NONCOMMUTATIVE DEFORMATIONS OF MODULES

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Dedicated to Jan Erik Roos.

Abstract. The classical deformation theory for modules on a k-algebra, where k is a field, is generalized. For any k-algebra, and for any finite family of r modules, we consider a deformation functor defined on the category of Artinian r-pointed (not necessarily commutative) k-algebras, and prove that it has a prorepresenting hull, which can be computed using Massey-type products in the *Ext*-groups. This is first used to construct k-algebras with a preassigned set of simple modules, and to study the moduli space of iterated extensions of modules. In forthcoming papers we shall show that this noncommutative deformation theory is a useful tool in the study of k-algebras, and in establishing a noncommutative algebraic geometry. See http://www.rmi.acnet.ge/hha/

Introduction.

In this paper I shall introduce a noncommutative deformation theory for modules on some k-algebra A, k a field. The basic idea of noncommutative deformation theory is very simple. Let $\underline{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R. The objects are the diagrams of k-algebras,

$$k^r \xrightarrow{\iota} R \xrightarrow{\rho} k^r$$

such that the composition of ι and ρ is the identity. Any such r-pointed k-algebra R is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of R is the bilateral ideal $Rad(R) := ker\rho$, such that $R/Rad(R) \simeq k^r$. The dual k-vectorspace of $Rad(R)/Rad(R)^2$ is called the tangent space of R.

For $r = 1$, there is an obvious inclusion of categories

$$\underline{l} \subseteq \underline{a}_1$$

where $\underline{l}$, as usual, denotes the category of commutative local artinian k-algebras with residue field k.

Fix a not necessarily commutative k-algebra A and consider a right A-module M. The ordinary deformation functor

$$Def_M : \underline{l} \to \underline{Sets}$$

1991 *Mathematics Subject Classification.* 16.40, 16.90, 16.50, 14A22..

Key words and phrases. Modules, deformations of modules, formal moduli, moduli of iterated extensions, Hochschild cohomology, Massey products, swarm of modules, algebra of observables, modular substratum, quivers.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

is then defined. Assuming $Ext^i_A(M,M)$ has finite k-dimension for $i=1,2$, it is well known, see [Sch], or [La 1], that Def_M has a noetherian prorepresenting hull H, *the formal moduli of* M. Moreover, the tangent space of H is isomorphic to $Ext^1_A(M,M)$, and H can be computed in terms of $Ext^i_A(M,M)$, $i=1,2$ and their *matric* Massey products, see [La 1], [La 2].

In the general case, consider a finite family $\mathcal{V}=\{V_i\}_{i=1}^r$ of A-modules. Assume that,

$$dim_k Ext^1_A(V_i,V_j)<\infty.$$

Any such family of A-modules will be called a swarm. Define a deformation functor,

$$Def_{\mathcal{V}}:\underline{a}_r\to\underline{Sets}$$

generalizing the functor Def_M above. Given an object $\rho:R=(R_{i,j})\to k^r$ of $\underline{a}_r$, consider the k-vectorspace and R-left module $(R_{i,j}\otimes_k V_j)$. ρ defines a k-linear and left R-linear map,

$$\rho(R):(R_{i,j}\otimes_k V_j)\to\oplus_{i=1}^r V_i,$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\rho}(R):(R_{i,j}\otimes_k Hom_k(V_i,V_j))\to\oplus_{i=1}^r End_k(V_i).$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras, $\eta_0:A\to\oplus_{i=1}^r End_k(V_i)$. Let

$$Def_{\mathcal{V}}(R)\in\underline{Sets}$$

be the isoclasses of homomorphisms of k-algebras,

$$\eta':A\to(R_{i,j}\otimes_k Hom_k(V_i,V_j))$$

such that,

$$\tilde{\rho}(R)\circ\eta'=\eta_0,$$

where the equivalence relation is defined by inner automorphisms in the k-algebra $(R_{i,j}\otimes_k Hom_k(V_i,V_j))$. One easily proves that $Def_{\mathcal{V}}$ has the same properties as the ordinary deformation functor:

Theorem 2.6. *The functor $Def_{\mathcal{V}}$ has a prorepresentable hull, i.e. an object H of the category of pro-objects $\hat{\underline{a}}_r$ of $\underline{a}_r$, together with a versal family,*

$$\tilde{V}=(H_{i,j}\otimes V_j)\in\varprojlim_{n\geq1}Def_{\mathcal{V}}(H/\mathfrak{m}^n)$$

such that the corresponding morphism of functors on $\underline{a}_r$,

$$\rho:Mor(H,-)\to Def_{\mathcal{V}}$$

is smooth (see §2), and an isomorphism on the tangent level. Moreover, H is uniquely determined by a set of matric Massey products of the form

$$Ext^1(V_i,V_{j_1})\otimes\cdots\otimes Ext^1(V_{j_{n-1}},V_j)\cdots\to Ext^2(V_i,V_j),$$

see [La 2] *and* [Siq] *for an exposition of the Massey product structure in the category of all O_X-modules for X a scheme defined on some field k.*

The right action of A on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j)),$$

and the k-algebra $O(\mathcal{V})$ acts on the family of A-modules $\mathcal{V} = \{V_i\}$, extending the action of A. If $dim_k V_i < \infty$, for all $i = 1, ..., r$, the operation of associating $(O(\mathcal{V}), \mathcal{V})$ to $(A, \mathcal{V})$ turns out to be a closure operation.

Moreover, if A is an object of $\underline{a}_r$, and if $\mathcal{V} = \{k_i\}_{i=1}^r$ is the correspondiong family of simple A-modules, then

$$\eta : A \longrightarrow H(\mathcal{V})$$

is an isomorphism.

In §3 we prove that there exists, in the noncommutative deformation theory, an obvious analogy to the notion of prorepresenting (modular) substratum H_0 of the formal moduli H. The tangent space of H_0 is determined by a family of subspaces

$$Ext^1_0(V_i, V_j) \subseteq Ext^1_A(V_i, V_j), \qquad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathcal{V}$, and by a subspace,

$$T_0(\Delta) \subseteq \prod_i Ext^1_A(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\mathcal{V} = \{V_i\}_{i=1}^r$, see [La 1]. *If $\mathcal{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some artinian k-algebra A, we show that the above notion of almost split sequence coincides with that of Auslander.*

In Remark (3.7) we associate to the family $\mathcal{V} = \{V_i\}$, and to any quotient R of H in $\underline{a}_r$ a quiver with vertices contained in the set $\{V_i\}$. The Auslander-Reiten quiver then turns out to correspond to $R = H_0/Rad(H_0)^2$.

Observe that, in general, the k-algebra H_0 and its corresponding modular family $\tilde{\mathcal{V}}_0$ contains much more information than what may be deduced from the tangent level.

In §4, we first prove,

Theorem (4.3) (A generalized Burnside theorem). *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of simple A-modules, then*

$$A \simeq O(\mathcal{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

Then we consider the general problem of classification of iterated extensions of a family of modules $\mathcal{V} = \{V_i,\}_{i=1}^r$, and the corresponding classification of filtered modules with graded components in the family $\mathcal{V}$. The remainder of the paragraph is concerned with properties of the O-construction, and some examples.

Based on this notion of noncommutative deformations, we have in [La 7] proposed a general definition of an affine noncommutative prescheme, and scheme, generalizing the classical notion of an affine algebraic scheme in the commutative case.

Earlier versions of this paper has appeared in the preprints [La 3-6].

1. Homological preparations.

Exts and Hochschild cohomology. Let k be a (usually algebraically closed) field, and let A be a k-algebra. Denote by A-mod the category of right A-modules and consider the exact forgetful functor

$$\pi : A - mod \longrightarrow k - mod$$

Given two A-modules M and N, we shall always use the identification

$$\sigma^i : Ext^i_A(M,N) \simeq HH^i(A, Hom_k(M,N)) \text{ for } i = 0,1,2,$$

where $Hom_k(M,N)$ is provided with the obvious left and right A-module structures. If L_* and F_* are A-free resolutions of M and N respectively, and if an element

$$\xi \in Ext^1_A(M,N)$$

is represented by the Yoneda cocycle,

$$\hat{\xi} = \{\xi_n\} \in \prod_n Hom_A(L_n, F_{n-1})$$

then $\sigma^1(\xi)$ is gotten as follows. Let σ be a k-linear section of the augmentation morphism

$$\rho : L_0 \longrightarrow M$$

and let for every $a \in A$ and $m \in M$, $\sigma(ma) - \sigma(m)a = d_0(x)$. Put,

$$\sigma^1(\hat{\xi})(a,m) = -\mu(\xi_1(x))$$

where

$$\mu : F_0 \longrightarrow N$$

is the augmentation morphism of F_*. Then,

$$\sigma^1(\hat{\xi}) \in Der_k(A, Hom_k(M,N))$$

and its class in $HH^1(A, Hom_k(M,N))$ equals $\sigma^1(\xi)$.

Recall the spectral sequence associated to a change of rings. If $\pi : A \longrightarrow B$ is a surjective homomorphism of commutative k-algebras, M a B-module and N an A-module, then $Ext^*_A(M,N)$ is the abuttment of the spectral sequence given by,

$$E_2^{p,q} = Ext^p_B(M, Ext^q_A(B,N)).$$

There is an exact sequence,

$$0 \longrightarrow E_2^{1,0} \longrightarrow Ext^1_A(M,N) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0},$$

which, for a B-module N, considered as an A-module, implies the exactness of

$$\begin{aligned} &0 \longrightarrow Ext^1_B(M,N) \longrightarrow Ext^1_A(M,N) \\ &\longrightarrow Hom_B(M, Hom_B(I/I^2, N)) \longrightarrow Ext^2_B(M,N) \end{aligned}$$

where I=ker π. The corresponding exact sequence,

$$0 \to HH^1(B, Hom_k(M,N)) \to HH^1(A, Hom_k(M,N))$$
$$\to Hom_{A\otimes A^{op}}(I, Hom_k(M,N))$$

in the noncommutative case is induced by the sequence

$$0 \to Der_k(B, Hom_k(M,N)) \to Der_k(A, Hom_k(M,N))$$
$$\to Hom_{A\otimes A^{op}}(I, Hom_k(M,N)).$$

Notice that in general we do not know that the last morphism is surjective. This, however, is true if B=A/rad(A), where rad(A) is the radical of A, and A is a finite dimensional, i.e. an artinian, k-algebra. In this case, B is semisimple and the surjectivity above follows from the Wedderburn-Malcev theorem. Notice also that in the commutative case,

$$Hom_{A\otimes A^{op}}(I, Hom_k(M,N)) \simeq Hom_B(I/I^2, Hom_B(M,N))$$

as it must, since for $\phi \in Hom_{A\otimes A^{op}}(I, Hom_k(M,N))$, $a \in A$, and $i \in I$, $ai = ia$, and therefore

$$a\phi(i) = \phi(ai) = \phi(ia) = \phi(i)a, \text{ i.e. } \phi(i) \in Hom_B(M,N).$$

This implies that for $B = A/\mathfrak{p}$, $M = A/\mathfrak{p}$, $N = A/\mathfrak{q}$, where $\mathfrak{p} \subseteq \mathfrak{q}$ are (prime) ideals of A,

$$Ext^1_A(A/\mathfrak{p}, A/\mathfrak{q}) \simeq Hom_A(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{q})$$

and, in particular

$$Ext^1_A(A/\mathfrak{q}, A/\mathfrak{q}) \simeq Hom_A(\mathfrak{q}/\mathfrak{q}^2, A/\mathfrak{q}) = N_{\mathfrak{q}},$$

the normal bundle of $V(\mathfrak{q})$ in Spec(A). If $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \neq \mathfrak{p}$ we find,

$$Ext^1_A(A/\mathfrak{p}, A/\mathfrak{q}) \simeq Ext^1_{A/\mathfrak{q}}(A/\mathfrak{p}, A/\mathfrak{q}).$$

In [La 1], chapter 1., we considered the cohomology of a category $\underline{c}$ with values in a bifunctor, i.e. in a functor defined on the category $mor\underline{c}$ of morphismes of $\underline{c}$. Recall that a morphism between the objects ψ and ψ' is a commutative diagram,

$$\begin{array}{ccc} c_1 & \xrightarrow{\psi} & c_2 \\ \uparrow & & \downarrow \\ c_1' & \xrightarrow{\psi'} & c_2'. \end{array}$$

It is easy to see that this cohomology is an immediate generalization of the projective limit functor and its derivatives, or if one likes it better, the obvious generalization of the Hochschild cohomology of a ring. In fact, for every small category $\underline{c}$ and for every bifunctor,

$$G : \underline{c} \times \underline{c} \longrightarrow Ab$$

contravariant in the first variable, and covariant in the second, one obtains a covariant functor,

$$G : mor\underline{c} \longrightarrow Ab.$$

Consider now the complex,

$$D^*(\underline{c}, G)$$

where,

$$D^p(\underline{c}, G) = \prod_{c_0 \to c_1 \cdots \to c_p} G(c_0, c_p)$$

where the indices are strings of morphisms $\psi_i : c_i \to c_{i+1}$ in $\underline{c}$, and the differential,

$$d^p : D^p(\underline{c}, G) \longrightarrow D^{p+1}(\underline{c}, G)$$

is defined as usual,

$$(d^p\xi)(\psi_1, \ldots, \psi_i, \psi_{i+1}, \ldots, \psi_{p+1}) = \psi_1\xi(\psi_2, \ldots, \psi_{p+1})$$
$$+ \sum_{i=1}^{p} (-1)^i \xi(\psi_1, \ldots, \psi_i \circ \psi_{i+1}, \ldots, \psi_{p+1}) + (-1)^{p+1}\xi(\psi_1, \ldots, \psi_p)\psi_{p+1}.$$

As shown in [La 1], the cohomology of this complex is the higher derivatives of the projective limit functor $\varprojlim^{(*)}_{mor\underline{c}}$ applied to the covariant functor

$$G : mor\underline{c} \longrightarrow Ab.$$

This is the "Hochschild" cohomology of the category $\underline{c}$, denoted

$$H^*(\underline{c}, G) := H^*(D^*(\underline{c}, G)).$$

Example 1.1. Let $\underline{c}$ be a multiplicative subset of a ring R, considered as a category with one object, and let $\tilde{R} : \underline{c} \times \underline{c} \longrightarrow Ab$ be the functor, defined for $\psi, \psi' \in \underline{c}$, by $\tilde{R}(\psi, \psi') = \psi^*\psi'_*$, where ψ^* is left multiplication on R by ψ, and where ψ'_* is right multiplication on R by ψ', then

$$H^0(\underline{c}, \tilde{R}) = \{\phi \in R |\ \phi\psi = \psi\phi \text{ for all } \psi \in \underline{c}\},$$

i.e. the commutant of $\underline{c}$ in R.

