PHASE SPACES AND DEFORMATION THEORY.

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Abstract. In the papers [La 5,6], we introduced the notion of (non-commutative) phase algebras (space) $Ph^n(A), n = 0, 1, ..., \infty$ associated to any associative algebra A (space), defined over a field k. The purpose of this paper is to prove that this construction is useful in non-commutative deformation theory for the construction of the versal family of families of modules, see [La 4]. In particular we obtain a much better understanding of the Massey products, introduced in [La 1, La 2], and used extensively in other texts.

Contents.

§0 Introduction.

In the paper, [La 5], we sketched a physical "toy model", where the space-time of classical physics became a section of a universal fiber space $\tilde{E}$, defined on the moduli space $\underline{H} = Simp(H)$, of the physical systems we chose to consider, in this case the systems composed of an observer and an observed, both sitting in Euclidean 3-space. This moduli space was called the *time-space*. Time, in this mathematical model, was defined to be a metric ρ on the time-space, measuring all possible infinitesimal changes of *the state* of the objects in the family we are studying. This gave us a model of relativity theory, in which the set of all velocities turned out to be a projective space. Dynamics was introduced into this picture, via the general construction of a *phase space* $Ph(A)$, for any associative algebra A. This is a universal pair of a homomorphism of algebras, $\iota : A \to Ph(A)$, and a derivation, $d : A \to Ph(A)$, such that for any homomorphism of A into a k-algebra R, the derivations of A in R are induced by unique homomorphisms $Ph(A) \to R$, composed with d. Iterating this $Ph(-)$-construction we obtained a limit morphism $\iota(n) : Ph^n(A) \to Ph^\infty(A)$ with image $Ph^{(n)}(A)$, and a universal derivation $\delta \in Der_k(Ph^\infty(A), Ph^\infty(A))$, the *Dirac*-derivation. A general *dynamical*

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structure of order n, is now a two-sided δ-ideal σ in $Ph^{\infty}(A)$ inducing a surjective homomorphism $Ph^{(n-1)}(A) \rightarrow Ph^{\infty}(A)/\sigma =: A(\sigma)$.

In [La 5], and later in [La 6], we have shown that, associated to any such *time space* H with a fixed dynamical structure $H(\sigma)$, there is a kind of "Quantum field theory". In particular, we have stressed the point that, if H is the affine ring of a moduli space of the objects we want to study, the ring $Ph^{\infty}(H)$ is the complete ring of observables, containing the parameters not only of the iso-classes of the objects in question, but also of all dynamical parameters. The choice made by fixing the dynamical structure σ, and reducing to the k-algebra $H(\sigma)$, would classically, correspond to the introduction of a parsimony principle, e.g. to the choice of some Lagrangian.

The purpose of this paper is to study this phase-space construction, in greater detail. There is a natural descending filtration of two-sided ideals, $\{\mathcal{F}_n\}_{0\leq n}$ of $Ph^{\infty}(A)$. The corresponding quotients $Ph^n(A)/\mathcal{F}_n$ are finite dimensional vector-spaces, and considered as affine varieties, these are our non-commutative Jet-spaces.

We shall first see, in §1, that we may extend the usual prolongation-projection procedure of Elie Cartan, to this non-commutative setting, and obtain a framework for the study of general systems of (non-commutative) PDE's.

In §2 we present a short introduction to non-commutative deformations of modules, and the generalized Massey products, as exposed in [La 1,La 2].

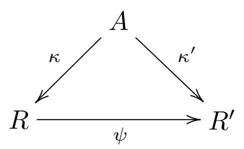
Then, in §3, follows the main part of the paper, the construction of the versal family of the non-commutative deformation functor of a finite family of A-modules, based on the phase-space of a *resolution* of the k-algebra A.

As a by-product we shall be able to make more precise the notion of *preparation* of an object, see loc. cit [La 5].

Notice that our $Ph^{\infty}(A)$ is a non-commutative analogue of the notion of higher differentials treated in a many texts, see [Itaka], and the more recent paper, [Laksov-Thorup] .

§1 Phase spaces and the Dirac derivation.

Given a k-algebra A, denote by $A/k-\underline{alg}$ the category where the objects are homomorphisms of k-algebras $\kappa : A \rightarrow R$, and the morphisms, $\psi : \kappa \rightarrow \kappa'$ are commutative diagrams,



and consider the functor,

$$Der_k(A,-) : A/k-\underline{alg} \longrightarrow \underline{Sets}.$$

It is representable by a k-algebra-morphism,

$$\iota : A \longrightarrow Ph(A),$$

with a *universal family* given by a universal derivation,

$$d : A \longrightarrow Ph(A).$$

It is easy to construct Ph (A). In fact, let $\pi : F \to A$ be a surjective homomorphism of algebras, with $F = k < t_1, t_2, ..., t_r >$, free generated by the t_i's, and put $I = ker\pi$. Let,

$$Ph(A) = k < t_1, t_2, ..., t_r, dt_1, dt_2, ..., dt_r > /(I, dI)$$

where dt_i is a formal variable. Clearly there is a homomorphism,

$$i_0' : F \to Ph(A)$$

and a derivation

$$d' : F \to Ph(A),$$

defined by putting, $d'(t_i) = cl(dt_i)$, the equivalence class of dt_i. Since i_0' and d' both kill the ideal I, they define a homomorphism,

$$i_0 : A \to Ph(A)$$

and a derivation,

$$d : A \to Ph(A).$$

To see that i_0 and d have the universal property, let $\kappa : A \to R$ be an object of the category $A/k - \underline{alg}$. Any derivation $\xi : A \to R$ defines a derivation $\xi' : F \to R$, mapping t_i to $\xi'\overline{(t_i)}$. Let $\rho_{\xi'} : k < t_1, t_2, ..., t_r, dt_1, dt_2, ..., dt_r > \to R$ be the homomorphism defined by,

$$\rho_{\xi'}(t_i) = \kappa(\pi(t_i)),\ \rho_{\xi'}(dt_i) = \xi(\pi(t_i)).$$

$\rho_{\xi'}$ sends I and dI to zero, so defines a homomorphism,

$$\rho_\xi : Ph(A) \to R,$$

such that the composition with $d : A \to Ph(A)$, is ξ. The unicity is a consequence of the fact that the images of i_0 and d generate $Ph(A)$ as k-algebra.

