n}(A)$ and put $\mathfrak{r} = \mathfrak{m}_{V_i} \cap \mathfrak{m}_{V_j}$. Then we have, for $m \geq 2$,*

$$Ext^1_A(V_i, V_j) \simeq Ext^1_{A/\mathfrak{r}^m}(V_i, V_j)$$

*(ii). Let $V \in Simp_n(A)$. Then,*

$$Ext^1_A(V, V) \simeq Ext^1_{A(n)}(V, V)$$

*Proof.* (i) follows directly from Lemma (2.5). To see (ii), notice that $Ext^1_A(V, V) \simeq Der_k(A, End_k(V))/Triv \simeq Der_k(A(n), End_k(V))/Triv \simeq Ext^1_{A(n)}(V, V)$. The second isomorphism follows from the fact that any derivation maps a standard n-commutator relation into a sum of standard n-commutator relations.

□

*Example 2.8.* Notice that, for distinct $V_i, V_j \in Simp_{\leq n}(A)$, we may well have,

$$Ext^1_A(V_i, V_j) \neq Ext^1_{A(n)}(V_i, V_j).$$

In fact, consider the matrix $k$-algebra,

$$A = \begin{pmatrix} k[x] & k[x] \\ 0 & k[x] \end{pmatrix},$$

and let $n = 1$. Then $A(1) = k[x] \oplus k[x]$. Put $V_1 = k[x]/(x) \oplus (0), V_2 = (0) \oplus k[x]/(x)$, then it is easy to see that,

$$Ext^1_A(V_i, V_j) = k,\ Ext^1_{A(1)}(V_i, V_j) = 0, i \neq j,$$

but,

$$Ext^1_A(V_i, V_i) = Ext^1_{A(1)}(V_i, V_i) = k, i = 1, 2.$$

**Lemma 2.9.** *Let $B$ be a $k$-algebra, and let $V$ be a vector space of dimension $n$, such that the $k$-algebra $B \otimes End_k(V)$ satisfies the standard $n$-commutator-relations, i.e. such that the ideal, $I(n) \subset B \otimes End_k(V)$ generated by the standard $n$-commutators $[x_1, x_2, .., x_{2n}]$, $x_i \in B \otimes End_k(V)$, is zero. Then $B$ is commutative.*

*Proof.* In fact, if $b_1, b_2 \in B$ is such that $[b_1, b_2] \neq 0$, then the obvious $n$-commutator,

$$(b_1 e_{1,1})(b_2 e_{1,1}) e_{1,2} e_{2,2} ... e_{n-1,n} . e_{n,n} - (b_2 e_{1,1})(b_1 e_{1,1}) e_{1,2} e_{2,2} ... e_{n-1,n} . e_{n,n}$$

is different from 0. Here $e_{i,j}$ is the $n \times n$ matrix with all elements equal to 0, except the one in the $(i, j)$ position, where the element is equal to 1.

□

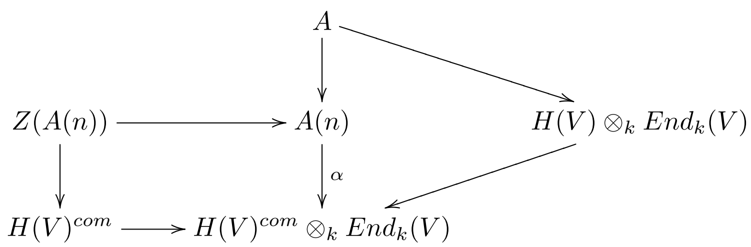
**Lemma 2.10.** *If $A$ is a finite type $k$-algebra, then any $V \in Simp_n(A)$ is an $A(n)$-module. Let $\mathcal{V} \subset Simp_n(A)$ be a finite family, then $H^{A(n)}(\mathcal{V})$ is commutative. In particular,*

1. *$Ext_{A(n)}(V_i, V_j) = 0$, for $V_i \neq V_j$*
2. *$H^{A(n)}(V) \simeq H^A(V)^{com} := H(V)/[H(V), H(V)]$.*

*Proof.* Since

$$A(n) \to O(\mathcal{V}) \simeq M_n(H^{A(n)}(\mathcal{V}))$$

is topologically surjective, we find using (Lemma 2.9), that $H^{A(n)}(\mathcal{V})$ is commutative. This implies (1) and the commutativity of $H^{A(n)}(V)$. Consider for $V \in Simp_n(A)$, the natural commutative diagram of homomorphisms of $k$-algebras,




where $Z(A(n))$ is the center of $A(n)$. The existence of $\alpha$ is a consequence of the ideal $I(n)$ of $A$ mapping to zero in $H(V)^{com} \otimes_k End_k(V) \simeq M_n(H(V)^{com})$. Therefore there are natural morphisms of formal moduli,

$$H^A(V) \to H^{A(n)}(V) \to H^A(V)^{com} \to H^{A(n)}(V)^{com}.$$

Since $H^{A(n)}(V) = H^{A(n)}(V)^{com}$ the composition,

$$H^{A(n)}(V) \to H^A(V)^{com} \to H^{A(n)}(V)^{com},$$

must be an isomorphism. Since, by Corollary (1.12), the tangent spaces of $H^{A(n)}(V)$ and $H^A(V)$ are isomorphic, the lemma is proved.

□

**Corollary 2.11.** *Let $A = k < x_1, .., x_d >$ be the free $k$-algebra on $d$ symbols, and let $V \in Simp_n(A)$. Then*

$$H^A(V)^{com} \simeq H^{A(n)}(V) \simeq k[[t_1, ..., t_{(d-1)n^2+1}]]$$

This should be compared with the results of [Procesi 1], see also [Formanek].

In general, the natural morphism,

$$\eta(n) : A(n) \to \prod_{V \in Simp_n(A)} H^{A(n)}(V) \otimes_k End_k(V)$$

is not an injection.

*Example 2.12.* In fact, let

$$A = \begin{pmatrix} k & k & k \\ k & k & k \\ 0 & 0 & k \end{pmatrix}.$$

The ideal $I(2)$ is generated by $[e_{1,1}, e_{1,2}, e_{2,2}, e_{2,3}] = e_{1,3}$. So

$$A(2) = \begin{pmatrix} k & k & k \\ k & k & k \\ 0 & 0 & k \end{pmatrix} / \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix} \simeq M_2(k) \oplus M_1(k).$$

However,

$$\prod_{V \in Simp_2(A)} H^{A(2)}(V) \otimes_k End_k(V) \simeq M_2(k),$$

therefore $ker\ \eta(2) = M_1(k) = k$.

Let $O(n)$, be the image of $\eta(n)$, then,

$$O(n) \subseteq \prod_{V \in Simp_n(A)} H^{A(n)}(V) \otimes_k End_k(V)$$

and for every $V \in Simp_n(A)$,

$$H^{O(n)}(V) \simeq H^{A(n)}(V).$$

Put $B = \prod_{V \in Simp_n(A)} H^{A(n)}(V)$. Choosing bases in all $V \in Simp_n(A)$, then

$$\prod_{V \in Simp_n(A)} H^{A(n)}(V) \otimes_k End_k(V) \simeq M_n(B),$$

Let $x_i \in A, i = 1, ..., d$ be generators of $A$, and consider their images $(x^i_{p,q}) \in M_n(B)$. Now, $B$ is commutative, so the $k$-sub-algebra $C(n) \subset B$ generated by the elements $\{x^i_{p,q}\}_{i=1,..,d;\ p,q=1,..,n}$ is commutative. We have an injection,

$$O(n) \to M_n(C(n)),$$

and for all $V \in Simp_n(A)$, with a chosen basis, there is a natural composition of homomorphisms of $k$-algebras,

$$\alpha : M_n(C(n)) \to M_n(H^{A(n)}(V)) \to End_k(V),$$

inducing a corresponding composition of homomorphisms of the centers,

$$Z(\alpha) : C(n) \to H^{A(n)}(V) \to k$$

This sets up a set theoretical injective map,

$$t : Simp_n(A) \longrightarrow Max(C(n)),$$

defined by $t(V) := ker Z(\alpha)$.

Since $A(n) \to H^{A(n)}(V) \otimes_k End_k(V)$ is topologically surjective, $H^{A(n)}(V) \otimes_k End_k(V)$ is topologically generated by the images of $x_i,\ i = 1, ..., d$. It follows that we have a surjective homomorphism,

$$\hat{C}(n)_{t(V)} \to H^{A(n)}(V).$$

Categorical properties implies, that there is another natural morphism,

$$H^{A(n)}(V) \to \hat{C}(n)_{t(V)},$$

which composed with the former is an automorphism of $H^{A(n)}(V)$. Since

$$M_n(C(n)) \subseteq \prod_{V \in Simp_n(A)} H^{A(n)}(V) \otimes_k End_k(V),$$

it follows that for $\mathfrak{m}_v \in Max(C(n))$, corresponding to $V \in Simp_n(A)$, the finite dimensional $k$-algebra $M_n(C(n)/{\mathfrak{m}_v}^2)$ sits in a finite dimensional quotient of some,

$$\prod_{V \in \mathcal{V}} H^{A(n)}(V) \otimes_k End_k(V).$$

where $\mathcal{V} \subset Simp_n(A)$ is finite. However, by Lemma (2.5), the morphism,

$$A(n) \longrightarrow \prod_{V \in \mathcal{V}} H^{A(n)}(V) \otimes_k End_k(V)$$

is topologically surjectiv. Therefore the morphism,

$$A(n) \longrightarrow M_n(C(n)/{\mathfrak{m}_v}^2)$$

is surjectiv, implying that the map

$$H^{A(n)}(V) \to \hat{C}(n)_{\mathfrak{m}_v},$$

is surjectiv, and consequently, $H^{A(n)}(V) \simeq \hat{C}(n)_{\mathfrak{m}_v}$.

We now have the following theorem, see Chapter VIII, §2, of the book [Procesi 2], where part of this theorem is proved.

**Theorem 2.13.** *Let $V \in Simp_n(A)$, correspond to the point $\mathfrak{m}_v \in Simp_1(C(n))$.*

*(i) There exist a Zariski neighborhood $U_v$ of $v$ in $Simp_1(C(n))$ such that any closed point $\mathfrak{m}'_v \in U$ corresponds to a unique point $V' \in Simp_n(A)$.*

*Let $U(n)$ be the open subset of $Simp_1(C(n))$, the union of all $U_v$ for $V \in Simp_n(A)$.*

*(ii) $O(n)$ defines a non-commutative structure sheaf $\mathcal{O}(n) := \mathcal{O}_{U(n)}$ of Azumaya algebras on the topological space $U(n)$ (Jacobson topology).*

*(iii) The center $\mathcal{S}(n)$ of $\mathcal{O}(n)$, defines a scheme structure on $Simp_n(A)$.*

*(iv) The versal family of $n$-dimensional simple modules, $\tilde{V} := C(n)) \otimes_k V$, over $Simp_n(A)$, is defined by the morphism,*

$$\tilde{\rho} : A \to O(n) \subseteq End_{C(n)}(C(n)) \otimes_k V) \simeq M_n(C(n)).$$

*(v) The trace ring $Tr\tilde{\rho} \subseteq \mathcal{S}(n) \subseteq C(n)$ defines a commutative affine scheme structure on $Simp_n(A)$. Moreover, there is a morphism of schemes,*

$$\kappa : U(n) \longrightarrow Simp_n(A),$$

*such that for any $v \in U(n)$,*

$$H^{A(n)}(V) \simeq \hat{\mathcal{S}}(n)_{\kappa(v)} \simeq (\hat{Tr\tilde{\rho}})_{\kappa(v)} \simeq \hat{C}(n)_v$$

*Proof.* Let $\rho : A \longrightarrow End_k(V)$ be the surjective homomorphism of $k$-algebras, defining $V \in Simp_n(A)$. Let, as above $e_{i,j} \in End_k(V)$ be the elementary matrices, and pick $y_{i,j} \in A$ such that $\rho(y_{i,j}) = e_{i,j}$. Let us denote by $\sigma$ the cyclical permutation of the integers $\{1, 2, ..., n\}$, and put,

$$s_k := [y_{\sigma^k(1),\sigma^k(2)}, y_{\sigma^k(2),\sigma^k(2)}, y_{\sigma^k(2),\sigma^k(3)} ... y_{\sigma^k(n),\sigma^k(n)}],\ s := \sum_{k=0,1,..,n-1} s_k \in A.$$

Clearly $s \in I(n-1)$. Since $[e_{\sigma^k(1),\sigma^k(2)}, e_{\sigma^k(2),\sigma^k(2)}, e_{\sigma^k(2),\sigma^k(3)} ... e_{\sigma^k(n),\sigma^k(n)}] = e_{\sigma^k(1),\sigma^k(n)} \in End_k(V)$, $\rho(s) := \sum_{k=0,1,..,n-1} \rho(s_k) \in End_k(V)$ is the matrix with non-zero elements, equal to 1, only in the $(\sigma^k(1), \sigma^k(n))$ position, so the determinant of $\rho(s)$ must be +1 or -1. The determinant $det(s) \in C(n)$ is therefore nonzero at the point $v \in Spec(C(n))$ corresponding to $V$. Put $U = D(det(s)) \subset Spec(C(n))$, and consider the localization $O(n)_{\{s\}} \subseteq M_n(C(n)_{\{det(s)\}})$, the inclusion following from general properties of the localization. Now, any closed point $v' \in U$ corresponds to a $n$-dimensional representation of $A$, for which the element $s \in I(n-1)$ is invertible. But then this representation cannot have a $m < n$ dimensional quotient, so it must be simple.

Since $s \in I(n-1)$, the localized $k$-algebra $O(n)_{\{s\}}$ does not have any simple modules of dimension less than $n$, and no simple modules of dimension $> n$ . In fact, for any finite dimensional $O(n)_{\{s\}}$-module $V$, of dimension $m$, the image $\hat{s}$ of $s$ in $End_k(V)$ must be invertible. However, the inverse $\hat{s}^{-1}$ must be the image of a polynomial (of degree $m-1$) in s. Therefore, if $V$ is simple over $O(n)_{\{s\}}$, i.e. if the homomorphism $O(n)_{\{s\}} \to End_k(V)$ is surjective, $V$ must also be simple over $A$. Since now $s \in I(n-1)$, it follows that $m \geq n$. If $m > n$, we may construct, in the same way as above an element in $I(n)$ mapping into a nonzero element of $End_k(V)$. Since, by construction, $I(n) = 0$ in $A(n)$, and therefore also in $O(n)_{\{s\}}$, we have proved what we wanted. By a theorem of M. Artin, see [Artin], $O(n)_{\{s\}}$ must be an Azumaya algebra with center, $S(n)_{\{s\}} := Z(O(n)_{\{s\}})$. Therefore $O(n)$ defines a presheaf $\mathcal{O}(n)$ on $U(n)$, of Azumaya algebras with center $\mathcal{S}(n) := Z(\mathcal{O}(n))$. Clearly, any $V \in Simp_n(A)$, corresponding to $\mathfrak{m}_v \in Max(C(n))$ maps to a point $\kappa(v) \in Simp(\mathcal{O}(n))$. Let $\mathfrak{m}_{\kappa(v)}$ be the corresponding maximal ideal of $\mathcal{S}(n)$. Since $O(n)$ is locally Azumaya, it follows that,

$$\hat{\mathcal{S}}(n)_{\mathfrak{m}_{\kappa(v)}} \simeq H^{O(n)}(V) \simeq H^{A(n)}(V).$$

The rest is clear.