Given a k-algebra A, and consider a subcategory $\underline{c}$ of the category of right A-modules. Let, as above $\pi : \underline{c} \to k - mod$ be the forgetful-functor, and consider the bifunctor,

$$Hom_\pi : \underline{c} \times \underline{c} \longrightarrow k - mod$$

defined by

$$Hom_\pi(V_i, V_j) = Hom_k(V_i, V_j).$$

Put,

$$O_0(\underline{c}, \pi) := H^0(\underline{c}, Hom_\pi).$$

It is clear that $O_0(\underline{c}, \pi)$ is a k-algebra, and that there is a canonical homomorphism of k-algebras,

$$\eta_0(\underline{c}, \pi) : A \longrightarrow O_0(\underline{c}, \pi),$$

see §5.

Example 1.2. Let A be a commutative k-algebra of finite type, k algebraically closed, and let $Spec(A)$ be the subcategory of A-mod consisting of the modules $A/\mathfrak{p}$, where $\mathfrak{p}$ runs through Spec(A), the morphisms being only the obvious ones. It is easy to see that the homomorphism

$$\eta_0(Spec(A), \pi) : A \longrightarrow O_0(Spec(A), \pi)$$

identifies $A/rad(A)$ with $O_0(Spec(A), \pi)$. If $rad(A) = 0$ we even find an isomorphism,

$$\eta_0(Simp^*(A), \pi) : A \simeq O_0(Simp^*(A), \pi).$$

Here $Simp^*(A)$ is the subcategory of A-mod where the objects are A and the simple A-modules, $A/\mathfrak{m}$, i.e. the closed points of $Spec(A)$, and where the morphisms are the obvious quotient morphisms $A \to A/\mathfrak{m}$. $\eta_0(Simp^*(A), \pi)$ is, however not, in general, an isomorphism. This is easily seen when A is a local k-algebra. To remedy this situation we shall in [La 7] introduce and study a generalization $O(\underline{c}, \pi)$ of $O_0(\underline{c}, \pi)$ defined in terms of the noncommutative deformation theory, see the next §.

§2. Noncommutative deformations.

The category $\underline{a}_r$, test algebras and liftings of modules.

Let $\underline{a}_r$ be the category of "r-pointed" artinian k-algebras. Recall that an object R of $\underline{a}_r$ is a diagram of morphism of artinian k-algebras,

$$\begin{tikzcd} k^r \arrow[r, "\iota"] \arrow[dr, "="'] & R \arrow[d, "\rho"] \\ & k^r \end{tikzcd}$$

Put, $Rad(R) := ker \rho$, such that,

$$R/Rad(R) \simeq \prod_{j=1}^{r} k_j, \qquad k_j \simeq k.$$

A morphism $\phi : R \to S$ of $\underline{a}_r$ is a morphism of such diagrams inducing the identity on k^r, implying that the induced map,

$$k^r \simeq R/Rad(R) \to S/Rad(S) \simeq k^r$$

is the identity. Pick idempotents $e_i \in k^r \subseteq R$ such that

$$\sum_{i=1}^{r} e_i = 1, \qquad e_i e_j = 0 \text{ } if \text{ } i \neq j.$$

For every (i, j), we shall consider the subspace $R_{ij} := e_i R e_j \subseteq R$, and the pairing

$$R_{ij} \otimes_k R_{jk} \to R_{ik}$$

given in terms of the multiplication in R.

Let

$$R' = (R_{ij})$$

be the matrix algebra, the elements of which are matrices of the form

$$(\alpha_{ij})$$

with $\alpha_{ij} \in R_{ij},\ i,j = 1,\cdots,r$. There is an obvious isomorphism of k-algebras

$$\phi : R \to R'$$

defined by

$$\phi(\alpha) = (e_i \alpha e_j).$$

identifying the sub k-algebra k^r of R with the algebra of diagonal $r \times r$-matrices. Now, for any pair (i,j), $i,j = 1,\ldots,r$, consider the symbol ϵ_{ij}, and let's agree to put all products of such symbols equal to zero. Then we define the (i,j)-test algebra $R(i,j)$ as the matrix algebra

$$R(i,j) = k^r \oplus \begin{array}{c c} & j \\ i & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & k \cdot \epsilon_{ij} & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array} \qquad \text{for } i \neq j$$

$$R(i,i) = \begin{array}{c c} & i \\ i & \begin{pmatrix} k & \vdots & 0 \\ \cdots & k[\epsilon_{ii}] & \cdots \\ 0 & \vdots & k \end{pmatrix} \end{array} \qquad \text{for } i = j$$

Denote by $HH^*(A,-)$ the Hochschild cohomology of the k-algebra A. If W is an A-bimodule denote by $Der_k(A,W)$ the k-vectorspace of derivations of A in W. Thus $\psi \in Der_k(A,W)$ is a linear map from A to W such that $\psi(a_1 \cdot a_2) = a_1\psi(a_2) + \psi(a_1)a_2$.

In particular, any element $w \in W$ determines a derivation $i(w) \in Der_k(A,W)$ defined by $i(w)(a) = aw - wa$. There is an exact sequence

$$0 \to HH^0(A,W) \to W \to Der_k(A,W) \to HH^1(A,W) \to 0$$

If V_i, V_j are right A-modules, then

$$Hom_k(V_i, V_j)$$

is an A-bimodule. In fact if $\phi \in Hom_k(V_i,V_j)$, then $a\phi$ is defined by $(a\phi)(v) = \phi(va)$, and ϕa is defined by $(\phi a)(v) = \phi(v)a$.

Moreover, we know that

$$HH^0(A, Hom_k(V_i,V_j)) = Hom_A(V_i,V_j)$$

$$HH^1(A, Hom_k(V_i,V_j)) = Ext^1_A(V_i,V_j).$$

Fix a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of right A-modules, and consider for every

$$\psi \in Der_k(A, Hom_k(V_i, V_j))$$

the left $R(i,j)$-module and right A-module,

$$V_{ij}(\psi) = \begin{array}{cc} & \begin{array}{ccccc} & & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ \\ \end{array} & \begin{pmatrix} V_1 & & & \vdots & \\ \cdots & V_i & \cdots & \epsilon_{ij}V_j & \cdots \\ & & & & \\ & & & V_j & \\ & & & \vdots & V_r \end{pmatrix} \end{array}$$

defined by

$$\begin{pmatrix} v_1 & & & \\ & v_i & \epsilon_{ij}v_j' & \\ & & v_j & \\ & & & v_r \end{pmatrix} \cdot a = \begin{pmatrix} v_1 a & & & \\ & v_i a & \epsilon_{ij}(\psi(a, v_i) + v_j' a) & \\ & & v_j a & \\ & & & v_r a \end{pmatrix}$$

and the obvious left $R(i,j)$-action. The $R(i,j)$- and the A-action commute, therefore we have got a $R(i,j) \otimes A$-module, such that

$$k^r \otimes_{R(i,j)} V_{ij}(\psi) \simeq \oplus_{i=1}^r V_i \,.$$

$V_{ij}(\psi)$ is called a lifting of $\mathcal{V}$ to $R(i,j)$. It is easy to see that if ψ maps to zero in $HH^1(A, Hom_k(V_i, V_j)) = Ext_A^1(V_i, V_j)$ then the lifting $V_{ij}(\psi)$ is trivial, i.e. isomorphic to the trivial one. Conversely, if $V_{ij}(\psi)$ is trivial, then ψ maps to zero in $Ext_A^1(V_i, V_j)$.

The noncommutative deformation functor.

We are now ready to start the study of noncommutative deformations of the family $\mathcal{V} = \{V_i\}_{i=1}^r$. We shall assume that $\mathcal{V}$ is a swarm, i.e. that for all $i, j = 1, 2, ..., r$,

$$dim_k Ext_A^1(V_i, V_j) < \infty.$$

Given an object $\rho : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the left R-module $(R_{i,j} \otimes_k V_j)$. ρ defines a k-linear and left R-linear map,

$$\rho(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i,$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\rho}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras,

$$\eta_0 : A \to \oplus_{i=1}^r End_k(V_i).$$

Definition 2.1. *The deformation functor*

$$Def_{\mathcal{V}} : \underline{a}_r \to Sets$$

is defined for every $R \in \underline{a}_r$, as the set,

$$Def_{\mathcal{V}}(R) \in \underline{Sets}$$

of isoclasses of homomorphisms of k-algebras,

$$\{\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))\}/\sim$$

such that,

$$\tilde{\rho}(R) \circ \eta' = \eta_0,$$

where the equivalence relation $\sim$ is defined by inner automorphisms in the k-algebra

$$End_R((R_{i,j} \otimes_k V_j)) = (R_{i,j} \otimes_k Hom_k(V_i, V_j)).$$

Any such isoclass $\tilde{\eta}'$ will be called a deformation or a lifting of V to R, and usually denoted V_R.

One easily proves that $Def_{\mathcal{V}}$ has the same properties as the ordinary deformation functor.

Let $\pi : R \to S$ be a morphism of $\underline{a}_r$, such that $Rad(R) \cdot \ker \pi = 0$. Morphisms like this will be called *small.* If $V_R \in Def_V(R)$ it is easy to see that $V_S := S \otimes_R V_R \in Def_V(S)$ and that $\overline{V} = \ker\{V_R \to S \otimes_R V_R\}$ is, as a left R-module, an $R/Rad(R) = k^r$-module. Put $\ker \pi = (K_{ij})$, then $\overline{V} = (\overline{V}_{ij})$ where $\overline{V}_{ij} = K_{ij} \otimes_k V_j$.

Consider now the k-vector spaces

$$E^d_{ij} = Ext^d_A(V_i, V_j)^*$$

i.e. the dual k-vectorspaces of $Ext^d_A(V_i, V_j)$, and consider the k-algebra of matrices,

$$T^d_2 = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix} + (\epsilon_{ij} E^d_{ij})$$

where as above, we assume all products of the ϵ_{ij}'s are equal to zero. Now let for every $i, j = 1, \dots, r$, and $d = 1, 2$,

$$\left\{ t^d_{ij}(\ell) \right\}_{\ell=1}^{e^d_{ij}}$$

be a basis of E^d_{ij}, and let $\{\psi^d_{ij}(\ell)\}_{\ell=1}^{e^d_{ij}}$ be the dual basis. Thus $e^k_{ij} = dim_k E^k_{ij}$. Consider the k-algebra

$$T^d = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix} + (\widetilde{E}^d_{ij})$$

freely generated as matrix algebra by the generators $\left\{t_{ij}^d(\ell)\right\}_{\ell=1}^{e_{ij}^d}$. An element of $\widetilde{E}_{ij}^d$ is then a matrix where the elements are linear combinations of elements of the form:

$$\begin{aligned}\tau_{ij} &= t_{ij_1}^d(l_1) \otimes t_{j_1j_2}^d(l_2) \otimes \cdots \otimes t_{j_{m-1}j_m}^d(l_m) \\ j &= j_m,\ 1 \leq l_s \leq e_{j_{s-1}j_s}^d,\ 1 \leq j_s \leq r,\ m \geq 1 \\ &\text{of } \ E_{ij_1}^d \otimes E_{j_1j_2}^d \otimes \cdots \otimes E_{j_{m-1}j}^d.\end{aligned}$$

Obviously

$$T_2^1 = T^1/Rad(T^1)^2.$$

where $Rad(T^1)$ is the two-sided ideal of T^1 generated by $(\widetilde{E}_{ij}^1)$.

Lemma 2.2. *Let R be an object of $\underline{a}_r$ and suppose that there exists a surjective homomorphism*

$$\phi_2 : T_2^1 \to R/Rad(R)^2,$$

then there exists a surjective homomorphism

$$\phi : T^1 \longrightarrow R$$

which lifts ϕ_2.

Definition 2.3. *For every object R of $\hat{\underline{a}}_r$, put*

$$T_R = (Rad(R)/Rad(R)^2)^*$$

and call it the tangent space of R.

Lemma 2.4. *Let $\phi : R \to S$ be a morphism of $\underline{a}_r$. Assume ϕ induces a surjective homomorphism*

$$\phi^1 : T_R^* \to T_S^*$$

(or an injective homomorphism on the tangent space level). Then ϕ is surjective.

Notice that if we pick any finite dimensional k-vectorspaces F_{ij}, then there is a unique maximal pro-algebra $F = F(F_{ij})$ in $\underline{a}_r$ with tangent space

$$T_F \simeq (F_{ij}^*)$$

F is defined in the same way as T^d, above, with E^d replaced by F.