Clearly $Ph(-)$ is a covariant functor on $k - \underline{alg}$, and we have the identities,

$$d_* : Der_k(A, A) = Mor_A(Ph(A), A),$$

and,

$$d^* : Der_k(A, Ph(A)) = End_A(Ph(A)),$$

the last one associating d to the identity endomorphisme of $Ph(A)$. In particular we see that i_0 has a cosection, $\sigma_0 : Ph(A) \to A$, corresponding to the trivial (zero) derivation of A.

Let now V be a right A-module, with structure morphism $\rho(V) =: \rho : A \to End_k(V)$. We obtain a universal derivation,

$$u(V) =: u : A \longrightarrow Hom_k(V, V \otimes_A Ph(A)),$$

defined by, $u(a)(v) = v \otimes d(a)$. Using the long exact sequence,

$$0 \to Hom_A(V, V \otimes_A Ph(A)) \to Hom_k(V, V \otimes_A Ph(A)) \to^\iota$$
$$Der_k(A, Hom_k(V, V \otimes_A Ph(A))) \to^\kappa Ext^1_A(V, V \otimes_A Ph(A)) \to 0,$$

we obtain the non-commutative Kodaira-Spencer class,

$$c(V) := \kappa(u(V)) \in Ext^1_A(V, V \otimes_A Ph(A)),$$

inducing the Kodaira-Spencer morphism,

$$g : \Theta_A := Der_k(A, A) \longrightarrow Ext^1_A(V, V),$$

via the identity d_*. If $c(V) = 0$, then the exact sequence above proves that there exist a $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ such that $u = \iota(\nabla)$. This is just another way of proving that $c(V)$ is the obstruction for the existence of a connection,

$$\nabla : Der_k(A, A) \longrightarrow Hom_k(V, V).$$

It is well known, I think, that in the commutative case, the Kodaira-Spencer class gives rise to a Chern character by putting,

$$ch^i(V) := 1/i!\ c^i(V) \in Ext^i_A(V, V \otimes_A Ph(A)),$$

and that if $c(V) = 0$, the curvature $R(\nabla)$ of the connection ∇, induces a curvature class,

$$R_\nabla \in H^2(k, A; \Theta_A, End_A(V)).$$

Any $Ph(A)$-module W, given by its structure map,

$$\rho(W)^1 =: \rho^1 : Ph(A) \longrightarrow End_k(W)$$

corresponds bijectively to an induced A-module structure $\rho : A \to End_k(W)$, together with a derivation $\delta_\rho \in Der_k(A, End_k(W))$, defining an element,

$$[\delta_\rho] \in Ext^1_A(W, W).$$

Fixing this last element we find that the set of $Ph(A)$-module structures on the A-module W is in one to one correspondence with,

$$End_k(W)/End_A(W).$$

Conversely, starting with an A-module V and an element $\delta \in Der_k(A, End_k(V))$, we obtain a $Ph(A)$-module V_δ. It is then easy to see that the kernel of the natural map,

$$Ext^1_{Ph(A)}(V_\delta, V_\delta) \to Ext^1_A(V, V),$$

induced by the linear map,

$$Der_k(Ph(A), End_k(V_\delta)) \to Der_k(A, End_k(V))$$

is the quotient,

$$Der_A(Ph(A), End_k(V_\delta))/End_k(V),$$

and the image is a subspace $[\delta_\rho]^\perp \subseteq Ext^1_A(V, V))$, which is rather easy to compute, see examples below.

Remark 1.1. Defining *time* as a metric on the moduli space, $Simp(A)$, of simple A-modules, in line with the philosophy of [La 5], noticing that $Ext^1_A(V, V)$ is the tangent space of $Simp(A)$ at the point corresponding to V, we see that the non-commutative space PhA also parametrizes the set of *generalised momenta*, i.e. the set of pairs of a point $V \in Simp(A)$, and a tangent vector at that point.

Example 1.2. (i) Let $A = k[t]$, then obviously, $Ph(A) = k < t, dt >$ and d is given by $d(t) = dt$, such that for $f \in k[t]$, we find $d(f) = J_t(f)$ with the notations of [La 4], i.e. the non-commutative derivation of f with respect to t. One should also compare this with the non-commutative Taylor formula of loc.cit. If $V \simeq k^2$ is an A-module, defined by the matrix $X \in M_2(k)$, and $\delta \in Der_k(A, End_k(V))$, is defined in terms of the matrix $Y \in M_2(k)$, then the $Ph(A)$-module V_δ is the $k < t, dt >$-module defined by the action of the two matrices $X, Y \in M_2(k)$, and we find

$$e^1_V := dim_k Ext^1_A(V,V) = dim_k End_A(V) = dim_k\{Z \in M_2(k)|\ [X,Z] = 0\}$$
$$e^1_{V_\delta} := dim_k Ext^1_{Ph(A)}(V_\delta, V_\delta) = 8 - 4 + dim\{Z \in M_2(k)|\ [X,Z] = [Y,Z] = 0\}.$$

We have the following inequalities,

$$2 \leq e^1_V \leq 4 \leq e^1_{V_\delta} \leq 8.$$

(ii) Let $A = k[t_1, t_2]$ then we find,

$$Ph(A) = k < t_1, t_2, dt_1, dt_2 > /([t_1, t_2], [dt_1, t_2] + [t_1, dt_2]).$$

In particular, we have a surjective homomorphism,

$$Ph(A) \to k < t_1, t_2, dt_1, dt_2 > /([t_1, t_2], [dt_1, dt_2], [t_i, dt_i] - 1),$$

the right hand algebra being the Weyl algebra. This homomorphism exists in all dimensions. We also have a surjective homomorphism,

$$Ph(A) \to k[t_1, t_2, \xi_1, \xi_2],$$

i.e. onto the affine algebra of the classical phase-space.