□

See the next § for a generalization of this result, to the case of a finite "swarm".

$Spec(C(n))$ is, in a sense, a compactification of $Simp_n(A)$. It is, however, not the correct *completion* of $Simp_n(A)$. In fact, the points of $Spec(C(n)) - Simp_n(A)$ may correspond to semi-simple modules, which do not deform into simple n-dimensional modules. We shall in the next § return to the study of the (notion of) completion, together with the degeneration processes that occur, at *infinity* in $Simp_n(A)$.

*Example 2.14.* Let us check the case of $A = k < x_1, x_2 >$, the free non-commutative $k$-algebra on two symbols. First, we shall compute $Ext^1_A(V,V)$ for a particular $V \in Simp_2(A)$, and find a basis $\{t_i^*,\}_{i=1}^5$, represented by derivations $\partial_i := \partial_i(V) \in Der_k(A, End_k(V))$, i=1,2,3,4,5. This is easy, since for any two $A$-modules $V_1, V_2$, we have the exact sequence,

$$\begin{aligned} 0 \to Hom_A(V_1,V_2) \to Hom_k(V_1,V_2) \to Der_k(A, Hom_k(V_1,V_2)) \\ \to Ext^1_A(V_1,V_2) \to 0 \end{aligned}$$

proving that, $Ext^1_A(V_1,V_2) = Der_k(A, Hom_k(V_1,V_2))/Triv$, where $Triv$ is the sub-vector space of trivial derivations. Pick $V \in Simp_2(A)$ defined by the homomorphism $A \to M_2(k)$ mapping the generators $x_1, x_2$ to the matrices

$$X_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =: e_{1,2}, \ X_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} =: e_{2,1}.$$

Notice that

$$X_1X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: e_{1,1} = e_1, \ X_2X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: e_{2,2} = e_2,$$

and recall also that for any $2 \times 2$-matrix $(a_{p,q}) \in M_2(k)$, $e_i(a_{p,q})e_j = a_{i,j}e_{i,j}$. The trivial derivations are generated by the derivations $\{\delta_{p,q}\}_{p,q=1.2}$, defined by,

$$\delta_{p,q}(x_i) = X_i e_{p,q} - e_{p,q} X_i.$$

Clearly $\delta_{1,1} + \delta_{2,2} = 0$. Now, compute and show that the derivations $\partial_i$, $i = 1,2,3,4,5$, defined by,

$$\partial_i(x_1) = 0, \text{for } i = 1,2, \ \partial_i(x_2) = 0, \text{for } i = 4,5,$$

by,

$$\partial_1(x_2) = e_{1,1}, \partial_2(x_2) = e_{1,2},\ \partial_3(x_1) = e_{1,2}, \partial_4(x_1) = e_{2,2}, \partial_5(x_1) = e_{2,1}$$

and by,

$$\partial_3(x_2) = e_{2,1},$$

form a basis for $Ext^1_A(V,V) = Der_k(A, End_k(V))/Triv$. Since $Ext^2_A(V,V) = 0$ we find $H(V) = k << t_1, t_2, t_3, t_4, t_5 >>$ and so $H(V)^{com} \simeq k[[t_1, t_2, t_3, t_4, t_5]]$. The formal versal family $\tilde{V}$, is defined by the actions of $x_1, x_2$, given by,

$$X_1 := \begin{pmatrix} 0 & 1+t_3 \\ t_5 & t_4 \end{pmatrix},\ X_2 := \begin{pmatrix} t_1 & t_2 \\ 1+t_3 & 0 \end{pmatrix}.$$

One checks that there are polynomials of $X_1, X_2$ which are equal to $t_i e_{p,q}$, modulo the ideal $(t_1, .., t_5)^2 \subset H(V)$, for all $i, p, q = 1, 2$. This proves that $\hat{C}(2)_v$ must be isomorphic to $H(V)$, and that the composition,

$$A \longrightarrow A(2) \longrightarrow M_2(C(2)) \subset M_2(H(V)))$$

is topologically surjective. By the construction of $C(n)$ this also proves that

$$C(2) \simeq k[t_1, t_2, t_3, t_4, t_5].$$

locally in a Zariski neighborhood of the origin. Moreover, the Formanek center, in this case. is cut out by the single equation:

$$f := det[X_1, X_2] = -((1+t_3)^2 - t_2t_5)^2 + (t_1(1+t_3) + t_2t_4)(t_4(1+t_3) + t_1t_5).$$

Computing, we also find the following formulas,

$$\begin{aligned} &trX_1 = t_4,\ trX_2 = t_1, \\ &detX_1 = -t_5 - t_3t_5,\ detX_2 = -t_2 - t_2t_3, \\ &tr(X_1X_2) = (1+t_3)^2 + t_2t_5 \end{aligned}$$

so the *trace ring* of this family is

$$k[t_1, t_2 + t_2t_3, 1 + 2t_3 + t_3^2 + t_2t_5, t_4, t_5 + t_3t_5] =: k[u_1, u_2, ..., u_5],$$

with,

$$u_1 = t_1,\ u_2 = (1+t_3)t_2,\ u_3 = (1+t_3)^2 + t_2t_5,\ u_4 = t_4,\ u_5 = (1+t_3)t_5,$$

and $f = -u_3^2 + 4u_2u_5 + u_1u_3u_4 + u_1^2u_5 + u_2u_4^2$. Moreover, the $k[\underline{u}]$ is algebraic over $k[\underline{u}]$, with discriminant, $\Delta := 4u_2u_5(u_3^2 - 4u_2u_5) = 4(1+t_3)^2t_2t_5((1+t_3)^2 - t_2t_5)^2$. From this follows that there is an étale covering

$$\mathbf{A}^5 - V(f\Delta) \to Simp_2(A) - V(\Delta).$$

Notice that if we put $t_1 = t_4 = 0$, then $f = \Delta$. See Example (3.7)

*Completions of $Simp_n(A)$.* In the example above it is easy to see that elements of the complement of $Simp_n(A)$ in the affine sub-scheme $Spec(C(n))$ will be represented by indecomposable, or decomposable representations. A decomposable representation $W$ will, however, not in general be deformable into a simple representation, since good deformations must conserve $End_A(W)$. Therefore, even though we have termed $Spec(C(n))$ a compactification of $Simp_n(A)$, it is a bad *completion*. The missing points *at infinity* of $Simp_n(A)$, should be represented as indecomposable representations, with $End_A(W) = k$. Any such is an iterated extension of simple representations $\{V_i\}_{i=1,2,..s}$, with representation graph $\Gamma$ (corresponding to an *extension type*, see [La 4]), and $\sum_{i=1}^{s} dim(V_i) = n$. To simplify the notations we shall write, $|\Gamma| := \{V_i\}_{i=1,2,..s}$. In [La 2,4], see also [Jö-La-Sl], we treat the problem of classifying all such indecomposable representations, up to isomorphisms. Let us recall the main ideas.

Assume that the simple modules $\{V_i\}$ we shall talk about are such that all $Ext^1_A(V_i, V_j)$ are finite dimensional as $k$-vector spaces. Let $\Gamma$ be an ordered graph with set of nodes $|\Gamma| = \{V_i\}$. Starting with the the first node of $\Gamma$, we can construct, in many ways, an extension of the corresponding module $V_{i_1}$ with the module $V_{i_2}$ corresponding to the end point of the first arrow of $\Gamma$, then continue, choosing an extension of the result with the module corresponding to the endpoint of the second arrow of $\Gamma$, etc. untill we have reached the endpoint of the last arrow. Any finite length module can be made in this way, the "oppositely ordered" $\Gamma$ corresponding to a decomposition of the module into simple constituencies, by peeling off one simple sub-module at a time, i.e. by picking one simple sub-module and forming the quotient, picking a second simple sub-module of the quotient and taking the quotient, and repeating the procedure untill it stops.

The "ordered" $k$-algebra $k[\Gamma]$ of the ordered graph $\Gamma$ is the quotient algebra of the usual algebra of the graph $\Gamma$ by the ideal generated by all admissible words which are not "intervals" of the ordered graph. Say $...\gamma_{i,j}(n-1)\gamma_{j,j}(n)\gamma_{j,k}(n+1)...$ is is an interval of the ordered graph, then $\gamma_{i,j}(n-1).\gamma_{j,k}(n+1) = 0$ in $k[\Gamma]$.

Now, let $H(|\Gamma|)$ be the formal moduli of the family $|\Gamma|$. We show in [La 4], see Proposition 2. above, that any iterated extension of the $\{V_i\}_{i=1}^{r}$ with *extension type*, i.e. graph, $\Gamma$ corresponds to a morphism in $\underline{a}_r$,

$$\alpha : H \longrightarrow k[\Gamma].$$

Moreover the set of isomorphism classes of such modules is parametrized by a quotient space of the affine scheme,

$$\underline{A}(\Gamma) := Mor_{\underline{a}_r}(H(|\Gamma|), k[\Gamma]).$$

Let $\alpha \in \underline{A}(\Gamma)$, and let $V(\alpha)$ denote the corresponding iterated extension module, then the tangent space of $\underline{A}(\Gamma)$ at $\alpha$ is,

$$T_{\underline{A}(\Gamma),\alpha} := Der_k(H(|\Gamma|), k[\Gamma]_\alpha),$$

where $k[\Gamma]_\alpha$ is $k[\Gamma]$ considered as a $H(|\Gamma|)$-bimodule via $\alpha$. The obstruction space for the deformation functor of $\alpha$ is $HH^2(H(|\Gamma|), k[\Gamma])$, and we may, as is explained in [La 0,1], compute the complete local ring of $\underline{A}(\Gamma)$ at $\alpha$. In particular we may decide whether the point is a smooth point of $\underline{A}(\Gamma)$, or not.

The automorphism group $G$ of $k[\Gamma]$, considered as an object of $\underline{a}_r$, has a Lie algebra which we shall call $\mathfrak{g}$. Obviously we have,

$$\mathfrak{g} = Der_k(k[\Gamma], k[\Gamma]).$$

Clearly an iterated extension $\alpha$ with graph $\Gamma$ will be isomorphic as $A$-module to $g(\alpha)$, for any $g \in G$. In particular, if $\delta \in \mathfrak{g}$, then $exp(\delta)(\alpha)$ is isomorphic to $\alpha$ as an iterated extension of $A$-modules, with the same graph as $\alpha$.

Consider the map,

$$\alpha^* : Der_k(k[\Gamma], k[\Gamma]) \to Der_k(H(\Gamma), k[\Gamma]_\alpha).$$

The image of $\alpha^*$ is the subspace of the tangent space of $\underline{A}(\Gamma)$ at $\alpha$ along which the corresponding module has constant isomorphism class.

Notice that if $\alpha$ is a smooth point, and $\alpha^*$ is not surjectiv then there is a positive-dimensional moduli space of iterated extension modules with graph $\Gamma$ through $\alpha$.

Clearly the kernel of $\alpha^*$ is contained in the Lie algebra of automorphisms of the module $V(\alpha)$, and should be contained in $End_A(V(\alpha))$. From this follows that if $V(\alpha)$ is indecomposable then $ker\alpha^* = 0$. The Euler type derivations, defined by,

$$\delta_E(\gamma_{i,j}) = \rho_{i,j}\gamma_{i,j}, \ \ \rho_{i,j} \in k$$

are the easiest to check! Notice however, that there may be discrete automorphisms in $G$, not of exponential type, leaving $\alpha$ invariant. Notice also that an indecomposable module may have an endomorphism-ring which is a non-trivial local ring.

Assume now that we have identified the *non-commutative* scheme of indecomposable $\Gamma$-representation, call it $Ind_\Gamma(A)$. Put $Simp_\Gamma(A) := Simp_n(A) \cup Ind_\Gamma(A)$. Now, repeat the basics of the construction of $Spec(C(n))$ above. Consider for every open affine subscheme $D(s) \subset Simp_\Gamma(A)$, the natural morphism,

$$A \to \varprojlim_{\underline{c} \subset D(s)} O(\underline{c}, \pi)$$

$\underline{c}$ running through all finite subsets of $D(s)$. Put $B_s(\Gamma) := \prod_{V \in D(s)} H^{A(n)}(V)^{com}$, and consider the homomorphism,

$$A \to A(n) \to \prod_{V \in D(s)} H^{A(n)}(V)^{com} \otimes_k End_k(V) \simeq M_n(B_s(\Gamma)).$$