To prove the existence of a hull for the deformation functor Def_V the basic tool is the obstruction calculus, which in this case is easily established:

Proposition 2.5. *Suppose $R \xrightarrow{\phi} S$ is a surjective small morphism of $\underline{a}_r$, i.e. suppose $ker\phi \cdot Rad(R) = 0$. Put $ker\phi = (I_{ij})$. Consider any $V_S \in Def_V(S)$. Then there exists an obstruction*

$$o(\phi, V_S) \in (I_{ij} \otimes_k Ext_A^2(V_i, V_j))$$

which is zero if and only if there exists a lifting $V_R \in Def_V(R)$ of V_S. The set of isomorphism classes of such liftings is a pseudotorsor under

$$(I_{ij} \otimes_k Ext^1_A(V_i, V_j))\,.$$

Proof. As a k-vectorspace $V_R = (R_{ij} \otimes V_j)$ maps onto $V_S = (S_{ij} \otimes V_j)$. Since the right action of A commutes with the left S-action the action of an element $a \in A$ on V_S is uniquely given in terms of a family of k-linear maps,

$$a_{ij} : V_i \to S_{ij} \otimes V_j.$$

We may of course lift these to k-linear maps

$$\sigma(a)_{ij} : V_i \to R_{ij} \otimes V_j$$

inducing a lift of the action of each element of A on

$$\bigoplus_{j=1}^{r} S_{ij} \otimes V_j$$

to a k-linear action on

$$\bigoplus_{j=1}^{r} R_{ij} \otimes V_,.$$

The obstruction for this to be an A-module structure is, as usual, the Hochschild 2-cocycle

$$\psi^2(a,b) = \sigma(ab) - \sigma(a) \cdot \sigma(b) \in (I_{ij} \otimes_k Hom_k(V_i, V_j))\,.$$

The fact that this is a 2-cocycle follows from

(1) $\sigma(c) \cdot \psi^2(a,b) = c \cdot \psi^2(a,b)$
(2) $\psi^2(a,b) \cdot \sigma(c) = \psi^2(a,b) \cdot c$

and the obvious relation

$$\begin{aligned}
&d\psi^2(a,b,c) = a\psi^2(b,c) - \psi^2(ab,c) + \psi^2(a,bc) - \psi^2(a,b) \cdot c \\
&= \sigma(a)(\sigma(bc) - \sigma(a)\sigma(c)) - (\sigma(abc) - \sigma(ab)\sigma(c)) + (\sigma(abc) - \sigma(a)\sigma(bc)) \\
&- (\sigma(ab) - \sigma(a)\sigma(b))\sigma(c) \equiv 0
\end{aligned}$$

Suppose the class of ψ^2 in $(I_{ij} \otimes_k Ext^2_A(V_i, V_j))$ is zero. This means that $\psi^2 = d\phi$, where $\phi \in Hom_k(A, (I_{ij} \otimes_k Hom_k(V_i, V_j))$, $\psi^2(a,b) = d\phi(a,b) = a\phi(b) - \phi(ab) + \phi(a)b$. Let $\sigma' = \sigma + \phi$ and consider,

$$\sigma'(ab) - \sigma'(a)\sigma'(b) = \sigma(ab) - \sigma(a)\sigma(b) + \phi(ab) - \sigma(a)\phi(b) - \phi(a)\sigma(b) - \phi(a)\phi(b)\,.$$

Since the matrix $\phi(a)\phi(b) = 0$ as $I_{ij} \cdot I_{jk} = 0$, $\forall i, j, k$ and since $\sigma(a)\phi(b) = a\phi(b)$, $\phi(a)\sigma(b) = \phi(a)b$ for the same reason, we find that $\sigma'(ab) - \sigma'(a)\sigma'(b) = 0$, i.e. there is a lifting of the A-module action to $V_R = (R_{ij} \otimes V_j)$.

If we have given one A-module action σ on V_R lifting the action on V_S, then for any other σ' we may consider the difference

$$\sigma' - \sigma : A \to (I_{ij} \otimes_k Hom_k(V_i, V_j))$$

Consider

$$d(\sigma' - \sigma)(a, b) = a(\sigma'(b) - \sigma(b)) - (\sigma'(ab) - \sigma(ab)) + (\sigma'(a) - \sigma(a))b$$

As above we may substitute $\sigma'(a)$ for a and $\sigma(b)$ for b, and the expression becomes zero. Thus $\sigma' - \sigma = \bar{\xi}$ defines a class

$$\xi \in (I_{ij} \otimes_k Ext^1_A(V_i, V_j))\,.$$

If $\xi = 0$, then $\bar{\xi} = d\phi$, $\phi \in (I_{ij} \otimes_k Hom_k(V_i, V_j))$ such that $\sigma'(a) - \sigma(a) = a\phi - \phi a$. Let $\phi = (\phi_{ij})$, then ϕ_{ij} defines an isomorphism

$$\bar{\phi} = id + \phi : \bigoplus_j R_{ij} \otimes V_j \to \bigoplus_j R_{ij} \otimes V_j$$

lifting the identity of $\bigoplus_j S_{ij} \otimes V_j$. Moreover

$$\begin{aligned} \sigma(a)(id + \phi)(v_i) &= \sigma(a)v_i + a\phi(v_i) \\ &= \sigma'(a)(v_i) + \phi(av_i) = (id + \phi)\sigma'(a)(v_i) \end{aligned}$$

since $\phi(\sigma'(a)v_i) = \phi(av_i)$.

Therefore the A-module structures on

$$V_R = (R_{ij} \otimes V_j)$$

defined by σ and σ' are isomorphic. The rest is clear. $\square$

Theorem 2.6. *The functor $Def_{\mathcal{V}}$ has a prorepresentable hull, or a formal moduli of V, $H \in \hat{\underline{a}}_r$, together with a versal family*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathcal{V}}(H/Rad(H)^n)$$

such that the corresponding morphism of functors on $\underline{a}_r$,

$$\rho : Mor(H, -) \to Def_{\mathcal{V}}$$

is smooth and an isomorphism on the tangent level. Moreover, H is uniquely determined by a set of matric Massey products defined on subspaces,

$$D_n \subset \bigoplus_{p=2}^{n} Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{p-1}}, V_j),$$

with values in $Ext^2(V_i, V_j)$.

Proof. Notice first that ρ being an isomorphism at the tangent level means that ρ is an isomorphism for all objects R of $\underline{a}_r$ for which $Rad(R)^2 = 0$.

Word for word we may copy the proof (4.2) of [La 1], and the proof of [La 2]. In particular $H/Rad(H)^2 \simeq T_2^1$ and

$$Mor(H, R(i,j)) \simeq Hom_k(E^1_{ij}, k) \simeq Ext^1_A(V_i, V_j) \simeq Def_{\mathcal{V}}(R(i,j)).$$

Notice that the universal lifting of V to T_2^1 is the $T_2^1 \otimes_k A$-module $\widetilde{V}_2$

$$\begin{pmatrix} V_1 & & 0 \\ & \ddots & \\ 0 & & V_2 \end{pmatrix} + (E^1_{ij} \otimes_k V_j)$$

with the obvious left T_2^1-action and the right A-action defined as,

$$((1 \otimes v_i) \cdot a)_{ii} = 1 \otimes v_i \cdot a + \sum_{\ell} t^1_{i,j}(\ell) \otimes (\psi^1_{i,j}(\ell)(a, v_i))$$

where $v_i \in V_i$, and where $\{t^1_{ij}(\ell)\}_{\ell=1}^{e_{ij}}$ is the chosen basis of E^1_{ij}. Recall that $\{\psi^1_{ij}(\ell)\}_{\ell=1}^{e_{ij}}$, the dual base, consists of elements $\psi^1_{ij}(\ell) \in Ext^1_A(V_i, V_j)$, which may be represented as elements of $Der_k(A, Hom_k(V_i, V_j))$.

To obtain H we kill obstructions for lifting $\widetilde{V}_2$ successively, to $T_3^1 := T^1/Rad(T^1)^3, T_4^1$ etc. just like in the commutative case. The proof of the existence of a prorepresentable hull for Def_V can, of course, also be modeled on the classical proof of M.Schlessinger [Sch]. This has been carried out by Runar Ile, see [Ile]. □

A general structure theorem for artinian k-algebras.

For every deformation $V_R \in Def_{\mathcal{V}}(R)$ there exists, by definition an, up to inner automorphisms, unique homomorphism of k-algebras,

$$\eta_{V_R} : A \to End_R(V_R) = (R_{ij} \otimes Hom_k(V_i, V_j)).$$

Definition 2.7. *Let $\mathcal{V} = \{V_i\}_{i=1}^r$ be any finite swarm of A-modules, and let $H := H(\mathcal{V})$ be the formal moduli for $\mathcal{V}$, and $\tilde{V}$ the versal family. The k-algebra of observables of the family $\mathcal{V}$ is the k-algebra,*

$$O(\mathcal{V}) := End_H(\tilde{V}) = (H_{ij} \otimes Hom_k(V_i, V_j))$$

We would like to describe the kernel and the image of the map,

$$\eta : A \to O(\mathcal{V})$$

To do this we need to consider the matric Massey products of the form,

$$D_n \to Ext^2_A(V_{i_1}, V_{i_n}),$$

the obvious generalizations of the matrix Massey products introduced in [La 2].

Here we shall describe these products using Hochschild cohomology. This is a more convenient way of describing the map η and maybe also an easier way of understanding the nature of the Massey products.

To simplify the notations, put

$$Ext^1_A(V) = (Ext^1_A(V_i, V_j))$$

For $l = 2$, the Massey product above is simply the cup product

$$Ext^1_A(V) \otimes Ext^1_A(V) \to Ext^2_A(V)$$

defined by: Let $(\psi^1_{ij}), (\psi^2_{ij}) \in Ext^1_A(V)$, and express ψ^k_{ij} as 1-Hochschild cocycles, i.e. $\bar{\psi}^1_{ij} \in Der_k(A, Hom_k(V_i, V_j)), \bar{\psi}^2_{ij} \in Der_k(A, Hom_k(V_i, V_j))$. The cup product $(\psi^1_{ij}) \cup (\psi^2_{ij}) \in \ Ext^2_A(V)$, now denoted

$$\langle (\psi^1_{ij}), (\psi^2_{ij}) \rangle \in \ Ext^2_A(V)$$

is defined by the 2-cocycle in the Hochschild complex

$$\langle (\psi^1_{ij}), (\psi^2_{ij}) \rangle_{ik}(a, b) = \sum_j \bar{\psi}^1_{ij}(a) \circ \bar{\psi}^2_{jk}(b) \in Hom_k(V_i, V_k)$$

Suppose $\langle (\psi^1_{ij}), (\psi^2_{ij}) \rangle = 0$, this means that there exists, for each pair (i, k) a 1-cochain ϕ^{12}_{ik} in the Hochschild complex, i.e. a map

$$\phi^{12}_{ik} \in Hom_k(A, Hom_k(V_i, V_k))$$

such that $d\phi^{12}_{ik} = \langle (\psi^1_{ij}), (\psi^2_{ij}) \rangle_{ik}$, i.e. such that for all $a, b \in A$,

$$a\phi^{12}_{ik}(b) - \phi^{12}_{ik}(ab) + \phi^{12}_{ik}(a)b = \sum_j \bar{\psi}^1_{ij}(a) \circ \bar{\psi}^2_{jk}(b)$$

Given classes $\psi^1 = (\psi^1_{ij}), \psi^2 = (\psi^2_{ij}), \psi^3 = (\psi^3_{ij}) \in Ext^1_A(V)$ such that $\langle \psi^1, \psi^2 \rangle = \langle \psi^2, \psi^3 \rangle = 0$ there exists $\phi^{12} = (\phi^{12}_{ik}), \phi^{23} = (\phi^{23}_{ik}) \in Hom_k(A, Hom_k(V_i, V_k))$ such that

$$d\phi^{12} = \langle \psi^1, \psi^2 \rangle, \quad d\phi^{23} = \langle \psi^2, \psi^3 \rangle \,.$$

Then there exists a matrix Massey product

$$\langle \psi^1, \psi^2, \psi^3 \rangle \in \ Ext^2_A(V)$$

defined by the 2-cocycle

$$\langle \psi^1, \psi^2, \psi^3 \rangle_{ik}(a, b) = \sum_j \phi^{12}_{ij}(a)\psi^3_{jk}(b) - \sum_j \psi^1_{ij}(a)\phi^{23}_{jk}(b)$$

in $Hom_k(A \otimes_k A, Hom_k(V_i, V_j))$.

As in [La 2] we may go on and obtain a sequence of *defining systems* $\{D_n\}^\infty_{n=2}$ and Massey products, computing the relations of $H(\mathcal{V})$.

Now if $a \in A$, denote by $\tilde{a}_i \in Hom_k(V_i, V_i)$ its action on V_i, $i = 1, \dots, d$. Let $End_0(V)$ be the diagonal matrix $(End_k(V_i, V_i))$, contained in the matrix $End_k(V) := (End_k(V_i, V_j))$. Put,

$$End(V)a = (\tilde{a}_1, \dots, \tilde{a}_d) \in End_0(V) \subseteq End(V)$$

If $a \in A$ is such that $End(V)a = 0$, this means that a acts trivially on each V_i. Let $\psi \in Ext^1_A(V)$ be represented by 1-cocycles $\psi_{ij} \in Der_k(A, End_k(V_i, V_j))$. If $End(V)a = End(V)b = 0$, we find,

$$\psi_{ij}(ab) = a\psi_{ij}(b) + \psi_{ij}(a)b = 0$$

This shows that $\psi \in Ext^1_A(V)$ defines a unique k-linear map,

$$\psi : \{a \in A \mid End(V)a = 0\} \to End_k(V)$$

vanishing on all squares.