The phase-space construction may, of course, be iterated. Given the k-algebra A we may form the sequence, $\{Ph^n(A)\}_{0 \leq n}$, defined inductively by

$$Ph^0(A) = A,\ Ph^1(A) = Ph(A), ..., Ph^{n+1}(A) := Ph(Ph^n(A)).$$

Let $i^n_0 : Ph^n(A) \to Ph^{n+1}(A)$ be the canonical imbedding, and let $d_n : Ph^n(A) \to Ph^{n+1}(A)$ be the corresponding derivation. Since the composition of i^n_0 and the derivation d_{n+1} is a derivation $Ph^n(A) \to Ph^{n+2}(A)$, there exist by universality a homomorphism $i^{n+1}_1 : Ph^{n+1}(A) \to Ph^{n+2}(A)$, such that,

$$d_n \circ i^{n+1}_1 = i^n_0 \circ d_{n+1}.$$

Notice that we compose functions and functors from left to right. Clearly we may continue this process constructing new homomorphisms,

$$\{i^n_j : Ph^n(A) \to Ph^{n+1}(A)\}_{0 \leq j \leq n},$$

with the property,

$$d_n \circ i^{n+1}_{j+1} = i^n_j \circ d_{n+1}.$$

Notice also that we have the "bi-gone", $i^0_0 i^1_0 = i^0_0 i^1_1$ and the "hexagone",

$$\begin{aligned} i^1_0 i^2_0 &= i^1_0 i^2_1 \\ i^1_1 i^2_0 &= i^1_0 i^2_2 \\ i^1_1 i^2_1 &= i^1_1 i^2_2, \end{aligned}$$

and, in general,

$$\begin{aligned} i^n_p i^{n+1}_q &= i^n_{q-1} i^{n+1}_p,\ p < q \\ i^n_p i^{n+1}_p &= i^n_p i^{n+1}_{p+1} \\ i^n_p i^{n+1}_q &= i^n_q i^{n+1}_{p+1},\ q < p, \end{aligned}$$

which is all easily proved by composing with i^{n-1}_0 and d_{n-1}. Thus, the $Ph^*(A)$ is a semi-simplicial algebra with a cosection onto A. Therefore, for any object $\kappa : A \to R \in A/k - \underline{alg}$ the semisimplicial algebra above induces a semisimplicial k-vectorspace,

$$Der_k(Ph^*(A), R),$$

and one should be interested in its homology.

The system of k-algebras and homomorphisms of k-algebras $\{Ph^n(A), i^n_j\}_{n,0\le j\le n}$ has an inductive (direct) limit, $Ph^\infty(A)$, together with homomorphisms,

$$i_n : Ph^n(A) \longrightarrow Ph^\infty(A)$$

satisfying,

$$i^n_j \circ i_{n+1} = i_n,\ j = 0, 1, .., n.$$

Moreover, the family of derivations, $\{d_n\}_{0\le n}$ define a unique derivation,

$$\delta : Ph^\infty(A) \longrightarrow Ph^\infty(A),$$

such that,

$$i_n \circ \delta = d_n \circ i_{n+1}.$$

Put

$$Ph^{(n)}(A) := im\ i_n \subseteq Ph^\infty(A)$$

The k-algebra $Ph^\infty(A)$ has a descending filtration of two-sided ideals, $\{\mathcal{F}_n\}_{0\le n}$ given inductively by:

$$\mathcal{F}_1 = Ph^\infty(A) \cdot im(\delta) \cdot Ph^\infty(A)$$

and,

$$\begin{aligned} \delta(\mathcal{F}_n) &\subseteq \mathcal{F}_{n+1} \\ \mathcal{F}_{n_1}\mathcal{F}_{n_2}...\mathcal{F}_{n_r} &\subseteq \mathcal{F}_n,\ n_1 + ... + n_r = n \end{aligned}$$

such that the derivation δ induces derivations,

$$\delta_n : \mathcal{F}_n \longrightarrow \mathcal{F}_{n+1}.$$

Using the canonical homomorphism $i_n : Ph^n(A) \longrightarrow Ph^\infty(A)$ we pull the filtration $\{\mathcal{F}_p\}_{0\le p}$ back to $Ph^n(A)$, not bothering to change the notation.

Definition 1.3. *Let* $\mathcal{D}(A) := \varprojlim_{n\geq 1} Ph^\infty(A)/\mathcal{F}_n$*, the completion of* $Ph^\infty(A)$ *in the topology given by the filtration* $\{\mathcal{F}_n\}_{0\leq n}$*. The* k*-algebra* $Ph^\infty(A)$ *will be referred to as the* k*-algebra of higher differentials, and* $\mathcal{D}(A)$ *will be called the* k*-algebra of formalized higher differentials. Put,*

$$\mathcal{D}_n := \mathcal{D}_n(A) := Ph^\infty(A)/\mathcal{F}_{n+1}$$

Clearly δ defines a derivation on $\mathcal{D}(A)$, and an isomorphism of k-algebras,

$$\epsilon := exp(\delta) : \mathcal{D}(A) \to \mathcal{D}(A).$$

and in particular, an algebra homomorphism,

$$\tilde{\eta} := exp(\delta) : A \to \mathcal{D}(A),$$

inducing the algebra homomorphisms,

$$\tilde{\eta}_n : A \to \mathcal{D}_n(A),$$

which, by killing, in the right hand algebra, the image of the maximal ideal, $\mathfrak{m}(\underline{t})$, of A corresponding to a point $\underline{t} \in Simp_1(A)$, induces a homomorphism of k-algebras,

$$\tilde{\eta}_n(\underline{t}) : A \to \mathcal{D}_n(A)(\underline{t}) := \mathcal{D}_n/(\mathcal{D}_n\mathfrak{m}(\underline{t})\mathcal{D}_n),$$

and an injective homomorphism,

$$\tilde{\eta}(\underline{t}) : A \to \varprojlim_{n\geq 1} \mathcal{D}_n(A)(\underline{t}).$$

see [La 5]. More generally, let A be a finitely generated k-algebra, and let $\rho : A \to End_k(V)$ be an n-dimensional representation, (e.g. a point of $Simp_n(A)$), corresponding to a two-sided ideal $\mathfrak{m} = ker\rho$ of A. Then $\tilde{\eta}$ induces a homomorphism,

$$\tilde{\eta}(\mathfrak{m}) : A \to \mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D}),$$

and we shall be interested in the image, see §3.