Let $x_i \in A, i = 1, ..., d$ be generators of $A$, and consider the images $(x^i_{p,q}) \in B_s(n) \otimes_k End_k(k^n)$ of $x_i$ via the homomorphism of $k$-algebras,

$$A \to B_s(\Gamma) \otimes M_n(k),$$

obtained by choosing bases in all $V \in Simp_\Gamma(A)$. Notice that since $V$ no longer is (necessarily) simple, we do not know that this map is topologically surjectiv. Now, $B_s(\Gamma)$ is commutative, so the $k$-sub-algebra $C_s(\Gamma) \subset B_s(\Gamma)$ generated by the elements $\{x^i_{p,q}\}_{i=1,..,d;\ p,q=1,..,n}$ is commutative. We have a morphism,

$$I_s(\Gamma) : A \to C_s(\Gamma) \otimes_k M_n(k) = M_n(C_s(\Gamma)).$$

Moreover, these $C_s(\Gamma)$ define a presheaf, $\mathcal{C}(\Gamma)$, on the Jacobson topology of $Simp_\Gamma(A)$. The rank $n$ free $C_s(\Gamma)$-modules with the $A$-actions given by $I_s(\Gamma)$, glue together to form a locally free $\mathcal{C}(\Gamma)$-Module $\mathcal{E}(\Gamma)$ on $Simp_\Gamma(A)$, and the morphisms $I_s(\Gamma)$ induce a morphism of algebras,

$$I(\Gamma) : A \to End_{\mathcal{C}(\Gamma)}(\mathcal{E}(\Gamma)).$$

As for every $V \in Simp_\Gamma(A)$, $End_A(V) = k$, the commutator of $A$ in $H^A(V)^{com} \otimes_k End_k(V)$ is $H^A(V)^{com}$. The morphism,

$$\zeta(V) : H^A(V)^{com} \to HH^0(A, H^A(V)^{com} \otimes_k End_k(V))$$

is therefore an isomorphism, and we may assume that the corresponding morphism,

$$\zeta : \mathcal{C}(\Gamma) \to HH^0(A, End_{\mathcal{C}(\Gamma)}(\mathcal{E}(\Gamma)))$$

is an isomorphism of sheaves. For all $V \in D(s) \subset Simp_\Gamma(A)$ there is a natural projection,

$$\kappa := \kappa(\Gamma) : \mathcal{C}_s(\Gamma) \otimes_k M_n(k) \to H^{A(n)}(V)^{com} \otimes_k End_k(V) \simeq M_n(H^{A(n)}(V)^{com}),$$

which, composed with $I_s(\Gamma)$ is the natural homomorphism,

$$A \longrightarrow H^{A(n)}(V)^{com} \otimes_k End_k(V).$$

$\kappa$ defines a set theoretical map,

$$t : Simp_\Gamma(A) \longrightarrow Spec(\mathcal{C}(\Gamma)),$$

and a natural surjective homomorphism,

$$\hat{\mathcal{C}}(\Gamma)_{t(V)} \to H^{A(n)}(V)^{com}.$$

Categorical properties implies, as usual, that there is another natural morphism,

$$\iota : H^{A(n)}(V) \to \hat{\mathcal{C}}(\Gamma)_{t(V)},$$

which composed with the former is the obvious surjection, and such that the induced composition,

$$A \longrightarrow H^{A(n)}(V)^{com} \otimes_k End_k(V) \to \hat{\mathcal{C}}(\Gamma)_{t(V)} \otimes_k End_k(V),$$

is $I(\Gamma)$ formalized at $t(V)$. From this, and from the definition of $C(\Gamma)$, it follows that $\iota$ is surjective, such that for every $V \in Simp_\Gamma(A)$ there is an isomorphism $H^{A(n)}(V)^{com} \simeq \hat{\mathcal{C}}(\Gamma)_{t(V)}$. For $V \in Simp_\Gamma(A)$ there is also a natural commutative diagram,

$$\begin{array}{ccc}
ZA(n) & \longrightarrow & \mathcal{C}(\Gamma) \\
\downarrow & & \downarrow \\
A(n) & \longrightarrow & End_{\mathcal{C}(\Gamma)}(\mathcal{E}(\Gamma)) \\
\downarrow & & \downarrow \\
H^{A(n)}(V) \otimes_k End_k(V) & \longrightarrow & \hat{\mathcal{C}}(\Gamma) \subset_{t(V)} (n) \otimes_k End_k(V)
\end{array}$$

Formally at a point $V \in Simp_\Gamma(A)$, we have therefore proved that the local, commutative structure of $\mathcal{S}imp_\Gamma(A)$ (as $A$ or $A(n)$-module), and the corresponding local structure of $Spec(\mathcal{C}(\Gamma))$ at $V$, coincide. We have actually proved the following,

**Theorem 2.15.** *The topological space $Simp_\Gamma(A)$, with the Jacobson topology, together with the sheaf of commutative $k$-algebras $\mathcal{C}(\Gamma)$ defines a scheme structure on $\mathcal{S}imp_\Gamma(A)$, containing an open subscheme, étale over $Simp_n(A)$. Moreover, there is a morphism,*

$$\pi(\Gamma) : \mathcal{S}imp_\Gamma(A) \to Spec(ZA(n)),$$

*extending the natural morphism,*

$$\pi_0 : Simp_n(A) \to Spec(ZA(n)).$$

*Proof.* As in Theorem (2.13) we prove that if $v = t(V), with\ V \in Simp_n(A) \subseteq Simp_\Gamma(A)$, then there exists an open subscheme of $Spec(\mathcal{C}(\Gamma))$ containing only simple modules of dimension $n$. If $v$ is indecomposables with $End_A(V) = k$ we may look at the homomorphism of $\mathcal{C}(\Gamma)$-modules,

$$End_A(\mathcal{C}(\Gamma)) \otimes End_k(V) \longrightarrow End_A(V) = k.$$

Clearly there is an open neighborhood of $v$ in $Spec(\mathcal{C}(\Gamma))$ containing only indecomposables of dimension $n$.

□

These morphisms $\pi(\Gamma)$ are our candidates for the possibly different completions of $Simp_n(A)$. Notice that for $W \in Spec(C(n)) - U_n$, the formal moduli $H^A(W)$ is not always pro-representing. If $W$ is semi-simple, but not simple then $End_A(W) \neq k$. The corresponding modular substratum will, locally, correspond to the semi-simple deformations of $W$, thus to a closed subscheme of $Spec(C(n)) - U_n \subset Spec(C(n))$. This follows from the fact that the substratum of modular deformations of any semisimple (but not simple) module $V$ will have a tangent space equal to the invariant space of the action of the $End_k(V)$ on $Ext^1_A(V, V)$, which must be the sum of the tangent spaces of the deformation spaces of the simple components of $V$.

$Spec(C(n))$ is, in a sense, a compactification of $U_n$. It is, however not the correct *completion* of $U_n$. In fact, the points of $Spec(C(n)) - U_n$ may correspond to semi-simple modules, which do not deform into simple n-dimensional modules. We shall in §4 return to the study of the (notion of) completion, in connection with the process of *decay* and *creation* of particles. Decay occur, at *infinity* in $Simp_n(A)$.

Above we have studied moduli spaces of representations of finitely generated $k$-algebras. We might as well have studied the Hilbert functor, $\mathcal{H}_A$, of the algebra $A$, or the moduli space $\mathcal{M}(A; R)$, of morphisms, $\kappa : A \to R$, for fixed algebras, $A$ and $R$. The difference is that whereas for finite $n$, the set $Simp_n(A)$ has a nice, finite dimensional scheme structure, this is, in general, no longer true for the sets, $\mathcal{H}_A$ or $\mathcal{M}(A; R)$. If, however, $R$ is Artinian of length $n$, then the corresponding Hilbert scheme, $\mathcal{H}^n_A$, does exist and has a nice structure, both as commutative and as non-commutative scheme. In particular, the toy model of relativity theory, referred to in the introduction, is modeled on $\mathcal{M}(k[x_1, x_2, x_3], k^2)$, i.e. on the set of surjective homomorphisms $k[x_1, x_2, x_3] \to R = k^2$. We shall now enrich these structures somewhat by introducing the notion of a *string*.

**Definition 2.16.** *A general string, a g-string, is an algebra $R$ together with a pair of Ph-points, i.e. a pair of homomorphisms $\epsilon_i : PhR \to k(p_i)$, corresponding to two points $k(p_i) \in Simp_1(R)$ each outfitted with a tangent $\xi_i$.*

We might have considered any two points $k(p_i) \in Simp_n(PhR)$, but since the main properties of the g-strings will be equally well understood restricting to the case $n = 1$, we shall postpone this generalization.

For any g-string, consider the *non-commutative tangent space* of the the pair of points,

$$T(R, p_1, p_2) := Ext^1_{PhR}(p_1, p_2).$$

We shall call it the *space of tensions*, between the two points of the string. The von Neumann condition on the string is now simply,

$$\epsilon_i \circ d = \xi_i = 0, \; i = 1 \vee i = 2,$$

which, if $x_j, j = 1, .., n$ and $\sigma_l, l = 1, .., p$ are parameters of $A$ respectively $R$, is equivalent to the condition,

$$\frac{\partial x_j}{\partial \sigma_{\;l}}(p_i) = 0, j = 1, ..., n, \; l = 1, .., p, \; i = 1 \vee 2.$$

Any homomorphism of algebras, $\kappa : A \to R$ induces a unique commutative diagram of algebras,

$$\begin{array}{ccc} A & \xrightarrow{\kappa} & R \\ \downarrow & & \downarrow \\ PhA & \xrightarrow{Ph\kappa} & PhR. \end{array}$$

Moreover, since any derivation $\xi \in Der_k(A, R)$ has a natural lifting to a derivation $\overline{\xi} \in Der_k(PhA, PhR)$ we find, using the general machinery of deformations of diagrams, see [La 0], that any family of morphisms $\kappa$ induces a family of the above diagram. If $\tau_k, \; k = 1, ..., d$ are parameters of such a family, $\mathbf{M} = Spec(M)$, then $d\tau_i \in PhM$ corresponds to a derivation, $\tau_i \in Der_k(A, R)$, and therefore to tangents $\eta_i, \; i = 1, 2$, of $Simp_1(A)$ at the two points $k(p_i)$. The Dirichlet condition on the string is now,

$$\eta_i = 0, \; i = 1 \vee 2,$$

which is equivalent to the condition,

$$\frac{\partial x_j}{\partial \tau_{\;l}}(p_i) = 0, j = 1, ..., n, \; l = 1, .., p, \; i = 1 \vee 2.$$

These conditions will define a new moduli space which we shall call $String_R(A)$. In the affine case the structure of this "space" is a problem, however we may of course do everything above for $A$ and $R$ replaced by projective schemes, and then all the moduli spaces exist as classical schemes. The volume form of the space the string is fanning out above a curve-element in $String_R(A)$, wil give us a Lagrangian, and then we go about defining the dynamics of the system, just like above.

*Example 2.17 (i) Let us go back to Example (1.1)(ii). It follows that the string of dimension 0,* $R = k^2, \; Ph(R) = k < x, dx > /((x^2 - r^2), (xdx + dxx))$*, has unique points,* $k(\pm r)$*. The space of tensions is of dimension 1, the von Neumann condition is automatically satisfied, and the moduli space of* $k^2$*-strings in* $A = k[x_1, x_2, x_3]$ *is nothing but* $\underline{H} := Speck[t_1, ..., t_6] - \underline{\Delta}$*. If we consider the string with* $R = k[x]/(x^2), \; PhR = k[x, dx]/(x^2, (xdx + dxx)$*, then we see that there is just one point* *of $R$, but a line of point for $PhR$, all corresponding to $x = 0$ in $R$. Therefore there is a 2-dimensional space of strings with the same $R$. Compare this with the blow-up $\underline{\tilde{H}}$, see* [La 6]. *(ii) In dimension 1 the simplest closed string is given by, $R = k[x, y]/(f)$, with $f = x^2 + y^2 - r^2$, such that $PhR = k < x, y, dx, dy > /(f, [x, y], d[x, y], df)$, and with the two points, $\epsilon_i : PhR \to k(p_i)$, defined by the actions on $k(p_i) := k$, given by, $x_i, y_i, (dx)_i, (dy)_i,\ i = 1, 2$. It is easy to see that the vectors, $\xi_i := ((dx)_i, (dy)_i)$ are tangent vectors to the circle at the points $p_i$, and if $p_1 \neq p_2$ we find that $Ext^1_{PhR}(k(p_1), k(p_2) = k$. The von Neumann condition is, $\xi_i = 0,\ i = 1\ \vee i = 2$, and this clearly means that $\frac{\partial x}{\partial \sigma} = \frac{\partial y}{\partial \sigma} = 0$ at one of the points $p_i$. The 1-dimensional open string is now left as an exercise..*

§**3. Geometry of time-spaces and the general dynamical law.** Given a natural number $n$, we have in §2 constructed a scheme $Simp_n(A)$, and a rank $n$ fiberbundle $\tilde{V}$ defined on $Simp_n(A)$. Moreover any element $a \in A$ will act on $\tilde{V}$, commuting with the action of $C(n)$, therefore as an endomorphism of the bundle defined on $Simp_n(A)$.

We would now like to use this theory for the $k$-algebra $Ph^\infty(A)$ of §1. However, the existence of deformations, and therefore the existence of the non-commutative, and commutative structure on $Simp_n(A)$ is dependent upon the finite type of the algebra $A$, see [La 3,4], and $Ph^\infty(A)$ is rarely of finite type. We shall therefore intoduce the notion of *dynamical structure*, and the *order* of a dynamical structure, to reduce the problem to a situation we can handle.

**Definition 3.1.** *A dynamical structure, $\sigma$, is a two-sided $\delta$-ideal $(\sigma) \subset Ph^\infty(A)$, such that*

$$\mathbf{A}(\sigma) := Ph^\infty(A)/(\sigma)$$

*is of finite type. A dynamical structure of order $n$ is now a section,*

$$\sigma : Ph^{(n)} \to Ph^{(n-1)}$$

*of the canonical homomorphism $\iota : Ph^{(n-1)}A \to Ph^{(n)}A$. If $A$ is generated by the coordinate functions, $\{t_i\}_{i=1,2,...,d}$ any section of order $n$ is defined by a system of equations,*

$$\delta^n t_p = \Gamma^p(\underline{t}_i, \underline{d}t_j, \underline{d}^2 t_k, .., \underline{d}^{n-1} t_l),\ p = 1, 2, ..., d.$$

*Put,*

$$\mathbf{A}(\sigma) := Ph^\infty(A)/(\delta^n t_p - \Gamma^p)$$

*where $\sigma := (\delta^n t_p - \Gamma^p)$ is the two-sided $\delta$-ideal generated by the defining equations of $\sigma$. Obviously $\delta$ induces a derivation $\delta_\sigma \in Der_k(\mathbf{A}(\sigma))$, also called the Dirac derivation, and usually just denoted $\delta$.*

Notice that if $\sigma_i,\ i = 1, 2$, are two order n dynamical systems, then we may well have,

$$\mathbf{A}(\sigma_1) \simeq \mathbf{A}(\sigma_2) \simeq Ph^{(n-1)}A/\sigma_*,$$

as $k$-algebras.

For any integer $n \geq 1$ consider the schemes $Simp_n(\mathbf{A}(\sigma))$ and $Spec(C(n))$, and the corresponding universal family,

$$\tilde{\rho} : \mathbf{A}(\sigma)) \to End_{Spec(C(n))}(\tilde{V}) \subseteq M_{n(C(n))}.$$

The Dirac derivation $\delta \in Der_k(\mathbf{A}(\sigma), \mathbf{A}(\sigma))$ defines, as explained above, a distrubution on $Simp_n(\mathbf{A}(\sigma))$. The reason why the Dirac derivation $\delta$, does not define a unique vector-field is, of course, that the structure maps of the simple modules can be scaled by any non-zero element of the field $k$. However, once we have chosen a (uni) versal family for the moduli space $Simp_n(A(\sigma))$, defined on $Spec(C(n))$, we find a well defined vector field $\xi$, in the distribution defined by $\delta$, induced by this versal family.