Let $a \in A$, $End(V)a = 0$, and put

$$Ext^1_A(V)a = 0$$

when $\psi(a) = 0$, $\forall \psi \in Ext^1_A(V)$. Consider the sub k-vector space of A

$$K_1 = \{a \in A \mid End(V)a = Ext^1_A(V)a = 0\}$$

Let $\sum \alpha_{ij}\psi^i \otimes \psi^j \in Ext^1_A(V) \otimes Ext^1_A(V)$ such that its Massey (cup-)product is zero, i.e. such that:

$$\sum \alpha_{ij}\langle \psi^i, \psi^j \rangle = 0$$

Then there exists a 1-cochain $\phi \in Hom_k(A, (Hom_k(V_i, V_j)))$ such that

$$d\phi = \sum_{ij} \alpha_{ij}\langle \psi^i, \psi^j \rangle$$

Since $d\phi = 0$ implies that ϕ represents an element of $Ext^1_A(V)$ it is clear that ϕ defines a unique k-linear map

$$\phi : K_1 \to End_k(V)\,.$$

Let us denote by

$$ker\langle Ext^1_A(V), Ext^1_A(V)\rangle$$

the subset of $Ext^1_A(V) \otimes Ext^1_A(V)$ for which the Massey product (i.e. the cup product) is zero. Then we may put

$$ker\langle Ext^1_A(V), Ext^1_A(V)\rangle a = 0$$

if for every $d\phi \in ker\langle Ext^1_A(V), Ext^1_A(V)\rangle$, $\phi(a) = 0$.

Let

$$K_2 = \{a \in A \mid End(V)a = Ext^1_A(V)a = ker\langle Ext^1_A(V), Ext^1_A(V)\rangle a = 0\}$$

Continuing in this way we find a sequence of ideals $\{K_n\}_{n\geq 0}$, where $K_0 = ker\{A \to End(V)\}$ and, in general, $K_n = \{a \in A \mid D_n a = 0\}$.

Theorem 2.8. *Let A be any k-algebra and let $\mathcal{V} = \{V_i\}_{i=1}^r$ be a swarm of A-modules. Then the kernel of the canonical map*

$$\eta : A \to O(\mathcal{V})$$

is determined by the matrix Massey product structure of $Ext^i_A(V)$, $i = 1, 2$. In fact

$$\ker \eta = \bigcap_{n \geq 0} K_n$$

Proof. By definition, the homomorphism of k-algebras

$$\eta : A \to O(\mathcal{V})$$

lifts the k-algebra homomorphism,

$$\eta_0 : A \to \prod_{i=1}^{d} End_A(V_i).$$

Modulo $Rad(H)^2$ η induces the homomorphism,

$$\eta_1 : A \to \prod_{i=1}^{d} End_k(V_i) \oplus (E^1_{ij} \otimes Hom_k(V_i, V_j))$$

with,

$$\eta_1(a)_{ij} = \delta_{ij} \otimes \eta_0(a)_i + \sum_l t_{ij}(l) \otimes \psi^1_{ij}(l)(a_i, -)\,, \qquad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Now, by construction H is the quotient of the formally free k-algebra T^1 generated by the independent variables $\{t_{ij}(l),\, l = 1, \ldots, l_{ij}\}$ as explained above. The relations of T^1 are generated by linear combinations of monomials in these variables of the form,

$$y_{ik} = \sum_{r=1}^{\infty} \sum_{\underline{j}, \underline{l}} \alpha^{l_1, \ldots, l_r}_{i, j_1, \ldots, j_{r-1}, k} t_{ij_1}(l_1) t_{j_1 j_2}(l_2) \cdots t_{j_{r-1}, k}(l_r),$$

corresponding to elements,

$$y_{ik} \in Ext^2_A(V_i, V_k)^*.$$

The coefficients α are expressed in terms of partially, but inductively well defined, matric Massey products,

$$\langle\ \rangle_{\underline{r}} : \quad D_{\underline{r}} \quad \longrightarrow \quad Ext^2_A(V)$$

such that, if the Massey product $\langle \psi^1_{ij_1}(l_1), \ldots, \psi^1_{j_{r-1},k}(l_r) \rangle$ is defined, then

$$y_{ik}(\langle \psi^1_{ij_1}(l_1), \ldots, \psi^1_{j_{r-1},k}(l_r) \rangle) = \alpha^{l_1, \ldots, l_r}_{ij_1, \ldots, j_{r-1}, k}.$$

We therefore obtain a basis for H, as k-vector space, by picking, in a coherent way, a k-basis for

$$coker\{Ext^2_A(V)^* \to D^*_{\underline{r}}\} = (\ker\langle\ \rangle_{\underline{r}})^*$$

Since $K_r = ker\langle\ \rangle_{\underline{r}}$, the conclusion of the Theorem follows. □

Remark 2.9. Let E_{ij} be an extension of V_i by V_j, then as a k-vector space $E_{ij} = V_j \oplus V_i$ and the right action by A is defined for $(v_j, v_i) \in E_{ij}$, $a \in A$ by,

$$(v_j, v_i)a = (v_j a + \psi^1_{ij}(a, v_i), v_i a),$$

where,

$$\psi^1_{ij} \in Der_k(A, Hom_k(V_i, V_j))$$

defines an element,

$$\bar{\psi}^1_{ij} \in Ext^1_A(V_i, V_j)$$

corresponding to E_{ij}. Suppose we consider an extension E_{ijk} of E_{ij} by V_k. Then as a k-vector space $E_{ijk} \simeq V_k \oplus E_{ij} = V_k \oplus V_j \oplus V_i$ and the action by A is defined by

$$(v_k, v_j, v_i)a = (v_k a + \phi(a, (v_j, v_i)), v_j a + \psi^1_{ij}(a, v_i), v_i a).$$

By additivity

$$\phi(a, (v_j, v_i)) = \phi(a, (v_j, 0)) + \phi(a, (0, v_i)).$$

Put

$$\psi^{1,0}_{ij}(a, v_i) = \psi^1_{ij}(a, v_i),\ \psi^{0,1}_{jk}(a, v_j) = \phi(a, (v_j, 0)), \quad \psi^{1,1}_{ik}(a, v_i) = \phi(a, (0, v_i)),$$

then the conditions on the action imply

$$\psi^{0,1}_{jk} \in Der_k(A, Hom_k(V_j, V_k))$$
$$\psi^{1,1}_{jk} \in Hom_k(A, Hom_k(V_i, V_k))$$

and

$$d\psi^{1,1}_{ik} = \psi^{0,1}_{jk} \circ \psi^{1,0}_{ij}.$$

This means that $\bar{\psi}^{0,1}_{jk} \in Ext^1_A(V_j, V_k)$ and that the cup product,

$$\bar{\psi}^{0,1} \cup \bar{\psi}^{1,0} \in Ext^2_A(V_i, V_k)$$

is zero.

Now, consider an extension E_{ijkl} of E_{ijk} by V_l. As before the action of A on E_{ijkl} is given by

$$(v_l, v_k, v_j, v_i) \cdot a$$
$$= (v_l \cdot a + \phi(a, v_k, v_j, v_i), v_k \cdot a + \psi^2_{ik}(a, v_i) + \psi^{0,1}_{jk}(a, v_j), v_j \cdot a + \psi^1_{ij}(a, v_i), v_i \cdot a).$$

Put, as above,

$$\psi^{1,0,0} = \psi^{1,0}$$
$$\psi^{0,1,0} = \psi^{0,1}$$
$$\psi^{0,0,1}_{kl}(a, v_k) = \phi(a, v_k, 0, 0)$$
$$\psi^{0,1,1}_{jl}(a, v_j) = \phi(a, 0, v_j, 0)$$
$$\psi^{1,1,1}_{ik}(a, v_i) = \phi(a, 0, 0, v_i)$$

The conditions on ϕ are expressed by:

$$\begin{aligned} d\psi_{kl}^{0,0,1} &= 0 \\ d\psi_{jl}^{0,1,1} &= \psi_{kl}^{0,0,1} \circ \psi_{jk}^{0,1,0} \\ d\psi_{il}^{1,1,1} &= \psi_{jl}^{0,1,1} \circ \psi_{ij}^{1,0,0} + \psi_{kl}^{0,0,1} \circ \psi_{ik}^{1,1,0} \end{aligned}$$

This means that $\bar{\psi}_{kl}^{0,0,1} \in Ext_A^1(V_k, V_l)$, that the cup product $\bar{\psi}_{kl}^{0,0,1} \cup \bar{\psi}_{jk}^{0,1,0} \in Ext_A^2(V_j, V_l)$ is zero, and that the Massey product

$$\langle \bar{\psi}_{kl}^{0,0,1}, \bar{\psi}_{jk}^{0,1,0}, \bar{\psi}_{ij}^{1,0,0} \rangle \in Ext_A^2(V_i, V_l)$$

is zero.

It is clear how to continue.

Corollary 2.10. *Suppose the k-algebra A is of finite dimension, and let the finite swarm $\mathcal{V} = \{V_i\}_{i=1}^r$ contain all simple representations, then*

$$\eta : A \to O(\mathcal{V})$$

is injective.

Proof. Let $a \in A$, and suppose $\eta(a) = 0$. Since A as a right A-module is an extension of the V'_is we may assume there are exact sequences of right A-modules

$$0 \longrightarrow Q_1 \longrightarrow A \longrightarrow \bigoplus_{i\in I_1} V_i \longrightarrow 0$$

$$0 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow \bigoplus_{i\in I_2} V_i \longrightarrow 0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$0 \longrightarrow Q_N \longrightarrow Q_{N-1} \longrightarrow \bigoplus_{i\in I_N} V_i \longrightarrow 0$$

with $Q_N = \bigoplus_{i\in I_{N+1}} V_i$, $Q_{N+1} = 0$. Since $End(V)a = 0$ it follows from the first exact sequence above that $1 \cdot a = a \in Q_1$. Consider the exact sequence

$$0 \longrightarrow \bigoplus_{i\in I_2} V_i \longrightarrow A/Q_2 \longrightarrow \bigoplus_{i\in I_1} V_i \longrightarrow 0$$

Since $Ext_A^1(V)a = 0$ it follows that $1 \cdot a = a \in Q_2$. In fact, multiplication by a is zero on V_i, $i = 1, \ldots, r$ and on A/Q_2 it is therefore given by the elements in $Ext_A^1(V)$. Continuing in this way, we consider the extensions of extensions,

$$0 \longrightarrow \bigoplus_{i\in I_3} V_i \longrightarrow A/Q_3 \longrightarrow A/Q_2 \longrightarrow 0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$0 \longrightarrow \bigoplus_{i\in I_{N+1}} V_i \longrightarrow A \longrightarrow A/Q_N \longrightarrow 0.$$

Refering to (2.9), we know that the multiplication by $a \in A$ on the right in the middle term is given inductively, by a family of cochains $\psi_{ij}^{\underline{\epsilon}} \in Hom_k(A, Hom_k(V_i, V_j))$, with $\underline{\epsilon} \in \{0,1\}^n$, for $2 \leq n$, such that

$$d\psi_{ik}^{\underline{\epsilon}} = \sum_{\substack{\underline{\epsilon}_1 + \underline{\epsilon}_2 = \underline{\epsilon} \\ j}} \psi_{ij}^{\underline{\epsilon}_1} \circ \psi_{jk}^{\underline{\epsilon}_2} .$$

Now, this means that all these extensions are defined in terms of a series of well defined Massey products each one containing 0. By the proof of Theorem (2.8), we find that for all i, j and all $\underline{\epsilon}$, $\psi_{ij}^{\underline{\epsilon}}(a, -) = 0$.

This means that the action of $a \in A$ must be 0, so $a = 0$. □

The same proof works for the following,

Corollary 2.10 bis. *Suppose the k-algebra A is an iterated extension of the objects in the finite swarm $\mathcal{V} = \{V_i\}_{i=1}^r$. Then*

$$\eta : A \to O(\mathcal{V})$$

is injective.

Corollary 2.11. *Suppose A is an object of $\underline{a}_r$, and let $\mathcal{V} = \{V_i\}_{i=1}^r$ be the family of simple representations, with $V_i \simeq k_i$. Then*

$$A \simeq H$$

Proof. Obviously A is a left A-and a right A-module, flat over A, therefore $A \in Def_V(A)$. Let $R \in \underline{a}_r$ and pick an element $V_R \in Def_{\mathcal{V}}(R)$. Since

$$End(V) = \begin{pmatrix} k & \cdots & k \\ \vdots & & \vdots \\ k & \cdots & k \end{pmatrix},$$

this amounts to a homomorphism of k-algebras $A \to End_R(V_R) = R$, implying that A is versal. But then the unicity of the hull of Def_V gives us an isomorphism:

$$\phi : H \to A$$

□

Remark 2.12 Reconstructing an ordered set Λ and $k[\Lambda]$, from the swarm of simple modules.

Let Λ be an ordered set, see §1, and let $A = k[\Lambda]$, $V = \{k_\lambda\}_{\lambda \in \Lambda}$. Then the Corollary above implies that $H \simeq k[\Lambda]$.