The k-algebras $Ph^n(A)$ are our generalized Jet spaces. In fact, any homomorphism of A-algebras,

$$P_n : Ph^n(A) \to A$$

composed with

$$\delta^n : A \to Ph^n(A),$$

is a usual differential operator of order $\leq n$ on A. Notice also the commutative diagram,

$$\begin{array}{ccccccccc}
A \xrightarrow{d^{n-1}} & Ph^{n-1}(A) & \xrightarrow{d} & Ph^n(A) & \xrightarrow{d} & Ph^{n+1}(A) & \longrightarrow & \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \\
& \mathcal{F}_{p-1} & \xrightarrow{d} & \mathcal{F}_p & \xrightarrow{d} & \mathcal{F}_{p+1} & \xrightarrow{d} & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \\
& \mathcal{F}_{p-1}/\mathcal{F}_p & \xrightarrow{d} & \mathcal{F}_p/\mathcal{F}_{p+1} & \xrightarrow{d} & \mathcal{F}_{p+1}/\mathcal{F}_{p+2} & \xrightarrow{d} & \cdots
\end{array}$$

Here the upper vertical morphisms are injective, the lower line being the sequence of *symbols*.

It is easy to see that the differential operators forms an associative k-algebra, $\mathcal{D}iff(A)$. In fact, given two differential operators,

$$P_m : Ph^m(A) \to A,\ P_n : Ph^n(A) \to A,$$

and consider the functorially defined diagram,

$$\begin{array}{ccccc}
A & \xrightarrow{d^m} & Ph^m(A) & \xrightarrow{d^n} & Ph^{m+n}(A) \\
 & & \downarrow{\scriptstyle P_m} & & \downarrow{\scriptstyle Ph^n(P_m)} \\
 & & A & \xrightarrow{d^n} & Ph^n(A) \\
 & & & & \downarrow{\scriptstyle P_n} \\
 & & & & A
\end{array}$$

then the product is defined as the composition,

$$P_m P_n = Ph^{(n)}(P_m) \circ P_n.$$

Let now V be, as above, a right A-module, with structure morphism $\rho(V) : A \to End_k(V)$. Consider the linear map,,

$$\iota_n := id \otimes (i_1 \circ ... \circ i_n) : V \otimes_A PhA \to V \otimes_A Ph^{n+1},\ n \geq 0.$$

Assume that the non-commutative Kodaira-Spencer class, defined above,

$$c(V) := \kappa(u(V)) \in Ext^1_A(V, V \otimes_A Ph(A)),$$

vanish. Then, as we know, there exist a connection, i.e. a linear map $\nabla_0 \in Hom_k(V, V \otimes_A Ph(A))$ such that $u = \iota(\nabla_0)$.

It is easy to see that this connection induces higher *order connections*, i.e. k-linear maps,

$$\nabla(n) \in Hom_k(V \otimes_A Ph^n(A), V \otimes_A Ph^{n+1}(A)),\ n \geq 0,$$

defined by,

$$\nabla(n)(v \otimes f) = \iota_n(\nabla_0(v))i_0(f) + v \otimes d_n(f).$$

In fact, we just have to prove that $\nabla(n)$ is well defined, i.e. we have to prove that,

$$\nabla(n)(va \otimes f) = \nabla(n)(v \otimes af), \forall\ a \in A,\ f \in Ph^n(A).$$

Noticing that,

$$\iota_n(v \otimes d_0(a)) = v \otimes d_n(a),$$

where we have put, $a := i_0 \circ ... \circ i_0(a)$, we find,

$$\begin{aligned}
\nabla(n)(va \otimes f) &= \iota_n(\nabla_0(va))i_0(f) + va \otimes d_n(f) \\
&= \iota_n(\nabla_0(v)i_0(a) + v \otimes da)i_0(f) + v \otimes ad_n f \\
&= \iota_n(\nabla_0(v))i_0(af) + v \otimes d_n a i_0(f) + v \otimes ad_n f \\
&= \nabla(n)(v \otimes af)
\end{aligned}$$

These higher order connections will induce a diagram,

$$
\begin{array}{ccccccc}
V\otimes_A Ph^{n-1}(A) & \xrightarrow{\nabla(n-1)} & V\otimes_A Ph^{n}(A) & \xrightarrow{\nabla(n)} & V\otimes_A Ph^{n+1}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
V\otimes_A \mathcal{F}_{p-1} & \xrightarrow{\nabla(n-1)} & V\otimes_A \mathcal{F}_{p} & \xrightarrow{\nabla(n)} & V\otimes_A \mathcal{F}_{p+1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
V\otimes_A \mathcal{F}_{p-1}/\mathcal{F}_p & \xrightarrow{\nabla(n-1)} & V\otimes_A \mathcal{F}_{p}/\mathcal{F}_{p+1} & \xrightarrow{\nabla(n)} & V\otimes_A \mathcal{F}_{p+1}/\mathcal{F}_{p+2} & \longrightarrow & \cdots,
\end{array}
$$

where the lower line is the sequence of *symbols*.

Notice that $\nabla^n \in Hom_k(V, V\otimes Ph^n(A))$, as given above, by definition has the property that for any $a\in A$ and any $v\in V$, we have,

$$\nabla^n(va) = \nabla^n(v)a + \nabla^{n-1}(v)da + ... + v\otimes d^n(a).$$

Assume, in particular, that V and the A-module W are free of rank p, q, respectively. Let $\{P_{i,j}\}_{i=1,...,p,j=1,...,q}$ be a family of A-homomorphisms $Ph^n(A)\to A$, defining a *generalized differential operator*,

$$\mathfrak{D} := \begin{pmatrix} P_{1,1} & .. & P_{1,p} \\ ... & & \\ P_{q,1}, & .. & P_{q,p} \end{pmatrix} \circ \nabla^n : V \to W.$$

The solution space of $\mathfrak{D}$, is by definition

$$\mathbf{S}(\mathfrak{D}) := ker\mathfrak{D}.$$

There are natural generalizations of this set-up, which we shall, hopefully, return to in a later paper, extending the classical prolongation-projection method of Elie Cartan to this non-commutative setting. See example (1.4) for the commutative analogue.

In [La 5], we introduced the notion of a *dynamical system* for a k-algebra A, as a two-sided δ-stable ideal $\sigma\subset Ph^\infty(A)$, or equivalently as the corresponding quotient $A(\sigma)$ of the δ-algebra $Ph^\infty(A)$. Any such $A(\sigma)$ will be given in terms of a sequence of ideals,

$$\sigma_n \subset Ph^n(A),\ n\geq 0,$$

with the property that $d(\sigma_n)\subset\sigma_{n+1}$. The *solution space* of such a system, should be considered as the non-commutative scheme parametrized by $A(\sigma)$, i.e. as the geometric system of all simple representations of $A(\sigma)$, see [La 3].