Moreover, for every $n$ the vectorfield $\xi \in \Theta_{Simp_n(\mathbf{A}(\sigma))}$ lifts to a field

$$[\delta] \in Der_k(A(\sigma), End_{C(n)}(\tilde{V})).$$

To see this, consider the classical isomorphism,

$$Ext^1_{C(n)\otimes\mathbf{A}(\sigma)}(\tilde{V}, \tilde{V}) \simeq Der_k(C(n) \otimes \mathbf{A}(\sigma), End_k(\tilde{V}))/\sim$$

where the equivalence relation is given in terms of trivial derivations. Recall that $\tilde{V} \simeq C(n) \otimes_k V$, as $C(n)$-module. Therefore

$$Ext^1_{C(n)}(\tilde{V}, \tilde{V}) = Der_k(C(n), End_k(\tilde{V}))/\sim = 0.$$

Moreover, the kernel of the natural surjective map,

$$Der_k(C(n) \otimes \mathbf{A}(\sigma), End_k(\tilde{V})) \to Der_k(C(n), End_k(\tilde{V}))$$

is,

$$Der_k(\mathbf{A}(\sigma), End_{C(n)}(\tilde{V})).$$

In fact, if $\tilde{\delta} \in Der_k(C(n)\otimes\mathbf{A}(\sigma), End_k(\tilde{V}))$ maps to zero in $Der_k(C(n), End_k(\tilde{V}))$, then $c\tilde{\delta}(a) = \tilde{\delta}(c)a + c\tilde{\delta}(a) = \tilde{\delta}(c \otimes a) = \tilde{\delta}(a)c + a\tilde{\delta}(c) = \tilde{\delta}(a)c$, since $c \otimes a = (c \otimes 1)(1 \otimes a) = (1 \otimes a)(c \otimes 1)$. Therefore,

$$Ext^1_{C(n)\otimes\mathbf{A}(\sigma)}(\tilde{V}, \tilde{V}) \simeq Der_k(\mathbf{A}(\sigma), End_{C(n)}(\tilde{V}))/\sim .$$

By construction of $C(n)$, or of universal families in general, we have an identification,

$$\theta_{U(n)} \simeq \mathcal{E}xt^1_{C(n)\otimes\mathbf{A}(\sigma)}(\tilde{V}, \tilde{V})|U(n).$$

Therefore, if $C(n)$ is smooth, we can find representatives,

$$\delta_i \in Der_k(\mathbf{A}(\sigma), End_{C(n)}(\tilde{V}))$$

of a (global) basis $\{\chi_i\}_{i=1,2,\dots,d}$ for the $C(n)$-module of vectorfields $\theta_{Simp_n(\mathbf{A}(\sigma))}$.

This, of course, is the way we construct the universal family,

$$\tilde{\rho} : \mathbf{A}(\sigma) \longrightarrow End_{C(n)}(\tilde{V})$$

in the first place.

Now, consider the Dirac derivation, $\delta \in Der_k(\mathbf{A}(\sigma), \mathbf{A}(\sigma))$, and the corresponding universal derivation,

$$\tilde{\rho}\delta : \mathbf{A}(\sigma) \longrightarrow End_{C(n)}(\tilde{V}).$$

We shall also denote this derivation $\delta$ and talk about it as the Dirac derivation. The context will, hopefully, avoid confusion. We have proved the following:

**Theorem 3.2.** *As operators on $\tilde{V}$, we must have,*

$$\delta = [\delta] + [Q, -]$$

*for some* $[\delta] = \sum_i \xi_i \delta_i \in Der_k(\mathbf{A}(\sigma), End_{C(n)}(\tilde{V}))$ *and some* $Q \in M_n(C(n))$. *This means that for every* $a \in \mathbf{A}(\sigma)$, *considered as an element* $\tilde{\rho}(a) \in M_n(C(n))$, $\delta(a)$ *acts on* $\tilde{V}$ *as*

$$\tilde{\rho}(\delta(a)) = [\delta](\tilde{\rho}(a)) + [Q, \tilde{\rho}(a)].$$

This $Q$ is the *Hamiltonian* of the system, also called the Dirac operator when $[\delta] = 0$, and sometimes denoted *delta slashed*, see e.g. [Schücker], or other texts on Connes' spectral tripples. In fact, a *spectral tripple* is composed of a vector space like $\tilde{V}$, together with a Dirac operator, like $Q$, and a complexification etc.

If $[\delta] = 0$, it is also easy to see that what we have observed implies that Heisenberg's and Schrödinger's way of doing quantum mechanics are strictly equivalent.

In general, $[\delta]$ induces the vector field,

$$\xi := \sum_i \xi_i \chi_i \in \theta_{Simp_n(\mathbf{A}(\sigma))}.$$

In line with our general philosophy, we shall consider $\xi$ as measuring *time* on $Simp_n(\mathbf{A}(\sigma))$.

Assume from now on that $k = \mathbf{R}$, the real numbers, and that our constructions go through, as if $k$ were algebraically closed. Let $v(\underline{\tau}_0) \in Simp_n(\mathbf{A}(\sigma))$ be an element, an *event*. Suppose there exist an integral curve $\gamma$ of $\xi$ through $v(\underline{\tau}_0) \in Simp_1(C(n))$, ending at $v(\underline{\tau}_1) \in Simp_1(C(n))$, given by the diffeomorphisms $e(\tau) := exp(\tau\xi)$, for $\tau \in [\tau_0, \tau_1] \subset \mathbf{R}$. The maximum $\tau$ for which the end point, $\underline{t}$, of $\gamma$ is in $Simp_n(\mathbf{A}(\sigma))$ should be called the *lifetime* of the particle. We shall see that it is relatively easy to compute these lifetimes, when the fundamental vector field $\xi$ has been computed.

This, however, is certainly not so easy, see the examples done by hand, (3.4)-(3,8).

Let now $\psi(\tau_0) \in \tilde{V}(v_0) \simeq V$ be a (classically considered) state of our *quantum system*, at the time $\tau_0$, and consider the (uni)versal family,

$$\tilde{\rho} : \mathbf{A}(\sigma) \longrightarrow End_{C(n)}(\tilde{V})$$

where $Simp_n(\mathbf{A}(\sigma)) \subseteq Simp_1(C(n))$. We shall consider $\mathbf{A}(\sigma)$ as our ring of *observables.*

What happens to $\psi(\tau_0) \in V(0)$ when *time* passes from 0 to $\tau$, along $\gamma$? This is obviously a question that has to do with whether we choose to consider the the Heisenberg or the Schrödinger picture. In fact, if we consider the formal flow $exp(t\delta)$ defined on the ring of observables, then putting,

$$U(t) := exp(t\nabla_\xi),$$

where,

$$\nabla_\xi := \xi + Q \in End_k(\tilde{V}),$$

we obtain for every $\psi \in \tilde{V}$, and every $a \in A(\sigma)$, that the equation,

$$U(t)(\tilde{\rho}(exp(-t\delta)(a)(\psi)) = \tilde{\rho}(a)(U(t)(\psi))$$

holds formally, at least up to first order. Thus, an element $\psi \in \tilde{V}$ which is *flat* with respect to the connection $\nabla_\xi$, above $\gamma$, has the property that,

$$\tilde{\rho}(\delta(a))\psi = 0,$$

for all $a \in A(\sigma)$, i.e. time-change of an observable does not change its action on $\psi$. It is therefore reasonable to consider any flat state, $\psi(t) \in \tilde{V}$, as the time development of $\psi(0) \in V(0)$. Clearly, the flat states $\psi \in \tilde{V}$, are solutions of the differential equation,

$$\xi(\psi) = -Q(\psi), i.e. \ \frac{\partial \psi}{\partial \tau} = -Q(\psi).$$

which, if we accept that time is the parameter $\tau$ of the integral curve $\gamma$, is the Schrödinger equation. Moreover, $\psi$ is completely determined by the value of $\psi(0)$, as we shall see now.

**Theorem 3.3.** *The evolution operator $U(\tau_0, \tau_1)$ that changes the state $\psi(\tau_0) \in \tilde{V}(v_0)$ into the state $\psi(\tau_1) \in \tilde{V}(v_1)$, where $\tau_1 - \tau_0$ is the length of the integral curve $\gamma$ connecting the two points $v_0$ and $v_1$, i.e. the time passed, is given by,*

$$\psi(\tau_1) = U(\tau_0, \tau_1)(\psi(\tau_0)) = exp[\int_\gamma Q(\tau)d\tau] \ (\psi(\tau_0)),$$

*where $exp \int_\gamma$ is the non-commutative version of the ordinary action integral, essentially defined by the equation,*

$$exp[\int_\gamma Q(\tau)dt] = exp[\int_{\gamma_2} Q(\tau)d\tau] \circ exp[\int_{\gamma_1} Q(\tau)d\tau]$$

*where $\gamma$ is $\gamma_1$ followed by $\gamma_2$.*

*Proof.* This is a well known consequence of the Schrödinger equation above. In classical quantum theory one uses a *chronological operator* $\tau$, to keep track of the *intermediate* time-steps that, in our case, are well defined by the integral curve $\gamma$, the existence of which we assume. The formula above is related to what the physicists call the Dyson series, see [Elbaz], Chapitre 6. Since we have given the real curve $\gamma$ parametrized by $\tau$ we may look at $\gamma$ as a closed interval of $\mathbf{R}$, $I := [0, \tau]$. Subdivide $I$ into m equal intervals, $[i\Delta\tau, (i+1)\Delta\tau]$, and see that the Schrödinger equation gives, formally,

$$\psi((i+1)\Delta\tau) = exp(\Delta\tau Q)(\psi(i\Delta\tau).$$

Writing out the power series in $\Delta\tau$, and summing up we find the formula above.

□

Notice that we find the same formulas if we extend the action of $A(\sigma)$ to $\tilde{V}_{\mathbf{C}} := \tilde{V} \otimes_{\mathbf{R}} \mathbf{C}$. This is what turns out to be the case in the physically reasonable situations, see again [La 5]. The result above may then be used to explain (partly) Feynman's formalism.

The usual interpretation of Feynman's integral, using the quantification of Weyl, with integral kernels, and without an intrinsic notion of time, is undoubtedly a good guess, but lacks any philosophical basis, see a good exposition for mathematicians in [Faddeev].

Here we see that the Schrödinger equation determines, uniquely, the outcome of the time-evolution, from every intermediate state $\psi(\tau),\ \tau \in [\tau_0, \tau_1]$, along $\gamma$. The problems leading to the path integrals, and Feynman's conjecture, is now related to the problem of determining the point of departure, i.e. $V(\tau_0)$, as an $A(\sigma)$-module, knowing only its structure as an $A$-module. This is, obviously, related to the Heisenberg's indeterminacy and I shall hopefully return to this in a later paper. Notice that we have not introduced any quadratic forms, or norms, on the modules we have studied. This is, of course necessary if we want to introduce probabilities, expectations etc.

In [La 5] we treated the case of a conservative system, i.e. where the vector field $\xi$ in $Simp_n(\mathbf{A}(\sigma)$ is singular, i.e. vanishes, at the point $v \in Simp_n(\mathbf{A}(\sigma)$ corresponding to the representation $V$, and where therefore the Hamiltonian $Q$ is both the time and energy operator, at the same "time". See examples (3.6) and (3.7) where we show how to compute these singularities in some classical cases.

We found, see §1, that there is a notion of *Planck's constant* $\hbar$, with the ordinary properties.

This is also true in general. In fact, since

$$[Q, \tilde{\rho}(\delta(f_\lambda))] = \tilde{\rho}\delta - [\delta]\tilde{\rho} : \mathbf{A}(\sigma) \longrightarrow End_{C(n)}(\tilde{V})$$

is a derivation, we show that the set,

$$\Lambda(\sigma) := \{\lambda \in C(n) | \exists f_\lambda \in \mathbf{A}(\sigma), f_\lambda \neq 0, [Q, \tilde{\rho}(\delta(f_\lambda))] = \lambda\tilde{\rho}(f_\lambda)\},$$

is a generalized additive monoid, i.e. if for $\lambda, \lambda' \in \Lambda(\sigma)$ the product $f_\lambda f_{\lambda'}$ is non-trivial, then $\lambda + \lambda' \in \Lambda(\sigma)$.

Let $\hbar_l \in k$ be "generators" of $\Lambda(\delta)$. These are our Planck's constants, see examples (3.7) and (3.8). Now, assume there exists a C(n)-module basis $\{\tilde{\psi}_i\}_{i\in I}$ of sections of $\tilde{V} = C(n) \otimes V$, formed by eigenfunctions for the Hamiltonian, i.e. such that

$$Q(\tilde{\psi}_i) = \kappa_i \tilde{\psi}_i,\ i \in I,$$

where $\kappa_i \in C(n)$. An element such as $\tilde{\psi}_i \in \tilde{V}$ is usually considered as a *pure state*, with *energy* $\kappa_i \in C(n)$, depending on time, i.e. depending on $\tau$, the length along the integral curve $\gamma$. It is also considered as as an *elementary particle* (since $\tilde{V}$ is, by assumption, simple). As in §1 we find,

$$\begin{aligned}
\tilde{\rho}(\lambda f_\lambda)(\tilde{\psi}_i) &= \lambda\tilde{\rho}(f_\lambda)(\tilde{\psi}_i) \\
&= \tilde{\rho}(\delta(f_\lambda))(\tilde{\psi}_i) - [\delta](\tilde{\rho}(f_\lambda))(\tilde{\psi}_i) \\
&= Q(\tilde{\rho}(f_\lambda) - \tilde{\rho}(f_\lambda))Q(\tilde{\psi}_i) \\
&= Q(\tilde{\rho}(f_\lambda)(\tilde{\psi}_i)) - \kappa_i\tilde{\rho}(f_\lambda)(\tilde{\psi}_i)
\end{aligned}$$

implying,

$$Q(\tilde{\rho}(f_\lambda)(\tilde{\psi}_i)) = (\kappa_i + \lambda)\tilde{\rho}(f_\lambda)(\tilde{\psi}_i).$$

By assumtion, if $\tilde{\rho}(f_\lambda)(\tilde{\psi}_i) \neq 0$ it must be an eigenvector of $Q$, with eigenvalue, say $\kappa_j = \kappa_i + \lambda$. It follows that we have,

$$\Lambda(\sigma) \subset \{\kappa_j - \kappa_i |\ i, j \in I\}.$$

To prove that the two sets are equal we need some extra conditions on the nature of $\mathbf{A}(\sigma)$ and $\tilde{\rho}$. If $\{\tilde{\rho}(f_\lambda\}_\lambda$ generate $End_{C(n)}(\tilde{V})$, then the equality must hold, since then $\{\tilde{\rho}(f_\lambda)(\psi(0)\}_\lambda$ must generate $\tilde{V}$ as $C(n)$-module, and therefore contain multiples of all $\psi_j$, so that any $\kappa_l$ must be equal to $\kappa_0+\lambda$ for some $\lambda$.

Notice that if $\hbar$ *goes to 0*, meaning that $[Q,\tilde{\rho}(a)]=0$, for all $a\in\mathbf{A}(\sigma)$, then all $a\in\mathbf{A}(\sigma)$ must commute with $Q$, and so act diagonally on the spectrum of $Q$.