1. By the general theory we know that $A = k[\Lambda]$ is the matrix algebra generated freely by the immediate relations $\lambda_1 \gg \lambda_2$, i.e. those for which $\{\lambda` \in \Lambda | \lambda_1 > \lambda` > \lambda_2\} = \emptyset$, modulo relations of the form

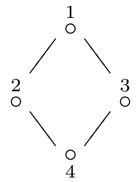
$$\begin{aligned} &(\lambda` > \lambda_2^1)(\lambda_2^1 > \lambda_3^1) \cdots (\lambda_{n_1}^1 > \lambda) \\ &= (\lambda` > \lambda_2^2)(\lambda_2^2 > \lambda_3^2) \cdots (\lambda_{n_2}^2 > \lambda) \end{aligned}$$

They correspond to the first obstructions, given by the n_i term well defined Massey products

$$Ext^1_A(k_{\lambda'}, k_{\lambda^1_2}) \otimes \cdots \otimes Ext^1_A(k_{\lambda^1_{n_1}}, k_\lambda) \to Ext^2_A(k_{\lambda'}, k_\lambda)$$
$$Ext^1_A(k_{\lambda'}, k_{\lambda^2_2}) \otimes \cdots \otimes Ext^1_A(k_{\lambda^2_{n_2}}, k_\lambda) \to Ext^2_A(k_{\lambda'}, k_\lambda)$$

There are as many relations as there are base elements of $Ext^2_A(k_{\lambda'}, k_\lambda)$.

2. Let us check this for the diamond, i.e. for Λ:



One easily computes the Ext's,

$$Ext^1_A(k_{\lambda_i}, k_{\lambda_j}) = \begin{cases} 0 & i = j \\ k & \text{for } \; i = 1,\; j = 2,3 \\ k & \text{for } \; i = 2,3,\; j = 4 \end{cases}$$

$$Ext^2_A(k_{\lambda_i}, k_{\lambda_j}) = \begin{cases} 0 & \text{for } \; (i,j) \neq (1,4) \\ k & \text{for } \; i = 1,\; j = 4 \end{cases}$$

The two cup-products

$$Ext^1_A(k_{\lambda_1}, k_{\lambda_j}) \otimes Ext^1_A(k_{\lambda_j}, k_{\lambda_4}) \to Ext^2_A(k_{\lambda_1}, k_{\lambda_4}) \quad \text{for } j = 2,3,$$

are non-trivial. At the tangent level we have:

$$H_2 = \begin{pmatrix} k & k & k & 0 \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}$$

Therefore H must be a quotient of the matrix ring,

$$T^1 = \begin{pmatrix} k & t_{12} \cdot k & t_{13} \cdot k & (t_{12}t_{24} \cdot k + t_{13}t_{34} \cdot k) \\ 0 & k & 0 & t_{24} \cdot k \\ 0 & 0 & k & t_{34} \cdot k \\ 0 & 0 & 0 & k \end{pmatrix}$$

The kernel of $T^1 \to H$ is given in terms of the cup products above. In fact, since we have $t^*_{13} \cup t^*_{34} = t^*_{12} \cup t^*_{24} = y^*$ where y^* is the generator of $Ext^2_A(k_{\lambda_1}, k_{\lambda_4})$, the kernel of $T^1 \to H$ is simply $t_{13} \otimes t_{34} + t_{12} \otimes t_{24}$ such that

$$H = \begin{pmatrix} k & k & k & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix} \simeq k[\Lambda]$$

as it should.

In general, we may reconstruct Λ from the tangent space T_H and the Massey-products above.

The corresponding problem for finite groups, i.e. reconstructing G from $k[G]$ is called the isomorphism problem. Due to some nice examples of Dade, we know that this is hopeless. In fact there are two non isomorphic finite groups such that their group algebras are isomorphic for all fields.

§3. Noncommutative modular deformations. Let V be any right A-module such that $dim_k Ext^1_A(V,V) < \infty$. Consider the formal moduli $H^A =: H$, the formal versal family $\tilde{V} = H \otimes V$, and the corresponding morphism of functors,

$$\rho : Mor_{\underline{a}_r}(H,-) \to Def_V.$$

We know that ρ is not, in general, injective. However, V is also a right $A \otimes End_A(V)$-module. As such it has a formal moduli $H^{A,End}$, and there is a natural k-algebra homomorphism, $H^A \to H^{A,End}$. Let H^A_0 be the unique maximal common quotient of H^A and $H^{A,End}$. Using the same construction as in [La, Pf], §2, we prove that the composition,

$$\rho_0 : Mor_{\underline{a}_r}(H_0,-) \to Mor_{\underline{a}_r}(H,-) \to Def_V$$

is injective.

At the tangent level, the homomorphisms,

$$H^A \to H^{A,End} \leftarrow H^{End},$$

looks like the canonical homomorphisms,

$$Ext^1_A(V,V) \leftarrow Ext^1_{A \otimes_k End}(V,V) \to Ext^1_{End}(V,V).$$

Representing elements of the Ext-groups as derivations, it is easy to see that the two images are contained in the subspace $Ext^1_A(V,V)^{End}$, respectively $Ext^1_{End}(V,V)^A$. Therefore the tangent space of H_0 must be contained in the subspace of invariants under $End_A(V)$ of the tangent space of H, $Ext^1_A(V,V)^{End}$.

The tangent space of the modular (prorepresenting) substratum, and almost split sequences.

Consider as above a swarm $\mathcal{V} = \{V_i\}_{i=1}^r$ of A-modules, and consider the k^r-algebra

$$End_A(V) = (Hom_A(V_i,V_j)).$$

Suppose from now on that the modules V_i are non-isomorphic, indecomposables, and that for each $i = 1,..,r$. $End_A(V_i)$ is a commutative local ring with maximal ideal $\underline{m}_i$.

Lemma 3.1. *Under the above assumptions, the radical of $End_A(V)$ has the form*

$$rad(V) = \begin{pmatrix} m_1 End_A(V_1) & & & \vdots & \\ \cdots & \cdots & \underline{m}_i End_A(V_i) & Hom_A(V_i,V_j) & \cdots \\ & & & \underline{m}_j End_A(V_j) & \\ & & & \vdots & \underline{m}_r End_A(V_r) \end{pmatrix}$$

Proof. We need only check that $rad(V)$ is an ideal, and this amounts to proving that if $\phi_{ij} \in Hom(V_i,V_j)$ $i \neq j$ and $\phi_{ji} \in Hom(V_j,V_i)$ then

$$\phi_{ji}\phi_{ij} \in \underline{m}_i \subseteq End_A(V_i).$$

Suppose $\phi_{ji}\phi_{ij}$ is not in $\underline{m}_i$, then $\phi_{ji}\phi_{ij}$ is an isomorphism, and we may as well assume that $\phi_{ji}\phi_{ij} = id_{V_i}$. But then $V_j \simeq V_i \oplus \ker \phi_{ji}$ which contradicts the indecomposability of V_j. $\square$

In particular this lemma proves that if A is artinian and all V_i are of finite type, then for some N,

$$rad(V)^N = 0$$

Obviously there is a left and a right action of $End_A(V)$ on

$$T_H = (Ext^1_A(V_i, V_j)).$$

The difference between these actions defines the action of the Lie algebra $End_A(V)$ on T_H. The invariants of T_H under the Lie algebra $rad(V)$, is equal to the invariants under $End_A(V)$, therefore equal to,

$$T_{H_0} := \{\xi \in T_H | \forall \phi \in End_A(V), \quad \phi\xi - \xi\phi = 0\},$$

containing the tangent space of the *modular*, or the *prorepresentable* substratum H_0 of H.

Lemma 3.2. *Let $\xi \in T_{H_0}$, with $\xi = (\xi_{i,j})$, then for all $\phi = (\phi_{k,l}) \in rad(V)$ we have for $i \neq j$, and all l,*

$$\phi_{l,i}\xi_{i,j} = 0$$
$$\xi_{i,j}\phi_{j,l} = 0.$$

Moreover, for all i, j

$$\phi_{i,j}\xi_{j,j} = \xi_{i,i}\phi_{i,j}$$

Proof. Just computation. □

Definition 3.3. *In the above situation, an extension $\xi \in Ext^1_A(V_i, V_j)$ is called a left almost split extension (resp. a right almost split extension), lase (resp. rase) for short, if for all $\phi_{ki} \in r(V)_{ki}$ (resp. $\phi_{jk} \in r(V)_{jk}$)*

$$\phi_{ki}\xi = 0 \quad (resp. \quad \xi\phi_{jk} = 0).$$

An extension ξ which is both a lase and a rase is called an ase, an almost split extension.

This, of course, is nothing but a trivial generalization of the notion of almost split sequence, due to Auslander, see [R].

Denote by $Ext^1_l(V_i, V_j)$ (resp. $Ext^1_r(V_i, V_j)$) the subspace of $Ext^1_A(V_i, V_j)$ formed by the lase's (resp. rase's), and put

$$T^l_H = (Ext^1_l(V_i, V_j)) \subseteq T_H$$
$$T^r_H = (Ext^1_r(V_i, V_j)) \subseteq T_H$$
$$T^a_H = T^l_H \cap T^r_H =: (Ext^1_a(V_i, V_j)) \subseteq T_H.$$

Observe that since the left and the right action of $End(V)$ on T_H commute, $End(V)$ acts at right on T^l_H and at left on T^r_H. Moreover, by the lemma above

$$T^a_H = T^l_H \cap T^r_H \subseteq T_{H_0}.$$

Observe also that if $End_A(V_i) = k \oplus \underline{m}_i$ the diagonal part of T_{H_0} is exactly the tangent space of the deformation functor of the full subcategory of mod_A generated by V, see [La 1].

The structure of the modular substratum, and the existence of almost split sequences for artinian k-algebras.

Assume that A is artinian, and that the V_i's are of finite type. Then T_H is a k-vectorspace of finite dimension, and the radical $rad(V)$ of $End(V)$ acts nilpotently on T_H.

Corollary 3.4. *Given $i \in \{1, \dots, r\}$, assume there exists one $j \in \{1, \dots, r\}$ such that $Ext^1_A(V_i, V_j) \neq 0$. Then there exists a $\tau(i) \in \{1, \dots, r\}$ such that,*

$$Ext^1_r(V_i, V_{\tau(i)}) \neq 0.$$

Proof. This is simply Engels theorem for the right action of $rad(V)$ on T_H. $\square$

Theorem 3.5. *Suppose $\mathcal{V}$ is such that every extension $\xi \in Ext^1_A(V_i, V_j)$ is of the form $0 \to V_j \to E \to V_i \to 0$ with E a direct sum of V_k's. Then, for every $i = 1, \dots, r$, such that there exists a $j = 1, \dots, r$ for which $Ext^1_A(V_i, V_j) \neq 0$, there is a unique ase of the form*

$$0 \to V_{\tau(i)} \to E_i \to V_i \to 0$$

Moreover, if we agree to put $\tau(i) = i$ for those i's for which $Ext^1_A(V_i, V_k) = 0$ for all k, then

$$\tau : \{1, \dots, r\} \to \{1, \dots, r\}$$

is a permutation.

Proof. We already know that there exists a rase of the form

$$\xi_i : 0 \to V_{\tau(i)} \to E_i \to V_i = 0.$$

We shall prove that ξ_i is also a lase. Let $\phi_{ki} \in Hom_A(V_k, V_i)$ for $k \neq i$, or pick an element $\phi_{ii} \in \underline{m}_i \subseteq End(V_i)$, and consider the commutative diagram,

$$\begin{tikzcd}
& 0 \arrow[d] \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[d] & E_i \arrow[r] & V_i \arrow[r] & 0, & \xi_i \\
& E_k = V_k \underset{V_i}{\times} E_i \arrow[ur, "\psi_{ki}"] \arrow[d] \\
& V_k \arrow[uurr, "\phi_{ki}"'] \arrow[d] \\
& 0
\end{tikzcd}$$

Suppose $V_{\tau(i)} \to E_k$ is not split, then

$$0 \to E_k \to E_k \overset{V_{\tau(i)}}{\oplus} E_i \to V_i \to 0$$

is split, since ξ_i is a rase. Let pr.: $E_k \overset{V_{\tau(i)}}{\oplus} E_i \to E_k$ be the splitting. But then the two following diagrams commute:

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[dl, shift left, "\psi_{ik}"] & V_i \arrow[r] \arrow[ddll, shift left, "\phi_{ik}"] & 0 \\
& E_k \arrow[d] \arrow[ur, shift left, "\psi_{ki}"] & & & \\
& V_k \arrow[d] \arrow[uurr, shift left, "\phi_{ki}"] & & & \\
& 0 & & &
\end{tikzcd}$$

Here ψ_{ik} is the composition of $E_i \to E_k \overset{V_{\tau(i)}}{\oplus} E_i$ and the projection $E_k \overset{V_{\tau(i)}}{\oplus} E_i \to E_k$ and ϕ_{ik} the induced map.

This means that $(\phi_{ki}\phi_{ik})\xi_i = \xi_i$ which is impossible since $(\phi_{ki}\phi_{ik})$ acts nilpotently on $Ext^1_A(V_i, V_{\tau(i)})$, and ξ is nonzero. Therefore $V_{\tau(i)} \to E_k$ splits and ξ_i is also a lase, therefore an ase.

The unicity and the permutation property follows immediately from the following: Assume there exist two ase's ξ_i and ξ'_i of the form:

$$\begin{tikzcd}
& & & 0 & & \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[ddrr, shift left, "\phi_i"] & E_i \arrow[r, "\rho_i"] \arrow[dr, shift left, "\psi_i"] & V_i \arrow[r] \arrow[u] & 0 & \xi_i : \\
& & & E'_i \arrow[u, "\rho'_i"] \arrow[ul, shift left, "\psi'_i"] & & \\
& & & V_{\tau(i')} \arrow[u] \arrow[uull, shift left, "\phi'_i"] & & \\
& & & 0 \arrow[u] & &
\end{tikzcd}$$

Then, since ρ'_i is not split, there exist liftings ψ_i, ψ'_i inducing morphisms ϕ_i, ϕ'_i. But then $(\phi_i\phi'_i)\xi_i = \xi_i$ which means that ξ_i is zero. Therefore an ase is unique and in particular, $\tau(i) = \tau(i')$. Dually we prove that $\tau(i) = \tau(i')$ implies $i = i'$, so that τ is a permutation.