This is, in a sense, dual to the classical theory of PDE's, as we shall show by considering the following example, leaving the general situation to the hypothetical paper referred to above.

Example 1.4. (i) Let $A = k[t_1, t_2, ..., t_n]$, and consider the situation, corresponding to a free particle, see [La 5], where in $A(\sigma)$ we have killed d^2t_i, for every

$i = 1, 2, ..., n$, then the commutativization $A(\sigma)_k^{com}$ of $A(\sigma)_k := Ph(A)/\mathcal{F}_{k+1}$ is a free A-module generated by the basis,

$$\{dt_{i_1} dt_{i_2} ... dt_{i_r}\}_{i_1 \leq i_2 \leq ... \leq i_r,\ r \leq k}.$$

Put $|\underline{i}| = r$, if $\underline{i} = \{i_1, i_2, ..., i_r\}$. The dual basis,

$$\{p_{\underline{i}}\}_{i_1 \leq i_2 \leq ... \leq i_r,\ r \leq k}$$

may be identified with a basis $D_{\underline{i}}$ of the A-module of all (classical higher-order) differential operators of order less or equal to k. In fact, for $f \in A$ we have,

$$p_{\underline{i}}(\tilde{\eta}(f)) = \frac{1}{\mu_1! \mu_2! ... \mu_s!} D_{j_1}^{\mu_1} D_{j_2}^{\mu_2} ... D_{j_r}^{\mu_r}(f),$$

where we assume $j_1 = i_1 = i_2 = ... = i_{\mu_1} < j_2 = i_{\mu_1+1} = i_{\mu_1+2} = ... = i_{\mu_1+\mu_2} < ... < i_{\mu_1+...+\mu_s} = j_r$, and where $D_{i_p}^{\mu_p}$ is μ_p'th order derivation with respect to t_{i_p}. If $\{i_1, i_2, ..., i_r\} = \emptyset$ we let $D_{\underline{i}}$ be the identity operator on A.

Now, consider the commutativization of $A(\sigma)$, as a k-linear space, and for every $k \geq 1$,

$$\mathcal{E}_k := A(\sigma)_k^{com}$$

as a family of affine spaces fibered over $Simp_1(A)$,

$$\pi_k : \mathcal{E}_k \to Spec(A).$$

This family is defined by the homomorphism of k-algebras,

$$A \to \mathcal{O}(\mathcal{E}_k) := A[p_{\underline{i}}],\ |\underline{i}] \leq k.$$

Let $P_q(\underline{t}, p_{\underline{i}}) \in \mathcal{O}(\mathcal{E}_{k_q}),\ q = 1, ..., d$, then the system of equations,

$$P_q = 0,\ q = 1, ..., d,$$

is a system of partial differential equations, a SPDE, for short. Suppose there is a *solution*, i.e. an $f \in A$, such that,

$$P_q(D_{\underline{i}}(f)) = 0,\ q = 1, ..., d,$$

then, for every j, we must have

$$D_j(P_q(D_{\underline{i}}(f))) = 0,$$

which amounts to extending the SPDE, by including together with $P \in \mathcal{O}(\mathcal{E}_{k_q})$, the polynomials,

$$D_j P := \frac{\partial P}{\partial t_j} + \sum_{\underline{i}} \frac{\partial P}{\partial p_{\underline{i}}}\ p_{\underline{i}+j} \in \mathcal{O}(\mathcal{E}_{k_q+1})$$

where it should be clear how to interpret the indices. Let us denote by $\mathcal{P}$ the extended family of plynomials,

$$\{D_{j_l} ... D_{j_1} P_q | P_q \in \mathcal{O}(\mathcal{E}_{k_q})\}_{j_l, q \geq 0}$$

and let

$$\mathfrak{p}_m \subset O(\mathcal{E}_m)$$

be the ideal, generated by the polynomials in $\mathcal{P}$, contained in $O(\mathcal{E}_m)$. Denote by,

$$S_m := S_m(\mathcal{P}) \subset \mathcal{E}_m$$

the corresponding subvariety. Clearly, the canonical map,

$$\mathcal{E}_{m+l} \to \mathcal{E}_m$$

induced by the trivial derivation of $Ph^m(A)$, has a canonical restriction,

$$pl_l : S_{m+l} \to S_m.$$

Denote also by,

$$\pi_k : S_k \to Spec(A)$$

the restriction,of the morphism, $\pi_k : \mathcal{E}_m \to Spec(A)$ defined above, to S_k. Classically, the system is called regular, if all π_k are fibre bundles, so smooth, for $k \geq 1$. Now, for any closed point of $Spec(A)$, i.e. for any point $\underline{t} \in Simp_1(A)$, consider the sequence of fibers over $\underline{t}$, and the corresponding sequence of maps,

$$pl_1(\underline{t}) : S_{m+1}(\underline{t}) \to S_m(\underline{t}).$$

An element

$$\tilde{f} \in \varprojlim_m S_m(\underline{t})$$

corresponds exactely to an element $\tilde{f} \in \hat{A}_{\underline{t}}$, for which,

$$P_q(D_{\underline{i}}(\hat{f})) = 0, \ q = 1, ..., d,$$

i.e. to a formal solution of the SPDE. Thus, the projective limit of schemes,

$$\mathcal{S}(\mathcal{P})(\underline{t}) := \varprojlim_m S_m(\underline{t}),$$

is the space of formal solutions of the SPDE at $\underline{t} \in Simp_1(A)$.

A fundamental problem in the classical theory of PDE is then the following:

Find necessary and sufficient conditions on the SPDE $\{P_q\}_{q=1,...,d}$ for $\mathcal{S}(\mathcal{P})(\underline{t})$ to be non-empty, and find, based on $\{P_l\}_l$, its structure. In particular, compute its dimension, $\sigma(\underline{t})$.

We shall not, here, venture into this vast theory, but just add one remark. The solution space is in fact a family, with parameter-space $Simp_1(A)$. Given any point $\underline{t} \in Simp_1(A)$ the (formal) scheme, $\mathcal{S}(\mathcal{P})(\underline{t})$ of formal solutions may have deformations. We might want to compute the formal moduli $\underline{H}$, and relate the given family to the corresponding mini-versal family.