Notice also that if, at a point $v\in\gamma$, $\hbar(v)\neq 0$ as an element of $k=\mathbf{R}$, then it is clearly reasonable to redefine $\delta$ and $Q(v)$ by dividing both with $\hbar(v)$. Then the *energy differences* of $1/\hbar(v)Q(v)$ will come up as integral values.

Using the theorem (3.3), we find that the evolution operator $U(\tau_0,\tau_1)$ maps any $\psi_i(\tau_0)$ to $\psi_i(\tau_1)=exp[\int_\gamma Q(\tau)d\tau]\ (\psi_i(\tau_0))=exp[\int_\gamma \kappa_i(\tau)d\tau]\ (\psi_i(\tau_0))$. In particular we find,

$$\frac{\partial\psi_i(\tau)}{\partial\tau}=\kappa_i exp(\int_\gamma \kappa_i(\tau)d\tau)(\psi_i(\tau_0))=Q(\psi_i(\tau)),$$

i.e. again the Schrödinger's equation, with $\tau$ as time.

Here our "time-space" is $Simp_n(\mathbf{A}(\sigma))$, and $Simp_1(A)$ is the analogue of the classical configuration space. Given an element $v\in Simp_n(\mathbf{A}(\sigma))$, corresponding to a simple module $V$ of dimension n, there are for every $a\in\mathbf{A}(\sigma)$, a set of $n$ possible values, namely its eigenvalues, as operator on $V$. Since $V$ is simple, the structure map,

$$\rho_V:\mathbf{A}(\sigma)\longrightarrow End_k(V)$$

is supposed surjective, and so in general (and, for order 2 dynamical systems always ) the operators $\tilde{\rho}(a)$ and $\tilde{\rho}(da), a\in A$, cannot all commute. In fact, if $dimV=\infty$, or $dimV$ "approaching" $\infty$, see example (3.7), (3.8), any one $a\in A(\sigma)$ would tend to have a conjugate, i.e. an element $b\in A(\sigma)$, such $[\tilde{\rho}(a),\tilde{\rho}(b)]=\mathbf{1}$. Therefore, if the values $q_i$ of $\tilde{\rho}(a))$ are determined, then the values $p_i$ of $\tilde{\rho}(b)$ will be totally biased, and vice versa, giving us the Heisenberg indeterminacy problem. In general there is no way of fixing a point of $Simp_1(\mathbf{A}(\sigma))$ as *representing* $V$ or finding natural morphisms,

$$Simp_n(A)\longrightarrow Simp_m(\mathbf{A}(\sigma)), m<n.$$

However, as we know, see [La 4] and [J-L-S], there are partially defined *decay* maps,

$$Simp_n(A)^\infty:=\overline{Simp_n(\mathbf{A}(\sigma))}-Simp_n(\mathbf{A}(\sigma))\rightarrow\oplus_{n>m\geq 1}Simp_m(A).$$

In the very special case, where $A=k[x_1,...,x_p]$ is a commutative polynomial algebra, there exists moreover, for every linear form $\int:V\rightarrow k$, and every state $\psi(\tau)\in\tilde{V}|\gamma$ a curve $\Psi(\gamma)\subset Spec(A)\simeq\mathbf{A}^p$ defined in the following way,

$$x_i=\int x_i\psi(\tau)/\int\psi(\tau),\ i=1,..,p.$$

If we are able to pick common eigenfunctions $\{\phi_j\in\tilde{V}\},\ j=1,...,n$ for $x_i,\ i=1,...,p$, generating $\tilde{V}$, with eigenvalues $\kappa_j^i(\tau),\ j=1,...,n,\ i=1,...,p$, and if $\psi(\tau)=\sum_j\lambda_j(\tau)\phi_j$, then picking the linear form defined by, $\int\phi_j=1,\ j=1,...,n$, we find,

$$x_i=\sum_j\lambda_j(\tau)\kappa_j^i(\tau)/\sum_j\lambda_j(\tau),$$

which is a general form of Ehrenfest's theorem.

Suppose now that we have a situation where there is a unique non-trivial positive (as a real function) Planck's constant, $\hbar \in C(n)$. Let $f := f_\hbar \in A(\sigma)$, and assume that there are among the $\{f_\lambda\}_\lambda$ a conjugate, i.e. a $f_\mu$ such that $[\tilde{\rho}(f), \tilde{\rho}(f\mu)] = \mathbf{1}$. This obviously cannot happen unless $dimV = \infty$, but see the examples (3.7) and (3.8) for what happens at the limit when $dimV$ goes to $\infty$.

Then we easily find that $\mu = -\hbar$. Moreover, if $\psi_0$ is an eigenvector for $Q$ with least energy (assumed always positive), $\kappa_0$, then $N := f_{-\hbar} f_\hbar$ is a *quanta-counting* operator, i.e. $N(\psi_i) = i$, when $\kappa_i = \kappa_0 + (i-1)\hbar$, is the $i-th$ energy level. It follows also that $[Q, f_{-\hbar} f_\hbar] = 0$. The algebra generated by $\{f_\hbar, f_{-\hbar}\}$ is a kind of a *Fock representation*, $\mathcal{F}$ on a *Fock space*. Its Lie algebra of derivations turns out to contain a *Virasoro*-like Lie-algebra. We shall return to this in the examples (3.7) and (3.8) at the end of this §, and in §5.

We have seen that starting with a finitely generated $k$-algebra $A$, and a dynamical system $\sigma$, we have created an infinite series of spaces $Simp_n(\mathbf{A}(\sigma)) \subset Simp_1(C(n))$ and a quantum field theory, defined on these spaces, with time being defined by the Dirac derivation.

Each $C(n)$ is commutative and $\tilde{V}$ is a universal bundle on $Simp_1(C(n))$, and the elements of $\mathbf{A}(\sigma)$, the observables, become sections of the bundle of operators, $End_{C(n)}(\tilde{V})$.

Clearly, if $D \subset Simp_1(C(n))$ is a subvariety, say a curve parametrized by some parameter $q$, then the universal family induces a homomorphism of algebras,

$$\tilde{\rho}_D : \mathbf{A}(\sigma) \longrightarrow End_D(\tilde{V}|D).$$

This is in many recent texts referred to as a *quantification* of the commutative algebra $A(\sigma)/[A(\sigma), A(\sigma)]$, or to a quantum deformation, and the parameter $q$ is sometimes confounded with Planck's constant. This is unfortunate, but probably unavoidable!

In quantum theory one attempts to treat the *second quantification* of an oscillator in dimension 1, as a certain representation on the Fock space, i.e. constructing observables acting on Fock space, with the properties one wants. This turns out to be related to the canonical representations of our $PhC := k < x, dx >$ on an $n$-bundle over the algebra, $R := k[[n]_{p,q}]$. Here the $p,q$-*deformed numbers* $[n]_{p,q}$ are introduced as,

$$[n]_{p,q} := q^{n-1} + pq^{n-2} + p^2 q^{n-3} + ... + p^{n-2} q + p^{n-1},$$

and we may as well consider $p, q$ as formal variables, so that $R \subset k[p,q]$. One obtains a homomorphism of $A(\sigma)$ into an endomorphism ring of the form $End_R(V \otimes_k R)$, see e.g. ([Elbaz], Appendice, on the "q-commutators"). Picking representatives for $x$ and $dx$ in $M_n(R)$, it turns out that, instead of getting the classical defining relations for an oscillator, i.e. a Hamiltonian $Q$, such that in $End_R(V \otimes_k R)$,

$$a_+ := x + dx,\ a_- : x - dx,\ [Q, x] = dx,\ [Q, dx] = x,\ [a_-, a_+] = 1$$

one finds,

$$a_+ := x + dx,\ a_- : x - dx,\ [Q, x]_q = dx,\ [Q, dx]_q = x,\ [a_-, a_+]_q = 1$$

where $[a,b]_q := ab - qba$ is the "quantized" commutator. This holds in particular for $p=1$, so for $R = k[q]$, defining a curve $D$ in $Simp_n(Phk[x])$.

However, this $k[q]$-parametrization is not parametrizing an integral curve of $\xi$ in $Simp_n(PhC)$. On the contrary, it is parametrizing a curve which is, transversal to $\xi$, and therefore represent a phenomenon which takes place instantaneously, see the examples (3.7), (3.8).

**Example 3.4.** Let $C$ be a finite type commutative $k$-algebra, say parametrizing an interesting moduli space, and assume it is non-singular, and pick a system of regular coordinates $\{t_1, t_2, ..., t_r\}$ in $C$. Let $L \in PhC$ be a Lagrangian, i.e. an element defining an "action" in $Simp_n(PhC)$, then the Euler-Lagrange differential equations

$$\delta(\frac{\partial L}{\partial dt_l}) - \frac{\partial L}{\partial t_l} = 0, \text{for all } 1 \leq l$$

determines a dynamical system $\sigma$. Suppose it is of order 2. Consider,

$$\mathbf{C}(\sigma) := Ph^{\infty}C/I(n) \simeq PhC/\sigma$$

with Dirac derivation $\delta$, determined by $\sigma$.

We could start with $L := g = \sum_{i,j=1,..,r} g_{i,j} dt_i dt_j \in PhC$, a Riemannian metric defined on $Simp_1(C)$. It is easy to compute the Euler-Lagrange equations. We find the *force laws*,

$$d^2 t_l = -\bar{\Gamma}^l := -\sum \bar{\Gamma}^l_{i,j} dt_i dt_j$$

where $\Gamma^l_{i,j} := 1/2(\bar{\Gamma}^l_{i,j} + \bar{\Gamma}^l_{j,i})$ are the Christoffel symbols for the Levi-Civita connection of $g$. This is clearly a dynamical system $\sigma := \sigma(g)$ of order 2, and so,

$$\mathbf{C}(\sigma(g)) := PhC/\sigma$$

as $k$-algebra. Since in $\mathbf{C}(\sigma)$ the Dirac derivation has the form,

$$\delta = \sum_l (dt_l \frac{\partial}{\partial t_l} + \Gamma^l \frac{\partial}{\partial dt_l}),$$

the corresponding fundamental vector field $\xi$ in $Simp_1(\mathbf{C}(\sigma)) = Spec(k[t_i, u_j], u_j := dt_j)$, is,

$$\xi = \sum_l (u_l \frac{\partial}{\partial t_l} + \Gamma^l \frac{\partial}{\partial u_l})$$

Its integral curves projects onto $Simp_1(C)$ to give the geodesics of the metric $g$.

Now consider the Levi-Civita-connection,

$$\nabla : \theta_H \longrightarrow End_k(\theta_H)$$

expressed in coordinates as,

$$\nabla_{\frac{\partial}{\partial t_i}} (\frac{\partial}{\partial t_j}) = \sum_l \Gamma^l_{i,j} \frac{\partial}{\partial t_l}$$

Classsically we define the curvature of a space, as the obstruction for $\nabla$ to be a Lie-algebra homomorphism. Put $\delta_i = \dfrac{\partial}{\partial t_i}$, then we find,

$$[\nabla_{\delta_i}, \nabla_{\delta_j}](\delta_k) = \sum_l R^l_{i,j,k}\delta_l.$$

This, obviously, is a commutative version of the more precise curvature

$$d^3 t_l = \bar{R}^l(\underline{t}, \underline{dt}).$$

Since in $PhH$ the $dt_j$'s and the $t_i$'s do not commute, we cannot, in general, write

$$d^3 t_l = \sum_{i,j,k} \bar{R}^l_{i,j,k}\delta_i\delta_j\delta_k.$$

Recall that the Ricci tensor is given as,

$$Ric_{i,k} = \sum_j R^j_{i,j,k}$$

and that,

$$R := \sum_{k,i} g^{k,i} Ric_{i,k}$$

is the scalar curvature of $g$, sometimes called $S$.

See [La 5]. for the special case of our "toy" model, the moduli of of an observer-observed in Euclidean 3-space.

**Example 3.5.** Let us go back to the case of $A = k < x_1, x_2 >$, the free non-commutative $k$-algebra on two symbols, and the dimension $n = 2$, see (2.14). We found,

$$C(2) \simeq k[t_1, t_2, t_3, t_4, t_5].$$

locally in a Zariski neighborhood of the origin. The versal family $\tilde{V}$, is defined by the actions of $x_1, x_2$, given by,

$$X_1 := \begin{pmatrix} 0 & 1+t_3 \\ t_5 & t_4 \end{pmatrix},\ X_2 := \begin{pmatrix} t_1 & t_2 \\ 1+t_3 & 0 \end{pmatrix}.$$

The Formanek center, in this case. is cut out by the single equation:

$$f := det[X_1, X_2] = -((1+t_3)^2 - t_2t_5)^2 + (t_1(1+t_3) + t_2t_4)(t_4(1+t_3) + t_1t_5).$$

and

$$\begin{aligned} &trX_1 = t_4,\ trX_2 = t_1, \\ &detX_1 = -t_5 - t_3t_5,\ detX_2 = -t_2 - t_2t_3, \\ &tr(X_1X_2) = (1+t_3)^2 + t_2t_5, \end{aligned}$$

so the *trace ring* of this family is

$$k[t_1, t_2 + t_2t_3, 1 + 2t_3 + t_3^2 + t_2t_5, t_4, t_5 + t_3t_5] =: k[u_1, u_2, u_3, u_4, u_5],$$

with,

$$u_1 = t_1,\ u_2 = (1+t_3)t_2,\ u_3 = (1+t_3)^2 + t_2t_5,\ u_4 = t_4,\ u_5 = (1+t_3)t_5,$$

and $f = -u_3^2 + 4u_2u_5 + u_1u_3u_4 + u_1^2u_5 + u_2u_4^2$. Moreover, $k[\underline{t}]$ is algebraic over $k[\underline{u}]$, with discriminant, $\Delta := 4u_2u_5(u_3^2 - 4u_2u_5) = 4(1+t_3)^2t_2t_5((1+t_3)^2 - t_2t_5)^2$, and there is an étale covering,

$$\mathbf{A}^5 - V(\Delta) \to Simp_2(A) - V(\Delta).$$

Notice that if we put $t_1 = t_4 = 0$, then $f|\Delta$.