We see that T^a_V looks like:

$$\begin{pmatrix} Ext^1_a(V_i, V_j) \end{pmatrix}$$

where

$$Ext^1_a(V_i,V_j) = \begin{cases} 0 & \text{if } j \neq \tau(i) \\ k & \text{if } j = \tau(i) \text{ and some } Ext^1_a(V_i,V_j) \neq 0 \end{cases}$$

□

Corollary 3.6. *With the assumptions of the theorem above, we find that*

$$T_{H_0} = \{(\alpha_{ij}) | \begin{cases} \alpha_{ij} \in k, \ \alpha_{ij} = 0 \ if \ j \neq \tau(i) \\ \alpha_{ii} \in Ext^1_A(V_i,V_i) \ \forall \phi_{ji} \in End_A(V_i), \ \phi_{ij}\alpha_{ii} = \alpha_{jj}\phi_{ji}, \ if \ i{=}j \end{cases} \}$$

Remark 3.7. Consider again a not necessarily finite swarm $\mathcal{V} = \{V_i\}_{i=1}^{\aleph}$ of noetherian A-modules. The k^r-algebra

$$End_A(V)_r := (Hom_A(V_i,V_j)), \ i,j \leq r$$

acts on

$$(Ext^1_A(V_i,V_j)), \ i,j \leq r,$$

in the way described above. Suppose that the modules V_i are non-isomorphic, indecomposables, and that for each i, $End_A(V_i)$ is a local ring with maximal ideal $\underline{m}_i$. Suppose moreover that any iterated extension is a direct sum of such V_i's. This is obviously the case when $\mathcal{V} = \{V_i\}_{i=1}^{\aleph}$ is the family of all indecomposible A-modules, but holds in many other interesting cases, see [R].

Let $H := H(\mathcal{V})$ and $\tilde{V}$ be the prorepresentable hull and the formal versal family, as defined in §2. For every quotient R of H in $\underline{a}_{\aleph}$, such that

$$dim_k Rad(R)/Rad(R)^2 < \infty$$

we consider the image $\tilde{V}(R) \in Def_{\mathcal{V}}(R)$ of $\tilde{V}$. Denote by $L_i(R)$ the i^{th}-line of $\tilde{V}(R)$. $L_i(R)$ is an A-module and a finite iterated extension of the V_j, therefore a finite sum of indecomposables $\{L_i(R,p)\}$, from our family. Obviously there is a canonical surjection,

$$L_i(R) \to V_i,$$

and a homomorphism of k-algebras,

$$\iota : H \to End_A(\tilde{V}(R)),$$

defined by left multiplication. Any element,

$$r_{i,j} \in R_{i,j} \subset R$$

defines a homomorphism of right A-modules,

$$r_{i,j*} : L_j(R) \to L_i(R).$$

In particular, if $r_{i,j}$ is in the socle of R, this morphism induces a homomorphism of right A-modules,

$$r_{i,j*} : V_j \to L_i(R).$$

Using this, we may consider different quiver-structures on the set of indecomposible modules, $\{V_i\}_{i=1}^{\aleph}$. The Auslander-Reiten quiver, see [R], is obtained by picking $R = H_0/Rad(H_0)^2$, a basis $\{h_{i,j}\}$ of $Rad(H_0)/Rad(H_0)^2$, the dual tangent space of H_0 and letting the arrows arriving at an indecomposible V_i be the compositions,

$$L_i(R,p) \to L_i(R) \to V_i,$$

and the arrows leaving an indecomposible V_j be the compositions,

$$V_j \stackrel{h_{i,j}}{\to} L_i(R) \to L_i(R,p).$$

For an arbitrary quotient R of H, we may construct another quiver containing more information than the Auslander-Reiten quiver. Consider representatives

$$\{r_{i,j}\} \in Rad(R)$$

of a basis of the dual tangent space $Rad(R)/Rad(R)^2$ of R, and let the arrows of the quiver be the compositions

$$L_j(R,q) \to L_j(R) \stackrel{r_{i,j*}}{\to} L_i(R) \to L_j(R,p).$$

There is a ring homomorphism,

$$R \to End_A(\tilde{V}) = (Hom_A(L_j,L_i)) = (Hom_A(V_q,V_p)^{n_{p,q}}).$$

If this homomorphism is surjective, or an isomorphism, we find that the arrows of the quiver generate, in an obvious way, all morphisms of the full sub-category of A-modules defined by the family of indecomposibles $\mathcal{V} = \{V_i\}$. In both case, it is easy to see that the relations in the quiver correspond to non-trivial cup and Massey products of $Ext_a^*(V_i,V_j)$. When A is artinian, and the family $\{V_i\}$ generates the category of A-modules, it turns out that $\tilde{V}$ is a projective generator. H defines a quiver, with vertices corresponding to the indecomposible projectives, and

$$H = End_A(\tilde{V})$$

is Morita equivalent to A. Moreover, H is determined by the quiver (with relations). In particular, if $\{V_i\}$ is the family of simple A-modules, we shall see in the next paragraph that,

$$A \to (H_{i,j} \otimes Hom_k(V_i,V_j))$$

is an isomorphism, and that

$$L_i = \oplus_{j=1,2,\ldots} H_{i,j} \otimes V_j$$

is a projective A-module, for $i = 1, 2, ..r$. Since therefore

$$H \simeq End_A(\tilde{V}) = (Hom_A(L_j,L_i))$$

is Morita-equivalent to A, the quiver of (projective) summands of $\tilde{V}$ determines the Morita-equivalence class of A.

We shall end this paragraph by proving the following easy result, see [K] for the notions of Frobenius extension and Frobenius bi-module.

Proposition 3.8. *Suppose the following conditions hold:*

(i) The family $\mathcal{V} = \{V_i\}$ of right A-modules are either finite dimensional as k-vector spaces, or such that,

$$Ext^p_A(V_i, V_j) = Ext^p_A(V_j^*, V_i^*).$$

(ii) The hull of the noncommutative deformation functor, $H(\mathcal{V}) = (H_{i,j})$ is a finite dimensional k-vectorspace.

(iii) For each i, the projective cover of V_i has a (finite) filtration with graded components contained in the family $\mathcal{V}$.

Then

$$\eta : A \to O(\mathcal{V})$$

is a Frobenius extension.

Proof. The assumption 1. implies that the versal family $P = \tilde{V}$ as a left H and right A-module has the duality property, ${}^*P = P^*$. The assumption 2. implies that P as left H-module is finite projective, and the assumption 3. garantees that P, as right A-module, is finite projective, therefore a Frobenius bi-module. □

§4. The generalized Burnside theorem.

In §2 we proved the following result,

Corollary 2.10. *Suppose the k-algebra A is of finite dimension and assume the swarm $\mathcal{V} = \{V_i\}_{i=1}^r$ contains all simple A-modules, then the natural k-algebra homomorphism*

$$\eta : A \to O(\mathcal{V}) = (H_{ij} \otimes_k Hom_k(V_i, V_j))$$

is injective.

Recall also the classical Burnside-Wedderburn-Malcev theorems, see [Lang], and [Curtis and Reiner].

Theorem (Burnside). *Let V be a finite dimensional k-vectorspace. Assume k is algebraically closed and let A be a subalgebra of $End_k(V)$. If V is a simple A-module, then $A = End_k(V)$.*

Theorem (Wedderburn). *Let A be a ring, and let V be a simple faithfull A-module. Put $D = End_A(V)$ and assume V is a finite dimensional D-vector space. Then $A \simeq End_D(V)$.*

Theorem (Wedderburn-Malcev). *Let A be a finite dimensional k-algebra, k-any field. Let $\mathfrak{r}$ be the radical of A, and suppose the residue class algebra $A/\mathfrak{r}$ is separable. Then there exists a semi-simple subalgebra $\mathcal{S}$ of A such that A is the semidirect sum of S and $\mathfrak{r}$. If S_1 and S_2 are subalgebras such that $A = S_i \oplus \mathfrak{r}$, $i = 1, 2$, then there exists an element $n \in \mathfrak{r}$, such that $S_1 = (1 - n) \cdot S_2 \cdot (1 - n)^{-1}$.*

In this § we shall prove a generalization of the theorem of Burnside. In fact, assuming the field k is algebraically closed and that $\mathcal{V} = \{V_i\}_{i=1}^r$ is the family of all simple A-modules we shall prove that the homomorphism η of the above Corollary (2.10), is an isomorphism.

When A is semi-simple we know that $Ext^1_A(V_i, V_j) = 0$ for all $i, j = 1, \ldots, r$, therefore the formal moduli H of V is isomorphic to k^r. This implies that

$$End_H(\tilde{V}) = \bigoplus_{i=1}^{r} End_k(V_i),$$

which is the classical extension of Burnsides theorem.

We shall need the following elementary lemma

Lemma 4.1. *Let the k-algebra A be a direct sum of the right-A-modules V_i, $i = 1, \dots, d$ of the family $\mathcal{V} = \{V\}_{i=1}^r$. Then left multiplication with an element $a \in A$ induces A-module homomorphisms*

$$a_{ij} \in Hom_A(V_i, V_j), \qquad i, j = 1, \dots, d\,.$$

Moreover, any k-linear map $x : A \to A$ expressed as $x = (x_{ij}) \in End_k(V) := (Hom_k(V_i, V_j))$, commuting with all $\varphi = (\varphi_{ij}) \in End_A(V) := (Hom_A(V_i, V_j))$ is necessarily a right multiplication by some element $\tilde{x} \in A$.

Proof. Trivial, since x commuting with all $\varphi \in (Hom_A(V_i, V_j))$ commutes with all left-multiplications by $a \in A$, and therefore $x(a) = a \cdot x(1)$, and we may put $\tilde{x} = x(1)$. $\square$

Corollary 4.2. *Assume that the family of right A-modules $\mathcal{V} = \{V_i\}_{i=1}^r$ is such that*

(i)
$$A \simeq \bigoplus_{i=1}^{m} V_i^{n_i}$$

(ii)
$$Hom_A(V_i, V_j) = 0 \textit{ for } i \neq j$$

Then the canonical morphism of k-algebras

$$\eta : A \to \bigoplus_{i=1}^{n_i} End_k(V_i)$$

is injective. Moreover, η induces an isomorphism

$$A \simeq \bigoplus_{i=1}^{n_i} End_{D_i}(V_i)$$

where $D_i = End_A(V_i)$.

This, in particular, implies the Wedderburn theorem for semisimple k-algebras A.

Theorem 4.3 (A generalized Burnside theorem). *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of simple A-modules, then*

$$A \simeq O(\mathcal{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

Proof. We know that the canonical map

$$\eta : A \to O(\mathcal{V})$$

is injective. Since $Rad(A)^n = 0$ for some n, we know that $\hat{A} = A$. The theorem therefore follows from the following lemmas.

Lemma 4.4. *Let A and B be finite type k-algebras and let $\varphi : A \to B$ be a homomorphism of k-algebras such that the induced morphism*

$$\varphi_2 : A \to B/Rad(B)^2$$

is surjective, then

$$\widehat{\varphi} : \hat{A} \to \hat{B}$$

is surjective.

Proof. Well-known. □

Lemma 4.5. *Let A be a finite dimensional k-algebra, k an algebraically closed field. Let $\mathcal{V} = \{V_i\}_{i=1}^r$ be the family of simple A-modules. Then the homomorphism*

$$\eta : A \to O(\mathcal{V})$$

induces an isomorphism

$$Rad(A)/Rad(A)^2 \simeq (Ext^1_A(V_i, V_j)^* \otimes_k Hom_k(V_i, V_j)).$$

Proof. The classical Burnside theorem implies that the canonical homomorphism of k-algebras

$$A \twoheadrightarrow \bigoplus_{i=1}^r End_k(V_i)$$

induces an isomorphism,

$$A/Rad(A) \simeq \bigoplus_{i=1}^r End_k(V_i).$$

According to the Wedderburn-Malcev theorem we may assume that $A/Rad(A)^2$ is a semidirect sum,

$$A/Rad(A) \oplus Rad(A)/Rad(A)^2.$$

Since $Rad(A)/Rad(A)^2$ is both a left and a right $\bigoplus_{i=1}^r End_k(V_i)$-module

$$Rad(A)/Rad(A)^2 = (E_{ij})$$

each E_{ij} being an $End_k(V_i)^{op} \otimes_k End_k(V_j)$-module. This, however, means that

$$E_{ij} \simeq Hom_k(V_i, V_j) \otimes k^{r_{ij}}$$

as a right $End_k(V_i)^{op} \otimes_k End_k(V_j)$-module. Since we already know that η is an injection, we must have,

$$E_{ij} \simeq Hom_k(V_i, V_j) \otimes k^{r_{ij}} \subset Ext^1_A(V_i, V_j) \otimes Hom_k(V_i, V_j).$$

We must show that this inclusion is an equality. Applying Hochschild cohomology as in §1, we find:

$$Ext^1_A(V_i, V_j) = HH^1(A, Hom_k(V_i, V_j)) = Der_k(A, Hom_k(V_i, V_j))/im\, d^\circ$$

where d° is the differential

$$Hom_k(V_i, V_j) \to Der_k(A, Hom_k(V_i, V_j)).$$

Clearly any derivation

$$\xi \in Der_k(A, Hom_k(V_i, V_j))$$

which is zero on $Rad(A)$ induces a derivation

$$\xi_0 \in Der_k(A/Rad(A), Hom_k(V_i, V_j))$$

which, since $A/Rad(A)$ is semisimple, obviously is a coboundary, i.e. an element of $im\, d^\circ$.