The tangent space of $\underline{H}$ is given as,

$$A^1(k, \mathcal{O}(\mathcal{S}(\mathcal{P})(\underline{t}), \mathcal{O}(\mathcal{S}(\mathcal{P})(\underline{t}))) = Hom_{\mathcal{O}(\mathcal{E})(\underline{t})}(\mathfrak{p}(\underline{t}), \mathcal{O}(\mathcal{S}(\mathcal{P})(\underline{t}))/Der,$$

see [La 0]. A tangent at the point $\underline{t}$ of $Simp_1(A)$ is the value at $\underline{t}$, of a linear combinations of the fundamental vectorfields, the derivations $\{D_j\}$ of A. The map between the tangent space of the given family and the tangent space of H, is then easily seen to be the following,

$$\eta : T_{Simp_1(A),\underline{t}} \to Hom_{\mathcal{O}(\mathcal{E})(\underline{t})))}(\mathfrak{p}(\underline{t}), \mathcal{O}(\mathcal{S}(\mathcal{P})(\underline{t}))))/Der,$$

where, $\eta(D_j)$ is the class of the map, associating a $P \in \mathfrak{p}$ to the class at $\underline{t}$ of $D_j(P)$. The image of the tangent at $\underline{t}$ of $Simp_1(A)$, corresponding to D_j, in the tangent space of H, is zero if this map is a derivation. Now, this is exactly what we have arranged, by together with any $P \in \mathfrak{p}$, also including

$$D_j P := \frac{\partial P}{\partial t_j} + \sum_{\underline{i}} \frac{\partial P}{\partial p_{\underline{i}}}\, p_{\underline{i}+j},$$

in the ideal $\mathfrak{p}$. Thus the map η is trivial, and the given pro-family is formally constant, as one maybe should have suspected! Moreover, it is easy to see that, if

$$pl_1 : S_{k+1}(\underline{t}) \to S_k(\underline{t})$$

has a local section, then,

$$\pi_k : S_k \to Spec(A)$$

is formally constant at $\underline{t} \in Spec(A)$. The basic problem is to find computable conditions under which the constancy of π_k implies the surjectivity of pl_1, and thereby the non-triviality of $\mathcal{S}(\mathcal{P})(\underline{t}))$.

We shall, hopefully, come back to these questions in a later paper.

(ii) Let $A = k[t]/(t^2)$, then

$$Ph(A) = k < t, dt > /(t^2, tdt + dtt),$$
$$Ph^{(2)}(A) = k < t, dt, d^2t > /(t^2, tdt + dtt, td^2t + 2dt^2 + d^2tt)$$
$$Ph^{\infty}(A) = k < t, dt, ..., d^nt, .. > /(t^2, tdt + dtt, ..., td^nt + ndtd^{n-1}t + ... + d^ntt, ...),$$

and it is easy to see that $\eta(t) = \sum_n 1/n!d^n(t)$ is non-zero in $\mathcal{D}/(\mathcal{D}\ (t)\mathcal{D})$, and, of course, $\eta(t)^2 = 0$. In particular, there is a homomorphism onto,

$$\mathcal{D}/(\mathcal{D}(t)\mathcal{D}) \to k[dt]/(dt^2) \simeq A$$

(iii) Let now $A = k[x, y]/(x^3 - y^2)$, and compute $\mathcal{D}$, and see that $dy^2 = 0$ in $\mathcal{D}/(\mathcal{D}(t)\mathcal{D})$, so that there is no natural surjectiv homomorphism,

$$\mathcal{D}/(\mathcal{D}(t)\mathcal{D}) \to A,$$

even though the map

$$\tilde{\eta} := exp(\delta) : A \to \mathcal{D}$$

is injective. The difference between examples (i) and (ii) is, of course, due to the fact that in the first case A is graded, and in the second it is not. See §3.

Remark 1.5. Preparation of a representation: Given a representation V of A, $\rho : A \to End_k(V)$, an extension of ρ to $Ph^{(n)}(A) \to End_k(V)$ corresponds to fixing the first n higher order "momenta of the point V", in the miniversal deformation space. In particular, $Ph^\infty(A)$ parametrizes, in this way, representations of A with a momentum, and any number of *higher order momenta*. If V_δ is any $Ph^\infty(A)$-module, notice, that, as above, the kernel of the natural map,

$$Ext^1_{Ph^\infty(A)}(V_\delta, V_\delta) \to Ext^1_A(V, V),$$

induced by the linear map,

$$Der_k(Ph^\infty(A), End_k(V_\delta)) \to Der_k(A, End_k(V))$$

is the quotient,

$$Der_A(Ph^\infty(A), End_k(V_\delta))/End_A(V),$$

and that the image is a subspace $[\delta_\rho]^\perp \subseteq Ext^1_A(V, V))$, which is easily computable.

§2. Non-commutative deformations of families of modules.

In [La 2], [La 3] and [La 4], we introduced non-commutative deformations of families of modules of non-commutative k-algebras, and the notion of *swarm* of right modules (or more generally of objects in a k-linear abelian category). Let $\underline{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R. The objects are the diagrams of k-algebras,

$$k^r \xrightarrow{\iota} R \xrightarrow{\pi} k^r$$

such that the composition of ι and π is the identity. Any such *r-pointed k-algebra* R is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of R is the bilateral ideal $Rad(R) := ker\pi$, such that $R/Rad(R) \simeq k^r$. The dual k-vector space of $Rad(R)/Rad(R)^2$ is called the tangent space of R.

For $r = 1$, there is an obvious inclusion of categories

$$\underline{l} \subseteq \underline{a}_1$$

where $\underline{l}$, as usual, denotes the category of commutative local Artinian k-algebras with residue field k.

Fix a (not necessarily commutative) associative k-algebra A and consider a right A-module M. The ordinary deformation functor

$$Def_M : \underline{l} \to \underline{Sets}$$

is then defined. Assuming $Ext^i_A(M, M)$ has finite k-dimension for $i = 1, 2$, it is well known, see [Sch], or [La 2], that Def_M has a pro-representing hull H, *the formal moduli of* M. Moreover, the tangent space of H is isomorphic to $Ext^1_A(M, M)$, and H can be computed in terms of $Ext^i_A(M, M)$, $i = 1, 2$ and their *matric* Massey products, see [La 2].