With this done, let us consider quantum theory in dimension 1. That is, we start with the $k$-algebra $C = k < x >= k[x]$, and consider the classical Lagrangians,

$$L = 1/2dx^2 + V(x) \in PhC.$$

The corresponding dynamical system $\sigma$ is given by the equation,

$$d^2x = \frac{\partial V}{\partial x},$$

and is of order 2, so the algebra of interest is,

$$\mathbf{C}(\sigma) = PhC = k < x, dx > \simeq k < x_1, x_2 >$$

Let us first compute the *particles* of rank 1 for some cases, and let us start with $V(x) = \pm 1/2\ x^2$, i.e. the classical oscillator. The fundamental equation of the dynamical system is,

$$\delta = [\delta] + [Q, -],$$

where, in dimension 1, the endomorphism $Q$ obviously commutes with the actions of $x_i,\ i = 1,2$. To solve the equation above, we may therefore forget about $Q$, so we are left with the vector fields,

$$[\delta] = \xi.$$

The space, $Simp_1(\mathbf{C}(\sigma_i))$, is just the ordinary phase space, $Simp_1(k[x, dx])$. Put as above, $x_1 := x, x_2 := dx$. We must therefore just solve the equations,

$$\begin{aligned} \delta(x) =& [\delta](x) = [\delta](x_1) \\ \delta^2(x) =& [\delta](dx) = [\delta](x_2) \end{aligned}$$

We can obviously pick,

$$\delta_i = \chi_i = \frac{\partial}{\partial x_i},$$

so we must have

$$[\delta] = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2}.$$

In the case of the potential, $V = 1/2x^2$, we get the equations,

$$\begin{aligned} x_2 =& [\delta](x) = [\delta](x_1) = \xi_1 \\ x_1 =& [\delta](dx) = [\delta](x_2) = \xi_2 \end{aligned}$$

Therefore the fundamental vector field is,

$$\xi = x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$$

i.e. we find hyperbolic motions in the phase space, with general solutions,

$$x = x_1 = r\cosh(t+c),\ dx = x_2 = r\sinh(t+c)$$

which is what we expected.

In the case of the oscillator, $V = -1/2x^2$, we get the equations,

$$\begin{aligned} x_2 =& [\delta](x) = [\delta](x_1) = \xi_1 \\ -x_1 =& [\delta](dx) = [\delta](x_2) = \xi_2 \end{aligned}$$

Therefore the fundamental vector field is,

$$\xi = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

i.e. we find circular motions in the phase space, with general solutions,

$$\gamma : x = x_1 = r\cos(t+c),\ dx = x_2 = -r\sin(t+c),$$

which is also what we expected. Consider now the versal family restricted to $\gamma$,

$$\tilde{\rho}_\gamma : k < x, dx > \rightarrow End_\gamma(\tilde{V}|\gamma),$$

and a state $\psi(t) \in \tilde{V}|\gamma$. If $Q$, restricted to $\gamma$, is multiplication by $\kappa(t)$, then the Schrödinger equation becomes,

$$\frac{\partial}{\partial t}\psi = \kappa(t)\psi$$

so that we should have,

$$\psi(t) = exp(\int_\gamma^t \kappa)!$$

This will turn out much nicer, if we extend the action of $k < x_1, x_2 >$ to $\tilde{V}_{\mathbf{C}}$, and put $Q$, restricted to $\gamma$, equal to multiplication by $\imath\kappa(t)$. Then we find the reasonable result,

$$\psi(t) = exp(\imath \int_\gamma^t \kappa).$$

See again [La 5].

In the repulsive, resp. attractive, Newtonian case, with $V = \pm 1/x$, we find,

$$\begin{aligned} x_2 =& [\delta](x) = [\delta](x_1) = \xi_1 \\ \epsilon(1/x_1^2) =& [\delta](dx) = [\delta](x_2) = \xi_2,\ \epsilon = +, -. \end{aligned}$$

Therefore the fundamental vector field is,

$$\xi = x_2 \frac{\partial}{\partial x_1} + \epsilon(1/x_1^2) \frac{\partial}{\partial x_2}$$

with the classical solution,

$$x = \epsilon(9/2)t^{2/3}.$$

**Example 3.6.** For the oscillator, in rank 2, things are more difficult. As we have computed above, we have found a partial iversal family of $Simp_2(Ph\ k[x])$, given by,

$$x = \begin{pmatrix} 0 & 1+t_3 \\ t_5 & t_4 \end{pmatrix}, dx = \begin{pmatrix} t_1 & t_2 \\ 1+t_3 & 0 \end{pmatrix}$$

and we may chose,

$$\chi_i = \frac{\partial}{\partial t_i},\ i = 1, 2, ..., 5.$$

and, obviously,

$$\delta_i = \frac{\partial}{\partial t_i},\ i = 1, 2, ..., 5.$$

The fundamental vector fields will have the form,

$$[\delta] = \sum \xi_i \delta_i,\ \xi = \sum \xi_i \frac{\partial}{\partial t_i},$$

with 5 unknowns, $\xi_i, i = 1, 2, .., 5$ Moreover,

$$Q = \begin{pmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{pmatrix},$$

with 4 unknowns $q_{i,j}, i = 1, 2, j = 1, 2$. Now, recall that $Q$ can only be determined up to a zentral element from $M_2(C)$, i.e. we have 8 essential unknowns, $\xi_i, i = 1, 2, 3, 4, 5$ and $(q_{1,1} - q_{2,2}), q_{1,2}, q_{2,1}$ in the two matrix equations,

$$\begin{aligned} \delta(x) &= dx = [\delta](x) + [Q, x] \\ \delta^2(x) &= \pm x = [\delta](dx) + [Q, dx] \end{aligned}$$

On the right hand side of the equations we have the terms,

$$\begin{aligned} [\delta](x) &= \sum \xi_i \delta_i (\begin{pmatrix} 0 & 1+t_3 \\ t_5 & t_4 \end{pmatrix}) = \begin{pmatrix} 0 & \xi_3 \\ \xi_5 & \xi_4 \end{pmatrix} \\ [\delta](dx) &= \sum \xi_i \delta_i (\begin{pmatrix} t_1 & t_2 \\ 1+t_3 & 0 \end{pmatrix}) = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & 0 \end{pmatrix} \end{aligned}$$

and the terms,

$$\begin{aligned} [Q, x] &= \begin{pmatrix} t_5 q_{1,2} - (1+t_3) q_{2,1} & (1+t_3) q_{1,1} + t_4 q_{1,2} - (1+t_3) q_{2,2} \\ t_5 q_{2,2} - t_5 q_{1,1} - t_4 q_{2,1} & (1+t_3) q_{2,1} - t_5 q_{1,2} \end{pmatrix} \\ [Q, dx] &= \begin{pmatrix} (1+t_3) q_{1,2} - t_2 q_{2,1} & t_2 q_{1,1} - t_1 q_{1,2} - t_2 q_{2,2} \\ t_1 q_{2,1} + (1+t_3) q_{2,2} - (1+t_3) q_{1,1} & t_2 q_{2,1} - (1+t_3) q_{1,2} \end{pmatrix}, \end{aligned}$$

and on the left side, we have,

$$\begin{aligned} \delta(x) &= dx = \begin{pmatrix} t_1 & t_2 \\ 1+t_3 & 0 \end{pmatrix} \\ \delta^2(x) &= \pm x = \pm \begin{pmatrix} 0 & 1+t_3 \\ t_5 & t_4 \end{pmatrix} \end{aligned}$$

Writing up the matrix for the corresponding linear equation, we find that the determinant of the $8 \times 8$ matrix turns out to be easily computed, it is,

$$D = 2(1+t_3)(t_2t_5 - (1+t_3)^2).$$

Notice that $D$ is a divisor in $\Delta$. Moreover we find,

$$\begin{aligned}
(q_{1,1}-q_{2,2})_+ &= D^{-1}(-(1+t_3)(t_1^2+t_4^2) + (t_2-t_5)(t_2t_5-(1+t_3)^2-t_1t_4)) \\
(q_{1,1}-q_{2,2})_- &= D^{-1}((1+t_3)(t_1^2-t_4^2) + (t_2-t_5)(t_2t_5-(1+t_3)^2-t_1t_4)) \\
&\quad - t_1t_4(1+t_3)) \\
q_{1,2} &= D^{-1}(2(1+t_3)(t_1t_2 \pm (1+t_3)t_4) \\
q_{2,1} &= D^{-1}(2(1+t_3)(t_4t_5 \pm t_1(1+t_3)) \\
\xi_1 &= t_2q_{2,1} - (1+t_3)q_{1,2} \\
\xi_2 &= -t_2(q_{1,1}-q_{2,2}) + t_1q_{1,2} \pm (1+t_3) \\
\xi_3 &= (1+t_3)(q_{1,1}-q_{2,2}) + t_1q_{2,1} \pm t_5 \\
\xi_4 &= t_5q_{1,2} - (1+t_3)q_{2,1} \\
\xi_5 &= t_5(q_{1,1}-q_{2,2}) + t_4q_{2,1} + (1+t_3)
\end{aligned}$$

One may check that for $V(x) = +1/2x^2$, the singularities of $\xi$ are given, uniquely, by,

$$\begin{aligned}
x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
dx &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
Q &= \begin{pmatrix} q_{1,1} & 0 \\ 0 & q_{1,1}+1 \end{pmatrix}.
\end{aligned}$$

corresponding to $t_1 = 0, t_2 = -1, t_3 = 0, t_4 = 0, t_5 = 1$. Notice that in this case (i.e. when $V(x) = 1/2x^2$), we find, in all dimensions, that $f_\hbar := \rho(x+dx)$, is an eigenvector for $[Q,-]$ with $f_{-\hbar} = \rho(x-dx)$ so that $N = f_{-\hbar}f_\hbar$ is the quantum counting operator.

Now, to find the integral curves of the vector field $\xi$, we must solve the obvious system of differential equations, $\dfrac{\partial t_i}{\partial \tau} = \xi_i, i = 1,..,5$. It turns out that we are mostly interested in the solutions for which $t_1 = t_4 = 0$. If they exist they look like,

$$\begin{aligned}
\frac{\partial t_1}{\partial \tau} &= \xi_1 = 0 \\
\frac{\partial t_2}{\partial \tau} &= \xi_2 = -t_2(t_2-t_5)(2+2t_3)^{-1} \pm (1+t_3) \\
\frac{\partial t_3}{\partial \tau} &= \xi_3 = 1/2(t_2-t_5) \pm t_5 \\
\frac{\partial t_4}{\partial \tau} &= \xi_4 = 0 \\
\frac{\partial t_5}{\partial \tau} &= \xi_5 = t_5(t_2-t_5)(2+2t_3)^{-1} + (1+t_3).
\end{aligned}$$

And these equations are obviously consistent with the conditions $t_1 = t_4 = 0$. Moreover, introducing new variables,

$$y_1 = (t_2 - t_5)$$
$$y_2 = (t_2 + t_5)$$
$$y_3 = (2 + 2t_3)$$

so that,

$$t_2 = 1/2\ (y_2 + y_1)$$
$$t_5 = 1/2\ (y_2 - y_1)$$
$$t_3 = 1/2\ y_3 - 1.$$

things look nicer. In fact, in the new coordinates the system of equations above reduces to,

$$y_1\frac{\partial y_1}{\partial \tau} - y_2\frac{\partial y_2}{\partial \tau} + y_3\frac{\partial y_3}{\partial \tau} = 0$$
$$y_1^{-1}\frac{\partial y_1}{\partial \tau} + y_3^{-1}\frac{\partial y_3}{\partial \tau} = 0.$$

The integral curves are therefore intersections of the form,

$$C(c_1, c_2) := V(y_1^2 - y_2^2 + y_3^2 = c_1) \cap V(y_1 y_3 = c_2).$$

Moreover, the stratum at infinity, given by $f = 0$, where $f$ is the Formanek center, is now easily computed, in terms of the new coordinates it is given as,

$$f = -1/16(y_1^2 - y_2^2 + y_3^2)^2$$

This shows that a particle corresponding to an integral curve $\gamma := C(c_1, c_2)$, with $c_1 \neq 0$ *lives eternally*, as it should. Its completion does not intersect the Formanek center, i.e. the stratum at infinity.

An easy calculation gives us,

$$16(y_1^2 - y_2^2 + y_3^2)^2 = -u_3^2 + 4u_2u_5,$$
$$y_1 y_3 = 2(u_2 - u_5)$$

so the integral curve of the harmonic oscillator will be plane conic curves in the part of $Simp_2(Phk[x])$, where $\Delta \neq 0,\ u_1 = u_4 = 0$, given by,

$$u_3^2 - 4u_2u_5 = c_3,\ (u_2 - u_5) = c_4.$$

Here $c_3 \neq 0, c_4$ are constants. Notice also that our special point, the singularity for $\xi$, given by $y_1 = 2, y_2 = 0, y_3 = 2$, sits on the curve defined by $c_1 = 4, c_2 = 4$, corresponding to $c_3 = -1,\ c_4 = 2$.

**Example 3.7.** (i) We shall not treat oscillators in rank$\geq$ 3, in general, but only look at the stationary, or singular points, in all ranks. This is all well known in physics, see [Elbaz], section 16, although in most books in physics, it is treated rather formally, in relation with the *second quantification* and the introduction of Fock-spaces, and their associated representations of the algebra of observables. We shall see that this second quantification is a natural quotient of the algebra of observables $PhC$, in line with the general philosophy of this paper. Although we may work in a very general setting, we shall, as above, restrict our attention to the classical oscillator ($V(x) = 1/2x^2$),in dimension 1.

As above we find,

$$d^2x = x$$

and the Dirac derivation has therefore,

$$a_+ := 1/2(x + dx),\ a_- := 1/2(x - dx)$$

as eigenvectors, with eigenvalues 1 and -1 respectively. Since $PhC = k < x, dx >$ is generated by the elements $a_+ := 1/2(x + dx),\ a_- := 1/2(x - dx)$, it is clear that Planck's constant $\hbar = 1$.

Using the methode above it is easy to see that for any rank $n = dimV$, a singular point $v \in Simp_n(PhC)$ corresponds to a $k < x, dx >$-module $V$, with $x$ and $dx$ acting as endomorphisms $X, dX \in End_k(V)$ for which there exists an endomorphism, the Hamiltonian, $Q \in End_k(V)$ with,

$$dX := \rho(dx) = [Q, \rho(x)] =: [Q, X]$$
$$X = \rho(d^2x) = [Q, \rho(dx)] =: [Q, dX]$$

Let $\psi_0$ be any eigenvector for $Q$ with eigenvalue $\kappa_0$. Since $V$ is simple, the family $\{a_+^m a_-^n(\psi_0)\}$ must generate $V$. Moreover, if $a_+^m a_-^n(\psi_0) \neq 0$, we know it must be an eigenvector for $Q$, with eigenvalue $\kappa_0 + (m - n)$. We can, by adding $\lambda\mathbf{1}$ to $Q$, assume that there is a basis for $V$ of eigenvectors for $Q$, with eigenvalues of this form. This means that $Q$ can be assumed to have the form,

$$Q = \begin{pmatrix} \kappa_0 & 0 & 0 & 0 & ... & 0 \\ 0 & \kappa_0 + \lambda_1 & 0 & 0 & ... & 0 \\ 0 & 0 & \kappa_0 + \lambda_2 & 0 & ... & 0 \\ . & . & . & . & ... & 0 \\ 0 & 0 & 0 & ... & 0 & \kappa_0 + \lambda_{n-1} \end{pmatrix},$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{n-1}$ are all integers. Moreover, since $V$ is simple, and $[Q, a_+] = a_+$, $[Q, a_-] = -a_-$, an easy computation shows that,

$$a_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & ... & 0 \\ a_{2,1} & 0 & 0 & 0 & ... & 0 \\ 0 & a_{3,2} & 0 & 0 & ... & 0 \\ . & . & . & . & ... & 0 \\ 0 & 0 & 0 & ... & a_{n,n-1} & 0 \end{pmatrix},$$

$$a_- = \begin{pmatrix} 0 & a_{1,2} & 0 & 0 & ... & 0 \\ 0 & 0 & a_{2,3} & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & ... & 0 \\ . & . & . & . & ... & a_{n-1,n} \\ 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix},$$

where all $a_{i,i-1}, a_{i,i+1} \neq 0$. We also find,

$$[a_+, a_-]$$
$$= \begin{pmatrix} -a_{1,2}a_{2,1} & 0 & 0 & 0 & ... & 0 \\ 0 & a_{2,1}a_{1,2} - a_{2,3}a_{3,2} & 0 & 0 & ... & 0 \\ 0 & 0 & a_{3,2}a_{2,3} - a_{3,4}a_{4,3} & 0 & ... & 0 \\ . & . & . & . & ... & 0 \\ 0 & 0 & 0 & ... & 0 & a_{n,n-1}a_{n-1,n} \end{pmatrix}$$

obviously with vanishing trace.