Moreover, any derivation $\xi \in Der_k(A, Hom_k(V_i, V_j))$ induces the zero map on $Rad(A)^2$ since $\xi(r_1 \cdot r_2) = r_i\xi(r_2) + \xi(r_1)r_2 = 0$ for $r_1, r_2 \in Rad(A)$, and any coboundary $\nu \in im\, d^\circ$ must vanish on $Rad(A)$ since $\nu(r) = \varphi r - r\varphi$, for some $\varphi \in Hom_k(V_i, V_j)$. Now, every $A^{op} \otimes_k A$-linear map $Rad(A)/Rad(A)^2 \to End_k(V_i, V_j)$ extends to a derivation of $Der_k(A/Rad(A)^2, Hom_k(V_i, V_j))$. In fact, let φ be an $A^{op} \otimes_k A$-linear map

$$Rad(A)/Rad(A)^2 \to End_k(V_i, V_j)$$

and define the map

$$\psi : A/Rad(A)^2 = A/Rad(A) \bigoplus Rad(A)/Rad(A)^2 \to End_k(V_i, V_j)$$

by

$$\psi(s, r) = \varphi(r) + \varphi(\rho(s))$$

where ρ is the 1-Hochschild cochain on $A/Rad(A)$ with values in $Rad(A)/Rad(A)^2$ that, according to the Wedderburn-Malcev theorem, defines the semidirect sum refered to above. Then,

$$\begin{aligned} &\psi((s_1, r_1) \cdot (s_2, r_2)) = \psi((s_1 \cdot s_2, s_1\rho(s_2) - \rho(s_1 \cdot s_2) + \rho(s_1)s_2 + s_1r_2 + r_1s_2)) \\ &= \varphi(s_1r_2 + r_1s_2 + s_1\rho(s_2) - \rho(s_1 \cdot s_2) + \rho(s_1) \cdot s_2) + \varphi(\rho(s_1 \cdot s_2)) \\ &= (s_1, r_1)\psi((s_2, r_2)) + \psi((s_1, r_1))(s_2, r_2) \end{aligned}$$

Therefore,

$$\begin{aligned} &Ext^1_A(V_i, V_j) = Hom_{A^{op}\otimes A}(Rad(A)/Rad(A)^2, Hom_k(V_i, V_j)) \\ &= \{\varphi : Rad(A)/Rad(A)^2 \to Hom_k(V_i, V_j)|\ \forall a \in A,\ r \in Rad(A),\ s.t. \\ &\varphi(a \cdot r) = a \cdot \varphi(r) \text{ and } \varphi(ra) = \varphi(r) \cdot a\} \end{aligned}$$

Since $Rad(A)/Rad(A)^2 \simeq (E_{ij})$ with

$$E_{ij} \simeq (V_i^* \otimes V_j)^{r_{ij}}$$

it is clear that

$$\begin{aligned} &Hom_{A^{op}\otimes A}(Rad(A)/Rad(A)^2, Hom_k(V_i, V_j)) \\ &\simeq Hom_{End_k(V_i)^{op}\otimes_k End_k(V_j)}((V_i^* \otimes V_j)^{r_{ij}}, (V_i^* \otimes V_j)) \\ &\simeq k^{r_{ij}} \end{aligned}$$

which means that

$$E_{ij} \simeq Ext^1_A(V_i, V_j)^* \otimes_k Hom_k(V_i, V_j).$$

□

Now suppose, as above, that A is a finite dimensional k-algebra, and let $\mathcal{V}_A = \{V_i\}_{i=1}^r$ be any family of finite dimensional A-modules. Obviously

$$dim_k Ext^p_A(V_i, V_j) < \infty$$

for all $p = 0, 1, 2, \dots$ and therefore the endomorphism ring

$$O(\mathcal{V}_A) := End_H(\tilde{V})$$

is a k-algebra such that

$$O(\mathcal{V})/Rad(O) = \bigoplus_{i=1}^r End_k(V_i).$$

This implies that $\mathcal{V} = \{V_i\}_{i=1}^r$ is the family of all simple $\mathcal{O}(V)$-modules. The generalized Burnside theorem applies also in this case, showing that the operation

$$(A, \mathcal{V}) \mapsto (O(\mathcal{V}), \mathcal{V})$$

is a closure operation. Moreover, we have the following,

Proposition 4.6. *Let $\tau : A \to B$ be any homomorphism of finite dimensional k-algebras. Consider a family $\mathcal{V}_B = \{V_i\}_{i=1}^r$ of finite dimensional B-modules and let $\mathcal{V}_A$ be the corresponding family of A-modules. Suppose moreover that $\mathcal{V}_B$ is the family of all simple B-modules. Then there exists an, up to isomorphisms, unique homomorphism of k-algebras*

$$O(\tau) : O(\mathcal{V}_A) \to O(\mathcal{V}_B) \simeq B$$

extending τ.

Proof. There is an obvious forgetful functor defining a morphism of functors on $\underline{a}_r$,

$$\tau^* : Def_{\mathcal{V}_B} \to Def_{\mathcal{V}_A}$$

which in its turn induces a k-algebra homomorphism

$$\eta : H(\mathcal{V}_B) \to H(\mathcal{V}_A)$$

unique up to isomorphisms, and therefore a k-algebra homomorphism

$$O(\mathcal{V}_A) := (H(\mathcal{V}_A)_{i,j} \otimes Hom_k(V_i, V_j)) \to (H(\mathcal{V}_B)_{i,j} \otimes Hom_k(V_i, V_j)) =: O(\mathcal{V}_B)$$

obviously extending τ. By the generalized Burnside theorem, $O(\mathcal{V}_B) \simeq B$, and the Proposition follows.

□

Remark 4.7. Up to now we have only considered finite families of A-modules such that

$$\dim_k Ext^p_A(V_i, V_j) < \infty, \qquad p = 1, 2.$$

Neither of these conditions are essential. Introducing natural topologies we may, as in [La 1], treat general families of finite type A-modules. Notice also that if $r_1 \leq r_2$, there is an obvious canonical morphism

$$\underline{a}_{r_1} \to \underline{a}_{r_2}$$

inducing a restriction morphism of functors

$$Def_{\mathcal{V}(2)} \to Def_{\mathcal{V}(1)}$$

where $\mathcal{V}(1) = \{V_i\}_{i=1}^{r_1}, \mathcal{V}(2) = \{V_i\}_{i=1}^{r_2}$. Therefore we obtain an up to isomorphisms unique k-algebra homomorphism

$$r_{2,1} : H_{A,V(2)} \to H_{A,V(1)}.$$

However, this *restriction* morphism is not, in general, unique. The resulting problems will be dealt with later.

Filtered modules and iterated extensions. Let as above $\mathcal{V} = \{V_i\}_{i=1}^r$ be a family of right A-modules, and let $E_{i_1,\ldots,i_s} : E_s \subset E_{s-1} \subset \cdots \subset E_1 = E$ be a filtered module such that $E_k/E_{k+1} \simeq V_{i_k}$. We shall, as before, refer to any such filtered module as an iterated extension of $\mathcal{V}$. Notice that for every p there is an extension,

$$\xi_{p,p+1} \in Ext^1_A(V_{i_p}, V_{i_{p+1}})$$

given by the exact sequence.

$$0 \to E_{p+1}/E_{p+2} \to E_p/E_{p+2} \to E_p/E_{p+1} \to 0.$$

Corresponding to the iterated extension $E_{i_1,\ldots,i_s}$ we shall associate two directed graphs, $\Gamma(\underline{i})$ and $\Gamma(E_{\underline{i}})$. The first is gotten as the graph with nodes in bijection with the modules of the family $\mathcal{V}$, and with arrows $\epsilon(i_p, i_{p+1})$ connecting the node i_p with the node i_{p+1}. The second, *the extension type* of the iterated extension, is obtained from the first identifying two arrows $\epsilon(i_p, i_{p+1})$ and $\epsilon(i_q, i_{q+1})$ if the corresponding extensions $\xi_{p,p+1}$ and $\xi_{q,q+1}$ coincide. The corresponding k-algebra is the object of $\underline{a}_r$ generated by the arrows $\epsilon(i_p, i_{p+1})$ with relations given by composable monomials of the form

$$\epsilon(j_1, j_2)\epsilon(j_3, j_4)\ldots\epsilon(j_w, j_{w+1})$$

where the sequence

$$\{j_1, j_2, j_3, \ldots, j_{w+1}\}$$

is not contained in the sequence

$$\{i_1, i_2, i_3, \ldots, i_s\}.$$

Example 4.8. Let us draw up all extension types for $r, s \leq 3$.

$r = 3, s = 3:$

$$\begin{tikzcd} ① \arrow[r] & ② \arrow[r] & ③ \end{tikzcd}$$

$r = 2, s = 3:$

$$\begin{tikzcd} ① \arrow[loop above] \arrow[r] & ② \end{tikzcd} \qquad \begin{tikzcd} ① \arrow[r] & ② \arrow[loop above] \end{tikzcd} \qquad \begin{tikzcd} ① \arrow[r, shift left] & ② \arrow[l, shift left] \end{tikzcd}$$

$r = 1, s = 3:$

$$\begin{tikzcd} ① \arrow[loop above] \arrow[loop below] \end{tikzcd} \qquad \begin{tikzcd} ① \arrow[loop above, "2"] \end{tikzcd}$$

The last example is a $\Gamma(E_{\underline{i}})$ corresponding to the situation, $i_1 = i_2 = i_3 = 1$, and $\xi_{i_1,i_2} = \xi_{i_2,i_3}$ The associated k-algebras are, respectively, the matrix algebras,

$$\begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \begin{pmatrix} k[\epsilon] & k[\epsilon] \\ 0 & k \end{pmatrix} \begin{pmatrix} k & k[\epsilon] \\ 0 & k[\epsilon] \end{pmatrix} \begin{pmatrix} k[t_{1,2}t_{2,1}] & kt_{1,2} \\ kt_{2,1} & k \end{pmatrix}$$

with the obvious relations, and the k-algebras,

$$k\{t, u\}/(t^2, u^2, ut), \quad k[t]/(t^3)$$

Lemma 4.9. *Let H be any object of $\hat{\underline{a}}_r$, and let $R \in \underline{a}_r$. Then Mor(H,R) has a natural structure of an affine algebraic scheme $Mor(H, R) = Spec(A(H, R))$, and there is a universal morphism,*

$$\tilde{\phi} : H \longrightarrow A(H, R) \otimes_k R$$

Proof. Put $(E_{i,j}) = rad(H)/rad(H)^2$, and consider the affine space,

$$\mathbf{A}^N = \prod_{i,j} E^*_{i,j} \otimes_k R_{i,j}$$

with coordinates $z_{i,j}(l, m) = t_{i,j}(l) \otimes x_{i,j}(m)$, where $\{t_{i,j}(l)\}_{i,j}$ is a basis of $E_{i,j}$ and $\{r_{i,j}(m)\}_{i,j}$ is a basis, and $\{x_{i,j}(m)\}_{i,j}$ is a dual basis of $R_{i,j}$. An element

$$(\alpha_{i,j}(l, m)) \in \mathbf{A}^N$$

corresponds to a morphism $\phi \in Mor(H, R)$ if, and only if, the corresponding map

$$t_{i,j}(l) \longmapsto \sum_m \alpha_{i,j}(l, m) r_{i,j}(m) \in R_{i,j}$$

satisfies the relations of H. Let these, modulo a high enough power of the radical, be polynomials in the generators $t_{i,j}(l)_{i,j}$ of the form

$$f_p(t_{i,j}(l)) = 0, \ p = 1, \dots s \ ,$$

and let the relations of R be expressed in terms of,

$$r_{i,j}(m) r_{j,k}(n) = \sum_p \beta^p_{i,j,k}(m, n) r_{i,k}(p), \ i, j = 1, \dots r.$$

Then we obtain equations for $Mor(H, R)$ given by (commutative) polynomial relations of the form,

$$F_p(z_{i,j}(l, m)) = 0, \ p = 1, \dots t.$$

But then

$$t_{i,j}(l) \longmapsto \sum_m z_{i,j}(l, m) r_{i,j}(m) \in A(H, R) \otimes_k R_{i,j}$$

where the coordinates $z_{i,j}(l, m)$ are subject to the conditions above, defines the universal morphism $\tilde{\phi}$. $\square$

Proposition 4.10. *Let A be any k-algebra, $\mathcal{V} = \{V_i\}_{i=1}^r$ any swarm of A-modules, i.e. such that,*

$$\dim_k Ext^1_A(V_i, V_j) < \infty \quad \textit{for all } i, j = 1, \dots, r.$$

(i): Consider an iterated extension E of $\mathcal{V}$, with directed graph Γ. Then there exists a morphism of k-algebras

$$\phi : H(\mathcal{V}) \to k[\Gamma]$$

such that

$$E \simeq k[\Gamma] \otimes_\phi \tilde{V}$$

in the above sense.

(ii): The set of equivalence classes of iterated extensions of $\mathcal{V}$ with extension type Γ, is a quotient of the set of closed points of the affine algebraic scheme

$$\underline{A}[\Gamma] = Mor(H(\mathcal{V}), k[\Gamma])$$

(iii): There is a versal family $\tilde{V}[\Gamma]$ of A-modules defined on $A[\Gamma]$, containing as fibres all the isomorphism classes of iterated extensions of $\mathcal{V}$s with extension type Γ.