In the general case, consider a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of right A-modules. Assume that,

$$dim_k Ext^1_A(V_i, V_j) < \infty.$$

Any such family of A-modules will be called a *swarm*. We shall define a deformation functor,

$$Def_{\mathcal{V}} : \underline{a}_r \to \underline{Sets}$$

generalizing the functor Def_M above. Given an object $\pi : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the k-vector space and left R-module $(R_{i,j} \otimes_k V_j)$. It is easy to see that $End_R((R_{i,j} \otimes_k V_j)) \simeq (R_{i,j} \otimes_k Hom_k(V_i, V_j))$. Clearly π defines a k-linear and left R-linear map,

$$\pi(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i,$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\pi}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras, $\eta_0 : A \to \oplus_{i=1}^r End_k(V_i) \subset (Hom_k(V_i, V_j)) =: End_k(V)$. Notice that this homomorphism also provides each $Hom_k(V_i, V_j)$ with an A- bimodule structure. Let

$$Def_{\mathcal{V}}(R) \in \underline{Sets}$$

be the set of isoclasses of homomorphisms of k-algebras,

$$\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

such that,

$$\tilde{\pi}(R) \circ \eta' = \eta_0,$$

where the equivalence relation is defined by inner automorphisms in the k-algebra $(R_{i,j} \otimes_k Hom_k(V_i, V_j))$ inducing the identity on $\oplus_{i=1}^r End_k(V_i)$. One easily proves that $Def_{\mathcal{V}}$ has the same properties as the ordinary deformation functor and we prove the following, see [La 2]:

Theorem 2.1. *The functor $Def_{\mathcal{V}}$ has a pro-representable hull, i.e. an object H of the category of pro-objects $\hat{\underline{a}}_r$ of $\underline{a}_r$, together with a versal family,*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathcal{V}}(H/\mathfrak{m}^n),$$

where $\mathfrak{m} = Rad(H)$, such that the corresponding morphism of functors on $\underline{a}_r$,

$$\kappa : Mor(H, -) \to Def_{\mathcal{V}}$$

defined for $\phi \in Mor(H, R)$ by $\kappa(\phi) = R \otimes_\phi \tilde{V}$, is smooth, and an isomorphism on the tangent level. Moreover, H is uniquely determined by a set of matric Massey products defined on subspaces,

$$D(n) \subseteq Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{n-1}}, V_k),$$

with values in $Ext^2(V_i, V_k)$.

The right action of A on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j)),$$

and the k-algebra $O(\mathcal{V})$ acts on the family of A-modules $\mathcal{V} = \{V_i\}$, extending the action of A. If $dim_k V_i < \infty$, for all $i = 1, ..., r$, the operation of associating $(O(\mathcal{V}), \mathcal{V})$ to $(A, \mathcal{V})$ turns out to be a closure operation.

Moreover, we prove the crucial result,

A generalized Burnside theorem 2.2. *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of all simple A-modules, then*

$$\eta : A \longrightarrow O(\mathcal{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

is an isomorphism.

We also prove that there exists, in the non-commutative deformation theory, an obvious analogy to the notion of pro-representing (modular) substratum H_0 of the formal moduli H, see [La 1]. The tangent space of H_0 is determined by a family of subspaces

$$Ext^1_0(V_i, V_j) \subseteq Ext^1_A(V_i, V_j), \qquad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathcal{V}$, and by a subspace,

$$T_0(\Delta) \subseteq \prod_i Ext^1_A(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\mathcal{V} = \{V_i\}_{i=1}^r$, see [La 1]. If $\mathcal{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some Artinian k-algebra A, we show that the above notion of *almost split sequence* coincides with that of Auslander, see [R].

Using this we consider, in [La 2], the general problem of classification of iterated extensions of a family of modules $\mathcal{V} = \{V_i, \}_{i=1}^r$, and the corresponding classification of filtered modules with graded components in the family $\mathcal{V}$, and extension type given by a directed representation graph Γ. The main result is the following, see [La 4],

Proposition 2.3. *Let A be any k-algebra, $\mathcal{V} = \{V_i\}_{i=1}^r$ any swarm of A-modules, i.e. such that,*

$$\dim_k Ext^1_A(V_i, V_j) < \infty \quad \textit{for all } i, j = 1, \dots, r.$$

(i): Consider an iterated extension E of $\mathcal{V}$, with representation graph Γ. Then there exists a morphism of k-algebras

$$\phi : H(\mathcal{V}) \to k[\Gamma]$$

such that

$$E \simeq k[\Gamma] \otimes_\phi \tilde{V}$$

as right A-algebras.

(ii): The set of equivalence classes of iterated extensions of $\mathcal{V}$ with representation graph Γ, is a quotient of the set of closed points of the affine algebraic variety

$$\underline{A}[\Gamma] = Mor(H(\mathcal{V}), k[\Gamma])$$

(iii): There is a versal family $\tilde{V}[\Gamma]$ of A-modules defined on $\underline{A}[\Gamma]$, containing as fibers all the isomorphism classes of iterated extensions of $\mathcal{V}$ with representation graph Γ.

To any, not necessarily finite, swarm $\underline{c} \subset \underline{mod}(A)$ of right-A-modules, we have associated two associative k-algebras, see [La 3] and [La 4], $O(|\underline{c}|, \pi) = \varprojlim_{\mathcal{V} \subset |\underline{c}|} O(\mathcal{V})$, and a sub-quotient $\mathcal{O}_\pi(\underline{c})$, together with natural k-algebra homomorphisms,

$$\eta(|\underline{c}|) : A \longrightarrow O(|\underline{c}|, \pi)$$

and,

$$\eta(\underline{c}) : A \longrightarrow \mathcal{O}_\pi(\underline{c})$$

with the property that the A-module structure on $\underline{c}$ is extended to an $\mathcal{O}$-module structure in an optimal way. We then defined an *affine non-commutative scheme* of right A-modules to be a swarm $\underline{c}$ of right A-modules, such that $\eta(\underline{c})$ is an isomorphism. In particular we considered, for finitely generated k-algebras, the swarm $Simp^*_{<\infty}(A)$ consisting of the finite dimensional simple A-modules, and the *generic* point A, together with all morphisms between them. The fact that this is a swarm, i.e. that for all objects $V_i, V_j \in Simp_{<\infty}$ we have $dim_k Ext^1_A(V_i, V_j) < \infty$, is easily proved. We have in [La 4] proved the following result, (see (4.1), loc.cit. for the definition of the notion of *geometric* k-algebra, and compare with Lemma (2.5).)