Now to have the classical formulas, see ([Elbaz], p.377-380), we just have to impose the condition that $a_+$ and $a_-$ be *conjugate* operators, i.e. that

$$[a_+, a_-] = \begin{pmatrix} -1 & 0 & 0 & ... & 0 \\ 0 & -1 & 0 & ... & 0 \\ 0 & 0 & -1 & ... & 0 \\ . & . & . & .... & 0 \\ 0 & 0 & 0 & ... & (n-1) \end{pmatrix}.$$

Then we find,

$$X = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & ... & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & ... & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & ... & 0 \\ . & . & . & . & ... & \sqrt{(n-1)} \\ 0 & 0 & 0 & ... & \sqrt{(n-1)} & 0 \end{pmatrix}$$

$$dX = \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & ... & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & ... & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & ... & 0 \\ . & . & . & . & ... & -\sqrt{(n-1)} \\ 0 & 0 & 0 & ... & \sqrt{(n-1)} & 0 \end{pmatrix}$$

with associated Hamiltonian,

$$Q = \begin{pmatrix} 1/2 & 0 & 0 & 0 & ... & 0 \\ 0 & 3/2 & 0 & 0 & ... & 0 \\ 0 & 0 & 5/2 & 0 & ... & 0 \\ . & . & . & . & ... & 0 \\ 0 & 0 & 0 & ... & 0 & (2n-1)/2 \end{pmatrix}.$$

Clearly, we cannot impose, $[a_-, a_+] = \mathbf{1}$, in finite rank. is not the classical representation on the Fock space. If, however, we let $n = dimV$ tend to $\infty$, then we find exactely the classical treatment of the oscillator in the "second quantification", see the reference above. In particular it follows that $[a_-, a_+]1$ is the only relation between the operators $a_-$ and $a_+]$ in this classical limit representation.

We might try to find functions, or formal power series, $[n] \in k[[\tau]]$ such that the representation,

$$x(n) = \begin{pmatrix} 0 & \sqrt{[1]} & 0 & 0 & ... & 0 \\ \sqrt{[1]} & 0 & \sqrt{[2]} & 0 & ... & 0 \\ 0 & \sqrt{[2]} & 0 & \sqrt{[3]} & ... & 0 \\ . & . & . & . & ... & \sqrt{[(n-1)]} \\ 0 & 0 & 0 & ... & \sqrt{[(n-1)]} & 0 \end{pmatrix}$$

$$dx(n)=\begin{pmatrix} 0 & -\sqrt{[1]} & 0 & 0 & ... & 0\\ \sqrt{[1]} & 0 & -\sqrt{[2]} & 0 & ... & 0\\ 0 & \sqrt{[2]} & 0 & -\sqrt{[3]} & ... & 0\\ . & . & . & . & ... & -\sqrt{[(n-1)]}\\ 0 & 0 & 0 & ... & \sqrt{[(n-1)]} & 0 \end{pmatrix}$$

with associated Hamiltonian,

$$Q=\begin{pmatrix} 1/2+[0] & 0 & 0 & 0 & ... & 0\\ 0 & 1/2+[1] & 0 & 0 & ... & 0\\ 0 & 0 & 1/2+[2] & 0 & ... & 0\\ . & . & . & . & ... & 0\\ 0 & 0 & 0 & ... & 0 & 1/2+[n-1] \end{pmatrix}$$

satisfiy the fundamental dynamical equation,

$$\delta = [\delta] + [Q, -].$$

We may, of course choose $[\delta] := \dfrac{\partial}{\partial\tau}$ as the generator of the vector fields on the $\tau$-line. We find the following system of differential equations,

$$\frac{\partial}{\partial\tau} f_n + (f_n^2 - f_{n-1}^2) f_n = f_n$$
$$\frac{\partial}{\partial\tau} f_n + (-f_n^2 - f + n - 1^2) f_n = -f_n$$

where $f_n := \sqrt{[n]}$, and with boundary conditions,

$$f_n(0)^2 = n.$$

These equations immediately lead to constant $f_n' s$, and therefore proves that the curve in $Simp_n(PhC)$ defined by the family $\{x(n), dx(n)\}$ is transversal to the fundamental vector field $\xi$. The introduction of the (p,q) commutators, and their treatment in physics, makes it possible to treat the fermions and the bosons in a common structure. Letting the parameter $q$ in the above family slide from 1 to -1, the $q$-commutator $[-,-]_q$ changes from the ordinary Lie product to the Jordan product. The computation above shows that this change takes place transversal to time, i.e. instantanously!

(ii) For the harmonic oscillator in dimension $n \geq 2$ we have $A = k[x_1, x_2]$, and, $PhA = k < x_1, x_2, dx_1, dx_2 > /([x_1, x_2], [x_1, dx_2] - [x_2, dx_1])$, and,

$$\mathbf{A}(\sigma) = k < x_1, x_2, dx_1, dx_2 > /([x_1, x_2], [x_1, dx_2] - [x_2, dx_1], [dx_1, dx_2]).$$

Moreover, in rank 2 we find a simple representation of $\mathbf{A}(\sigma)$, given by,

$$X_1 = \begin{pmatrix} 1 & 0\\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0\\ 0 & 1 \end{pmatrix}$$
$$dX_1 = \begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix}, dX_2 = \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix}$$

with,

$$[X_1, dX_1] = [X_2, dX_2] = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The problem of integrating the differential equations above, i.e. finding algebraic geometric formulas for the integral curves of $\xi$, is a classical problem, and we may use a technique already well known to Hamilton and Jacobi. In fact, assuming that $A = k[t_1, ..., _n]$, and that $\sigma$ is determined by the following *force-laws*,

$$d^2 t_i = \Gamma^i(t_1, ..., t_n, dt_1, ..., dt_n)$$

we have that,

$$\mathbf{A}(\sigma) = PhA/\sigma, \ \delta = \sum_{i=1}^{n} (dt_i \frac{\partial}{\partial t_i} + \Gamma^i \frac{\partial}{\partial dt_i}).$$

We may try to solve the equation,

$$\delta\theta = 0$$

in the ring $\mathbf{A}(\sigma)$. Obviously the set of solutions form a sub-ring of $\mathbf{A}(\sigma)$, the ring of invariants, and we have the following easy result,

**Proposition 3.8.** *Let $\Theta = ker\delta$, be the subring of invariants in $\mathbf{A}(\sigma)$.*

*(i) Suppose $\rho : \mathbf{A}(\sigma) \to End_k(V)$ is a representation for which the tangent space of $V$, $Ext^1_{\mathbf{A}(\sigma)}(V, V) = 0$, or suppose $V$ corresponds to a point $\underline{t} \in Simp_n(\mathbf{A}(\sigma))$ for which $\xi(\underline{t}) = 0$, then any $\theta \in \Theta$ is constant in $V$, i.e. $[Q, \rho(\theta)] = 0$, so that the eigenvectors of $Q$ are eigenvectors for $\theta$.*

*(ii) Consider for any $n$ the universal family,*

$$\tilde{\rho} : \mathbf{A}(\sigma) \to End_{C(n)}(\tilde{V}).$$

*and let $\theta \in \Theta$, then*

$$trace\tilde{\rho}(\theta) \in C(n)$$

*is constant along any integral curve of $\xi$ in $Simp_n(\mathbf{A}(\sigma))$.*

*Proof.* (i) Suppose $\delta(\theta) = 0$, and consider the dynamical equation,

$$\delta = [\delta] + [Q, -],$$

where we may assume $[\delta] = \xi$. If the tangent space of $V$ is trivial then, obviously $[\delta] = 0$ therefore $\delta(\theta) = 0$ implies $[Q, \rho(\theta)] = 0$.

(ii) If $\delta(\theta) = 0$, we must have, in $End_{C(n)}(\tilde{V})$,

$$0 = trace\xi(\tilde{\rho}(\theta)) + trace[Q, \tilde{\rho}(\theta)] = trace\xi(\tilde{\rho}(\theta)) = \xi(trace\tilde{\rho}(\theta)).$$

□

On the basis of the examples above, in particular (3.6), it is tempting to conjecture that all integral curves of $\xi$ are intersections of hypersurfaces of $Spec(C(n))$, of the form $trace\xi(\tilde{\rho}(\theta)) = 0$. However, this is not true, as we can see by going back to (3.6). Here we have

$$A = k[x],\ \mathbf{A}(\sigma) = PhA = k < x, dx >= k < x, y >,\ y = dx,\ \delta = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

There are only two obvious invariants, $\theta_1 = x^2 - y^2$ and $\theta_2 = xy - yx$. Moreover the universal family on $C(2) = k[t_1, .., t_5]$, is given by,

$$\tilde{\rho}(x) = \begin{pmatrix} 0 & 1+t_3 \\ t_5 & t_4 \end{pmatrix}, \tilde{\rho}(y) = \begin{pmatrix} t_1 & t_2 \\ 1+t_3 & 0 \end{pmatrix}.$$

We find,

$$\begin{aligned} trace(\tilde{\rho}(\theta_1)) =& - u_1 - 2u_2 + u_4 + 2u_5 \\ det(\tilde{\rho}(\theta_1)) =&(u_5 - u_2 - u_1^2)(u_5 - u_2 + u_4^2) - u_4^2u_5 + u_1u_3u_4 - u_1^2u_2. \\ det(\tilde{\rho}(\theta_2)) =& - u_3^2 + 4u_2u_5 + u_1u_3u_4 + u_1^2u_5 + u_2u_4^2 \\ det(\tilde{\rho}(\theta_1)\tilde{\rho}(\theta_2)) =&0. \end{aligned}$$

If we put $t_1 = t_4 = 0$, we find the result of (3.6), namely $trace(\tilde{\rho}(\theta_1)) = y_1y_3 = 2(u_5 - u_2)$, $det(\tilde{\rho}(\theta_1)) = 1/4(y_1^2y_3^2) = (u_5 - u_2)^2$, $det(\tilde{\rho}(\theta_2)) = -1/16(y_1^2 - y_2^2 + y_3^2)^2 = -u_3^2 + 4u_2u_5$. However the fact that $det(\tilde{\rho}(\theta_1)\tilde{\rho}(\theta_2)) = 0$ indicates that there are non-algebraic integral curves sitting on an algebraic surface of $\mathbf{A}^5$. Notice also that the "non-commutative invariant" $\theta_2$ is essential in the integration of $\xi$ in this case. Notice also that in the case $A = k[x_1, x_2, x_3]$, if the Lagrangian $L = 1/2(dx_1^2 + 1/2(dx_2^2 + 1/2(dx_3^2) + U$, has a potential U, such that $\frac{\partial U}{\partial x}_i x_j = \frac{\partial U}{\partial x}_j x_i$, i.e. concerns a *central force* then the kinetic moments $L_{i,j} := x_idx_j - x_jdx_i$, are constants, i.e. $\delta(L_{i,j}) = 0$, which of course have the classical consequences one knows. Combining this with the representations discussed in the Example(1.1), (iii), we find interesting results, see a forthcoming paper on this subject.

**Example 3.9.** Now, let us consider the infinite rank case. In particular we may consider the representation given in the above example, when $n = dimV$ "becomes" $\infty$. Notice that this is given as the limit case of the singular simple representation of the classical oscillator in dimension $n$, with an obvious conjugation condition imposed. For $k = \mathbf{R}$, we have a real Planck's constant which we obviously may assume equal to $\hbar = 1$.

Moreover, we now have, $([a_+, a_-]) = 1$, and we have a representation of $PhC$ onto the algebra $\mathcal{F}$, generated by $\{a_+, a_-\}$. Notice that in each finite rank, this algebra generate the whole $End_k(V)$. The commutation relations is given by a classical formula,

$$\begin{aligned} a_-^m a_+^n =& a_+^n a_-^m + mn\, a_+^{n-1}a_-^{m-1} + 1/2!m(m-1)n(n-1)\, a_+^{n-2}a_-^{m-2} \\ &+ 1/3!m(m-1)(m-2)n(n-1)(n-2)\, a_+^{n-3}a_-^{m-3} + ... \end{aligned}$$

and the Lie algebra $\mathfrak{f}$, of derivations of $\mathcal{F}$ are easily seen to be generated by the derivations $\{\delta_{p,q}\}_{p,q}$, defined as,

$$\delta_{p,q}(a_+) = a_+^p a_-^q,\ \delta_{p,q}(a_-) = -p/(q+1)a_+^{p-1}a_-^{q+1}.$$

If we put, for $m, n \geq 0$

$$\chi_{m,n} := \delta_{m+1,n}, \;\; \chi_m := \chi_{m,0}$$

then we find the Witt-algebra, with the classical relations,

$$[\chi_m, \chi_n] = (n-m)\chi_{m+n}.$$

Moreover we find,

$$[\chi_0, \chi_{m,n}] = (m-n)\chi_{m,n} =: deg(\chi_{m,n})\chi_{m,n}.$$

Clearly the Lie algebra $Der_k(\mathcal{F})$ has an ascending filtration with respect to the degree, $deg$, defined above, and it is easy to see that the corresponding graded Lie algebra $\mathfrak{g} := gr(Der_k(\mathcal{F}))$ has the following products,

$$[\chi_{p,q}, \chi_{r,s}] = (r - p + (s+1)^{-1}(r+1)q - (q+1)^{-1}(p+1)s)\chi_{p+r,q+s}.$$

In particular the degree zero component of $\mathfrak{g}$ is Abelian.