Proof. Any morphism $\varphi : H \to k[\Gamma]$ in $\underline{a}_r$ correspond to an iterated extension of the V_i's. This may be expressed in the following way. As vector spaces, we have an isomorphism,

$$k[\Gamma] \otimes_\varphi \tilde{V} \simeq V(\Gamma) \simeq V_{i_1} \times V_{i_2} \times \cdots \times V_{i_s}$$

An A-module structure on this vectorspace, corresponding to an iterated extension with extension type Γ, is given by a homomorphism of k-algebras,

$$\psi : A \longrightarrow End_k(V_{i_1} \times V_{i_2} \times \cdots \times V_{i_s})$$

inducing a family of linear maps

$$\psi_{p,p+1,\dots,p+q} : A \longrightarrow End_k(V_{i_p}, V_{i_{p+q})}$$

for $0 \leq p < p+q \leq s$.

Consider these maps as 1-cochains in the Hochschild complex

$$HC^*(A, Hom_k(V(\Gamma), V(\Gamma)))$$

The maps $\psi_{p,p+1}$ correspond to the extensions $\xi_{p,p+1}$ above, and must therefore be 1-cocycles, or derivations. To obtain an A-module structure, corresponding to an iterated extensions of $\mathcal{V}$ with extension type Γ, the conditions on these cochains are: For all $a, b \in A$,

$$\psi_{p,p+1}(a)\psi_{p+1,p+2}(b) = d\psi_{p,p+1,p+2}(a,b)$$
$$\psi_{p,p+1}(a)\psi_{p+1,p+2,p+3}(b) + \psi_{p,p+1,p+2}(a)\psi_{p+2,p+3}(b) = d\psi_{p,p+1,p+2,p+3}(a,b)$$
$$\dots$$
$$\sum_{m=2,\dots,s-1} \psi_{1,2,\dots,m}(a)\psi_{m,\dots,s}(b) = d\psi_{1,2,\dots,s}(a,b),$$

which means that all Massey products of the form

$$< \xi_{i_p,i_{p+1}}, \xi_{i_{p+1},i_{p+2}}, \ldots, \xi_{i_{p+q-1},i_{p+q}} >$$

are defined and contain zero.

Now (i) follows from the very definition of H, generated as it is by a basis of the dual Ext^1's, with relations exactly expressing the vanishing of the above Massey products. (ii) and (iii) then follows from deformation theory, together with the Lemma (4.9), above. $\square$

Example 4.11. Consider the extension E_{ijk} of length 3 given as the composite extension of $\xi_{i,j} : 0 \leftarrow V_i \leftarrow E_{ij} \leftarrow V_j \leftarrow 0$ and $\xi_{i,j,k} : 0 \leftarrow E_{ij} \leftarrow E_{ijk} \leftarrow V_k \leftarrow 0$. Take the pullback $\xi_{i,k}$ of $\xi_{i,j,k}$ via $V_j \rightarrow E_{ij}$ and consider the diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 & V_i \arrow[l] \arrow[u] & E_{ij} \arrow[l] \arrow[u] & V_j \arrow[l] \arrow[u] & 0 \arrow[l] \\
 & E_{ik} \arrow[u] & E_{ijk} \arrow[l] \arrow[u] & E_{jk} \arrow[l] \arrow[u] & \\
 & V_k \arrow[u] \arrow[r, equal] & V_k \arrow[u] \arrow[r, equal] & V_k \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Let $\psi_{i,j} \in Der_k(A, Hom_k(V_i, V_j))$ be a Hochschild cocycle representing the class $\xi_{i,j}$. The multiplication with $a \in A$ on E_{ij}, identified with $V_j \times V_i$ as k-vectorspace, is given by

$$(v_j, v_i)a = (v_j a + \psi_{i,j}(a, v_i), v_i a)$$

and the multiplication with $a \in A$ on E_{ijk} identified with $V_k \times V_j \times V_i$ as k-vectorspace, must be such that,

$$(v_k, v_j, 0)a = (v_k a + \psi_{j,k}(a, v_j), v_j a, 0).$$

If there exists an action of A on E_{ijk} consistent with the above, then one proves the existence of a Hochschild cochain

$$\psi_{i,j,k} \in Hom_k(A, Hom_k(V_i, V_k))$$

such that

$$d\psi_{i,j,k}(a, b) = \psi_{i,j}(a)\psi_{j,k}(b).$$

From this follows that the cup-product $\xi_{ij} \cup \xi_{jk}$ is 0 in $Ext^2_A(V_i, V_k)$. This is also the criterion for the existence of $\xi_{i,j,k}$. Moreover if ξ and ξ' are two extensions

$$\xi : 0 \leftarrow E_{ij} \leftarrow E_{ijk} \leftarrow V_k \leftarrow 0$$
$$\xi' : 0 \leftarrow E_{ij} \leftarrow E'_{ijk} \leftarrow V_k \leftarrow 0$$

with the same pullback ξ_{jk}, then there is an extension

$$\xi_{ik} : 0 \leftarrow V_i \leftarrow E_{ik} \leftarrow V_k \leftarrow 0$$

such that its pullback, via $E_{ij} \to V_i$, is the difference $\xi - \xi'$.

Consider for the iterated extensions E_{ijk}, the extension diagram

$$\Gamma : \qquad i \longrightarrow j \longrightarrow k,$$

the first one of the example (4.8) above, then the corresponding k-algebra is given by,

$$k[\Gamma] = \begin{array}{cc} & \begin{array}{ccccc} & i & j & k & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} k & 0 & 0 & 0 & 0 \\ \ddots & \cdots & \cdots & \cdots & \vdots \\ & k & k & k & 0 \\ & \ddots & \cdots & \cdots & \vdots \\ & & k & k & 0 \\ & & \ddots & \cdots & \vdots \\ & & & k & 0 \\ & & & \ddots & \cdots \\ & & & & k \end{pmatrix} \end{array}$$

Notice that E_{ijk} then corresponds to a morphism

$$\phi : H \to k[\Gamma]$$

determined, modulo $Rad^2(k[\Gamma])$ (i.e. the radical squared), by

$$\xi_{i,j} = \phi|E_{i,j} \to k, \text{ and } \xi_{j,k} = \phi|E_{j,k} \to k$$

Here

$$Rad^2(k[\Gamma]) = \begin{array}{cc} & \begin{array}{ccccc} & i & j & k & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \ddots & \cdots & \cdots & \cdots & \vdots \\ & 0 & 0 & k & 0 \\ & \ddots & \cdots & \cdots & \vdots \\ & & 0 & 0 & 0 \\ & & \ddots & \cdots & \vdots \\ & & & 0 & 0 \\ & & & \ddots & \cdots \\ & & & & 0 \end{pmatrix} \end{array}$$

and ϕ is, according to the analysis above, "calibrated" by the morphism $\phi_{i,k} : E_{i,k} \to k$, i.e. by $Ext^1_A(V_i, V_k)$, as it should.

Corollary 4.12. *(i): Given any finitely generated module M on a Noetherian ring A, there is a finite set of primes $E(M)$, containing the set of associated primes Ass(M), such that the module M is an iterated extensions of the corresponding modules $A/\mathfrak{p}$ for $\mathfrak{p} \in E(M)$. The extension type of such an iterated extension is an ordered directed graph $\Gamma(M)$ the nodes of which is E(M).*

(ii): For any finite ordered directed graph Γ, with nodes corresponding to a set of primes $\mathcal{P} \subset Spec(A)$, there is an affine versal family of A-modules $\tilde{M}$ with extension type Γ, and $E(M) = \mathcal{P}$.

Proof. Obvious $\square$

Example 4.13. Given any scheme $\underline{H} = Spec(H)$, say the 2-dimensional affine space given by $H = k[x_1, x_2]$. We shall be interested in the noncommutative moduli space parametrizing subschemes of length 2 of $\underline{H}$. We may do this by simply considering a point in the space $Spec(H)$ together with a tangent direction, i.e. the right H-module of the form,

$$V = k[x_1, x_2]/(x_1^2, x_2),$$

and compute the formal moduli of V.

Lemma 4.14. *The formal moduli, $H(V)$ of the H-module $V = H/(x_1^2, x_2)$, is given as the completion of the k-algebra,*

$$\Omega = k\{t_1, t_2, \omega_1, \omega_2\}/(y_1, y_2)$$

where

$$y_1 = [t_1, t_2] - t_1[\omega_1, \omega_2] \quad y_2 = [t_1, \omega_2] - [t_2, \omega_1] - \omega_1[\omega_1, \omega_2],$$

and where the family of left Ω-and right H-modules,

$$\Omega \otimes_k k^2$$

is defined by the actions of x_1 and x_2, given by,

$$x_1 = \begin{pmatrix} 0 & t_1 \\ 1 & \omega_1 \end{pmatrix}, \; x_2 = \begin{pmatrix} t_2 & t_1\omega_2 \\ \omega_2 & t_2 + \omega_1\omega_2 \end{pmatrix}$$

Proof. Consider the obvious free resolution of $V := H/(x_1^2, x_2)$ as an H-module,

$$\begin{tikzcd} V & H \arrow[l, "\rho"'] & H^2 \arrow[l, "d_0"'] & H \arrow[l, "d_1"'] & 0 \arrow[l, "d_2"'] \end{tikzcd}$$

where we have,

$$d_0 = (x_1^2, x_2), \; d_1 = \begin{pmatrix} x_2 \\ -x_1^2 \end{pmatrix}.$$

Consider the Yoneda complex, and pick a basis

$$\{\hat{t}_1, \hat{t}_2; \hat{\omega}_1, \hat{\omega}_2, \}$$

of $Ext^1_H(V, V)$ represented by the morphisms of the diagram,

$$\begin{tikzcd}
V & H \arrow[l, "\rho"'] & H^2 \arrow[l, "d_0"'] \arrow[dl, "\hat{\omega}_j"', "\hat{t}_i"] & H \arrow[l, "d_1"'] \arrow[dl, "\hat{\omega}_j^2"', "\hat{t}_i^2"] & 0 \arrow[l] \\
V & H \arrow[l, "\rho"'] & H^2 \arrow[l, "d_0"'] \arrow[dl, "\hat{\omega}_j"', "\hat{t}_i"] & H \arrow[l, "d_1"'] \arrow[dl, "\hat{\omega}_j^2"', "\hat{t}_i^2"] & 0 \arrow[l] \\
V & H \arrow[l, "\rho"] & H^2 \arrow[l, "d_0"] & H \arrow[l, "d_1"] & 0 \arrow[l]
\end{tikzcd}$$

Here,

$$\hat{t}_1^1 = (1,0),\ \hat{t}_2^1 = (0,1,);$$
$$\hat{\omega}_1^1 = (x_1,0),\ \hat{\omega}_2^1 = (0,x_1)$$

and,

$$\hat{t}_1^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \hat{t}_2^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and finally,

$$\hat{\omega}_1^2 = \begin{pmatrix} 0 \\ x_1 \end{pmatrix},\ \hat{\omega}_2^2 = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix}.$$

Using this it is easy to see that ,

$$\hat{t}_i \cup \hat{t}_i = 0,\ \hat{t}_1 \cup \hat{t}_2 = -\hat{t}_2 \cup \hat{t}_1 = \hat{y}_1,$$

and that

$$\hat{t}_1^1\hat{\omega}_2^2 = \hat{\omega}_1^1 t_2^2 = -\hat{y}_2,\ \hat{\omega}_i^1\hat{t}_i^2 = 0,\ \hat{\omega}_i^1\hat{\omega}_j^2 = 0,\ \hat{t}_2^1\hat{\omega}_1^2 = \hat{\omega}_2^1\hat{t}_1^2 = \hat{y}_2,$$

where,

$$\{\hat{y}_1, \hat{y}_2\}$$

is a basis for of $Ext_H^2(V,V)$ represented by the morphisms of the diagram,

$$\begin{tikzcd}
V & H \arrow[l, "\rho"'] & H^2 \arrow[l, "d_0"'] & H \arrow[l, "d_1"'] \arrow[dll, "\hat{y}_1", "\hat{y}_2"'] & 0 \arrow[l] \\
V & H \arrow[l, "\rho"'] & H^2 \arrow[l, "d_0"'] & H \arrow[l, "d_1"'] & 0 \arrow[l]
\end{tikzcd}$$

where $\hat{y}_1 = (1)$ and $\hat{y}_2 = (x_1)$. Therefore

$$-\hat{y}_2 = \hat{t}_1 \cup \hat{\omega}_2 = \hat{\omega}_1 \cup \hat{t}_2 = -\hat{t}_2 \cup \hat{\omega}_1 = -\hat{\omega}_2 \cup \hat{t}_1,\ \hat{\omega}_i \cup \hat{\omega}_j = \hat{t}_i \cup \hat{t}_j = 0.$$

Now, consider the dual basis $\{t_1, t_2; \omega_1, \omega_2\}$ generating the hull of the deformation functor $Def_{k[\epsilon]}$, we find after a simple computation of the 3. order Massey products the formulas we want.

Notice that we just have to compute the tangent situation and check that our formulas give us a lifting of the quadratic relations and of the corresponding H-action, to know that our result holds.

□

By a simple computation one checks that the k-points of Ω form an open dense part of $Hilb^2\mathbf{A}^2$ containing V. $Hilb^2\mathbf{A}^2$ is the blow-up of $(\mathbf{A}^2 \times \mathbf{A}^2)/\mathbf{Z}_2$ along the diagonal. However, there are other simple representations of Ω. The homomrphism,

$$\Omega \to k|t_1, t_2, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}]$$

maping ω_i to $\dfrac{\partial}{\partial t_i}$, shows that $k|t_1, t_2]$ is a simple representation of Ω.

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