**Example 3.10.** Finally let $C := \mathbf{R}[x]$, and let $\mathcal{C} := C \otimes_{\mathbf{R}} \mathbf{C}$, and consider some representation on $V = \mathcal{C}$ of $PhC = \mathbf{R} < x, dx >$. Clearly,

$$Ext^1_C(V, V) = 0,$$

but, in general,

$$Ext^1_{Ph\mathbf{C}}(V, V)$$

is infinite dimensional.

(i) Consider the free particle, i.e. the dynamical system, $\sigma$ given by,

$$L = 1/2 \; dx^2, \;\; \sigma : \delta^2 x = 0,$$

and let $V$ be defined by letting $dx$ act as the identity. Then we find that,

$$[\delta] = 0, Q = \frac{\partial}{\partial x}.$$

This means that $[\delta]$ does not move $V$ in the moduli space of $V$. The Hamiltonian $Q$ defines time, and

$$exp(tQ)(f(x)) = f(x+t).$$

(ii) Consider the same dynamical system, and let $V$ be defined by letting $dx$ act as $\frac{\partial}{\partial x}$. Then we find that,

$$[\delta] = 0, Q = (\frac{\partial}{\partial x})^2.$$

As above, $[\delta]$ does not move $V$ in the moduli space. The Hamiltonian $Q$ defines time, and the time evolution looks like,

$$U(t, \psi) = exp(tQ)(\psi).$$

Introducing the Fourier transformed $\hat{\psi}$, we obtain a time evolution given by,

$$U(t,\hat{\psi}) = exp(tp^2)(\hat{\psi}).$$

(iii) Consider again the harmonic oscillator, and let the representation $V := k[x^{-1}]$ be defined by letting $x$ act as multiplication by $x^{-1}$, and $dx$ act as $\frac{\partial}{\partial x}$. Then we find that,

$$[\delta] = 0, Q = (x\frac{\partial}{\partial x}).$$

As above, $[\delta]$ does not move $V$ in the moduli space. The eigenvectors of the Hamiltonian $Q$ are the monomials $x^{-n},\ n \geq 0$, with eigenvalues $-n$, and the time evolution looks like,

$$U(t, x^{-n}) = exp(-nt)x^{-n}.$$

Notice that,

$$[x, dx] = x^2,$$

as operators on $V$. Notice also that $V$ in this case is not simple. It is, however, a limit of the finite representations, $V_n := k[x^{-1}]/(x^{-1})^n$. The representation $V_2$ is given by the actions,

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$dx = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where we have chosen the basis $\{1, x^{-1}\}$ in $V_2$. It is clearly not simple, but it sits as a point at infinity, $t_1 = t_2 = 1 + t_3 = t_4 = 0,\ t_5 = 1$, for the (almost) versal family,

$$x = \begin{pmatrix} 0 & 1+t_3 \\ t_5 & t_4 \end{pmatrix}$$

$$dx = \begin{pmatrix} t_1 & t_2 \\ 1+t_3 & 0 \end{pmatrix}.$$

For a thorough introduction to the classical dynamical systems, see the Master thesis of Olav Gravir Imenes, [GI].

§**4. Interactions.** Given a dynamical system $\sigma$ of, order 2. A particle, i.e. a component given by its universal family, i.e. the bundle $\tilde{V}$ on $Simp_n(\mathbf{A}(\sigma)) \subset Simp_1(C(n))$, that we know "happened" at time $\underline{t} \in Simp_n(\mathbf{A}(\sigma))$, will after some time $t$ have develloped into the particle sitting at a point on the integral curve $\gamma$ defined by the vectorfield $\xi$ of $\sigma$, with distance $t$ in $Simp_n(\mathbf{A}(\sigma))$. Now, this may well be a point on the border of $Simp_n(\mathbf{A}(\sigma))$, i.e. in $\Gamma_n = Simp(C(n)) - Simp_n(\mathbf{A}(\sigma))$, where it *decays* into an indecomposable, or into a semi-simple, but not simple representation, i.e. into two or more new particles $\{V_i\}$.

What happens now, is taken care of by the following scenario: If the different particles we have produced are not interacting, each of the new particles should be considered as an independent object, with its own dynamical system defined by a Lagrangian $L_i$ and a resulting Dirac derivation $\delta_i$. However, if the particles we have produced are interacting, we have a different situation.

Notice first that for $n = 1$, we have a canonical morphism of schemes,

$$Simp_1(\mathbf{A}(\sigma)) \longrightarrow Simp_1(A)$$

and a canonical vector-field $\xi$ in $Simp_1(\mathbf{A}(\sigma))$, the *phase space*. Given any point of $Simp_1(A)$, the *configuration space*, and any tangent-vector at this point, there is an integral curve of $\xi$ in $Simp_1(\mathbf{A}(\sigma))$, through the corresponding point, projecting down to a fundamental curve in the configuration space.

For $n \geq 2$ the spaces $Simp_n(\mathbf{A}(\sigma))$ and $Simp_n(A)$ are, however, totally different and without any easy relations to each other.

Let now $v_i \in Simp_{n_i}(\mathbf{A}(\sigma)),\ i = 1, 2$ be two points of $Simp(\mathbf{A}(\sigma))$, maybe in different components, and/or ranks. Consider their components, i.e. the universal families in which they are contained,

$$\tilde{\rho}_i : \mathbf{A}(\sigma_i) \longrightarrow End_{C(n_i)}(\tilde{V}_i)$$

The Dirac derivation, $\delta_i$ defines derivations,

$$[\delta_i] : \mathbf{A}(\sigma) \longrightarrow End_{C(n_i)}(\tilde{V}_i)$$

and therefore also the fundamental vector-fields,

$$\partial_i \in Ext^1_{\mathbf{A}(\sigma)\otimes_k C(n_i)}(\tilde{V}_i, \tilde{V}_i), and$$

$$\xi_i \in Der(\tilde{V}_i, \tilde{V}_i).$$

**4.1 Definition.** *Let $B$ be any finitely generated $k$-algebra. We shall say that the components, $C_1 \subseteq Simp_{n_1}(B)$, $C_2 \subseteq Simp_{n_2}(B)$, or the corresponding particles $\tilde{V}_i$, i=1,2, are non-interacting if*

$$Ext^1_B(V_1, V_2) = 0, \forall v_1 \in C_1, \forall v_2 \in C_2.$$

*Otherwise they interact.*

Suppose now that the points $v_1$ and $v_2$, sit in $Simp_{n_1}(\mathbf{A}(\sigma))$ and $Simp_{n_2}(\mathbf{A}(\sigma))$, respectively. "Physically", we shall consider this as an "observation" of two particles, $\tilde{V}_1$ and $\tilde{V}_2$ in the "state" $V_1$ and $V_2$, at some instant. If the two particles are non-interacting, the resulting entity, considered as the the sum $V := V_1 \oplus V_2$, of dimension $n := n_1 + n_2$, as module over $\mathbf{A}(\sigma)$, will stay, "as time passes", a sum of two simples.

If $V_1$ and $V_2$ interact, this may change. To explain what may happen, we have to take into consideration the non-commutativity of the geometry of $PhA$. In particular, we have to consider the non-commutative deformation theory, see [La 2,3,4]. Consider the deformation functor,

$$Def_{\{V_1,V_2\}} : \underline{a}_2 \longrightarrow \underline{Sets},$$

or, if we want to deal with more points, say a finite family $V_i,\ i = 1, 2, ..., r$ the deformation functor,

$$Def_{\{V_i\}} : \underline{a}_2 \longrightarrow \underline{Sets},$$

and its formal moduli,

$$H := \begin{pmatrix} H_{1,1} & ... & H_{1,r} \\ . & ... & . \\ H_{r,1} & ... & H_{r,r} \end{pmatrix}.$$

together with the versal family, i.e. the essentially unique homomorphism of $k$-algebras,

$$\tilde{\rho} : \mathbf{A}(\sigma) \longrightarrow \begin{pmatrix} H_{1,1} \otimes End_k(V_1) & ... & H_{1,r} \otimes Hom_k(V_1, V_r) \\ . & ... & . \\ H_{r,1} \otimes Hom_k(V_r, V_1) & ... & H_{r,r} \otimes End_k(V_r) \end{pmatrix}.$$

We need a way of specifying which interactions we want to consider. This is the purpose of the following, tentative, definition,

**4.2 Definition.** *Let $B$ be any finitely generated $k$-algebra. We shall say that we have specified an interaction mode in $Simp(B)$, if we have given a quiver $\Gamma$, with vertices $\{v_i\}$, $i = 1, ..., r$, and, for any choice of a finite family of simple $B$-modules $\{V_i\}$, i=1,...,r, a homomorphism,*

$$\phi : H(\{V_i\}) \longrightarrow k\Gamma.$$

An interaction mode is a kind of *higher order preparation*, see (1.5). In fact, an interaction mode consists of a rule, telling us, for each family of $r$ points $\{v_i\}, v_i \in Simp_{n_i}(B)$ and each prescribed sequence of $B$-modules $\{V_i\}_{i=1,...,n}$ representing some of these these points, how to *prepare* their interactions. The morphism $\phi$ fixes all relevant *higher order* momenta, i.e. $\phi$ evaluates all the tangents between the modules, and creates a new $B$-module.

In fact, an interaction mode induces a homomorphism,

$$\tilde{\rho}_\alpha : B \longrightarrow \begin{pmatrix} k\Gamma_{(1,1)} \otimes End_k(V_1) & ... & k\Gamma_{(1,r)} \otimes Hom_k(V_1, V_r) \\ & ... & \\ k\Gamma_{(r,1)}\ k \otimes Hom_k(V_r, V_1) & ... & k\Gamma_{(r,r)} \otimes End_k(V_r) \end{pmatrix}.$$

And the last matrix algebra is naturally embedded in,

$$End_k(V) \simeq M_k(n),$$

where $V := \oplus_{i=1,...,r} V_i$. Thus, we have constructed a new $B$-module, which may be decomposable, indecomposable or simple, depending on the interaction mode we choose, and, of course, depending upon the tangent structure of the moduli space $Simp(B)$.

**Example 4.3.** Let $B$ be the free $k$-algebra on two non-commuting symbols, $B = k < x_1, x_2 >$, and see Example (2.14). Let $P_1$ and $P_2$ be two different points in the $(x_1, x_2)$-plane, and let the corresponding simple $B$-modules be $V_1, V_2$. Then, $Ext^1_B(V_1, V_2) = k$. Let $\Gamma$ be the quiver,

$$V_1 \bullet \longrightarrow \bullet\ V_2,$$

then an interaction mode is given by the following elements: First the formal moduli of $\{V_1, V_2\}$,

$$H := \begin{pmatrix} k < u_1, u_2 > & < t_{1,2>} \\ < t_{2,1>} & k < v_1, v_2 > \end{pmatrix},$$

then the $k$-algebra,

$$k\Gamma := \begin{pmatrix} k & k \\ 0 & k \end{pmatrix},$$

and finally a homomorphism,

$$\phi : H \longrightarrow k\Gamma$$

specifying the value of $\phi(t_{1,2}) \in Ext^1_B(V_1, V_2)$. Since $Hom_k(V_i, V_j) = k$, we obtain $V = k^2$, and we may choose a representation of $\phi(t_{1,2})$ as a derivation, $\psi_{1,2} \in Der_k(B, Hom_k(V_1, V_2))$, such that the $B$-module $V = k^2$ is defined by the actions of $x_1, x_2$, given by,

$$X_1 := \begin{pmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{pmatrix},\ X_2 := \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

where $P_1 = (\alpha_1, \beta_1)$ and $P_2 = (\alpha_2, \beta_2)$. $V$ is therefore an indecomposable $B$-module, but not simple.

If we had chosen the following quiver,

$$V_1 \bullet \underset{\epsilon_{1,2}}{\overset{\epsilon_{2,1}}{\rightleftarrows}} \bullet\ V_2,\ \ \epsilon_{i,j}\epsilon_{j,i} = 0,\ i, j = 1, 2,$$

then the resulting $B$-module $V = k^2$ would have been simple, represented by,

$$X_1 := \begin{pmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{pmatrix},\ X_2 := \begin{pmatrix} \beta_1 & 0 \\ 1 & \beta_2 \end{pmatrix}.$$

In general, if $B = \mathbf{A}(\sigma)$, and if we have given a Lagrangian, and therefore a Dirac derivation, any interaction mode producing a simple module $V$, thus a point $v \in Simp(\mathbf{A}(\sigma))$, represents a *creation* of a new particle from the information contained in the interacting constituencies. Moreover, any family of state-vectors $\psi \in V_i$, produces a corresponding state -vector $\psi := \sum_{i=1,...} \psi_i \in V$, and Theorem (3.3) then tells us how the evolution operator acts on this new state-vector. If the created new particle $V$ is not simple, the Dirac derivation $\delta \in Der_k(\mathbf{A}(\sigma))$, will induce a tangent vector $[\delta](V) \in Ext^1_{PhA}(V, V)$ which may or may not be modular, or pro-representable, which means that the *particles* $V_i$, when integrated in this direction, may or may not continue to exist as distinct particles, with a non-trivial endomorphism ring, or, with a Lie algebra of automorphisms, equal to $k^2$. If they do, this situation is analogous to the case which in physics is referred to as the "super-selection rule".

Or, if $[\delta] \in Ext^1_{\mathbf{A}(\sigma)}(V, V)$ does not sit (or stay) in the modular stratum, the particle $V$ looses automorphismes, and may become indecomposible, or simple, instantaneously.

We may thus create new particles, and we have in Example (3.6) discussed the notion of lifetime for a given particle. In particular we found that the harmonic oscillator had ever-lasting particles of dimension 2. If, however the Lagrangian of the system had been different, we may produce particles of finite lifetime.

**Example 4.4.** Let, as in (3.6) $\mathbf{A}(\sigma) = PhA = k < x_1, x_2 >$, and consider the curve of two-dimensional simple $PhA$-modules,

$$X_1 = \begin{pmatrix} 0 & 1+t \\ 0 & t \end{pmatrix}$$
$$X_2 = \begin{pmatrix} t & 0 \\ 1+t & 0 \end{pmatrix}.$$

Computing the Formanek center $f$, see (3.6), we find,

$$f(t) = t^2(1+t)^2 - (1+t)^4.$$

The corresponding particle, born at $t = 0$, decays after $t = -1/2$, and thus has a finite lifetime.

Given a $k$-algebra $A$ we have therefore forged a framework for *standard models*, for all finite families of particles of dynamical systems of order 2. We "just" take all interesting particles in (components of) $Simp_n(\mathbf{A}(\sigma))$, $n \geq 1$, call them $\{V_i\}_{i=1,2,..}$, choose corresponding relative "Lagrangians, or dynamical systems of order 2, determining their Dirac derivations $\delta_i \in Der_k(\mathbf{A}(\sigma))$, and then we compute all interactions, all decays, all possible creations, etc. and then, maybe, cook up *new common* Lagrangians, for the particles the different interaction modes create. We shall return to concrete examples of such "standard models" in a later paper.

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