Proposition 2.4. *Let A be a geometric k-algebra, then the natural homomorphism,*

$$\eta(Simp^*(A)) : A \longrightarrow \mathcal{O}_\pi(Simp^*_{<\infty}(A))$$

is an isomorphism, i.e. $Simp^*_{<\infty}(A)$ is a scheme for A.

In particular, $Simp^*_{<\infty}(k < x_1, x_2, ..., x_d >)$, is a scheme for $k < x_1, x_2, ..., x_d >$. To analyze the local structure of $Simp_n(A)$, we need the following, see [La 4],(3.23),

Lemma 2.5. *Let $\mathcal{V} = \{V_i\}_{i=1,..,r}$ be a finite subset of $Simp_{<\infty}(A)$, then the morphism of k-algebras,*

$$A \to O(\mathcal{V}) = (H_{i,j} \otimes_k Hom_k(V_i, V_j))$$

is topologically surjective.

Proof. Since the simple modules $V_i,\ i = 1, .., r$ are distinct, there is an obvious surjection, $\eta_0 : A \to \prod_{i=1,..,r} End_k(V_i)$. Put $\mathfrak{r} = ker\eta_0$, and consider for $m \geq 2$ the finite-dimensional k-algebra, $B := A/\mathfrak{r}^m$. Clearly $Simp(B) = \mathcal{V}$, so that by the generalized Burnside theorem, see [La 2], (2.6), we find, $B \simeq O^B(\mathcal{V}) := (H^B_{i,j} \otimes_k Hom_k(V_i, V_j))$. Consider the commutative diagram,

$$\begin{array}{ccccc} A & \longrightarrow & (H^A_{i,j} \otimes_k Hom_k(V_i, V_j)) =: O^A(\mathcal{V}) & & \\ \downarrow & & \downarrow & \searrow & \\ B & \longrightarrow & (H^B_{i,j} \otimes_k Hom_k(V_i, V_j)) & \xrightarrow{\alpha} & O^A(\mathcal{V})/\mathfrak{m}^m \end{array}$$

where all morphisms are natural. In particular α exists since $B = A/\mathfrak{r}^m$ maps into $O^A(\mathcal{V})/rad^m$, and therefore induces the morphism α commuting with the rest of the morphisms. Consequently α has to be surjective, and we have proved the contention.

□

Example 2.6. As an example of what may occur in rank infinity, we shall consider the invariant problem, $\mathbf{A}^1_{\mathbf{C}}/\mathbf{C}^*$. Here we are talking about the algebra $A = \mathbf{C}[x](\mathbf{C}^*)$ crossed product of $\mathbf{C}[x]$ with the group $\mathbf{C}^*$. If $\lambda \in \mathbf{C}^*$, the product in A is given by $x \times \lambda = \lambda \times \lambda^{-1}x$. There are two "points", i.e. orbits, modeled by the obvious origin $V_0 := A \to End_{\mathbf{C}}(\mathbf{C}(0))$, and by $V_1 := A \to End_{\mathbf{C}}(\mathbf{C}[x, x^{-1}])$. We may also choose the two points $V_0 := \mathbf{C}(0), V_1 := \mathbf{C}[x]$, in line with the definitions of [La 3]. Obviously $\mathbf{C}[x]$ correspond to the closure of the orbit $\mathbf{C}[x, x^{-1}]$. This choice is the best if one want to make visible the adjacencies in the quotient, and we shall therefore treat both cases.

We need to compute,

$$Ext^p_A(V_i, V_j), p = 1, 2,\ i, j = 1, 2.$$

Now,

$$Ext^1_A(V_i, V_j) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_i, V_j))/Triv,\ i, j = 1, 2,$$

and since x acts as zero on V_1, and $\mathbf{C}^*$ acts as identity on V_1 and as homogenous multiplication on V_0, we find,

$$Der_k(A, Hom_k(V_0, V_0))/Triv, = Der_k(A, Hom_k(V_0, V_0)) = Der_{\mathbf{C}}(A, \mathbf{C}(0)).$$

Any $\delta \in Der_k(A, \mathbf{C}(0))$, is determined by its values, $\delta(x), \delta(\lambda) \in \mathbf{C}(0))|\lambda \in \mathbf{C}^*$. Moreover since in A we have, $(\lambda) \times (\lambda^{-1}x) = x \times (\lambda)$, we find,

$$\begin{aligned} \delta(\lambda\mu) =& \delta(\lambda) + \delta(\mu) \\ \delta((\lambda) \times (\lambda^{-1}x)) =& \delta(x \times (\lambda)). \end{aligned}$$

The left hand side of the last equation is $\delta((\lambda^{-1}x)) = \lambda^{-1}\delta(x)$, and the right hand side is $\delta(x)$, and since this must hold for all $\lambda \in \mathbf{C}^*$, we must have $\delta(x) = 0$. Moreover, since $\delta(\lambda\mu) = \delta(\lambda) + \delta(\mu)$, it is clear that continuity of δ, implies that δ must be equal to $\alpha ln(|.|)$, for some $\alpha \in \mathbf{C}$. (To simplify the writing, we shall put $log := ln(|.|)$.)Therefore,

$$Ext^1_A(V_0, V_0) = Der_k(A, Hom_{\mathbf{C}}(V_0, V_0)) = \mathbf{C}$$

The cup-product of this class, $log \cup log$, sits in $HH^2(A, \mathbf{C}(0)) = Ext^2_a(V_0, V_0)$, and is given by the 2-cocycle,

$$(\lambda, \mu) \to log(\lambda) \times log(\mu).$$

This is seen to be a boundary, i.e. there exist a map $\psi : \mathbf{C}^* \to \mathbf{C}(0)$, such that for all, $\lambda, \mu \in \mathbf{C}^*$ we have,

$$log(\lambda) \times log(\mu) = \psi(\lambda) - \psi(\lambda\mu) + \psi(\mu).$$

Just put $\psi_{1,1} := \psi_2 = -1/2log^2$. Therefore the cup product is zero, and if we, in general, put

$$\psi_n := \psi_{1,1,\ldots,1} = (-)^{n+1}1/(n!)log^n,\ n \geq 1$$

where n is the number of 1's in the first index, then computing the Massey products of the element $log \in Ext^1_A(V_0, V_0)$, we find the n'th. Massey product,

$$[log, log, ..., log] = \{(\lambda, \mu) \to \sum_{p=1,...,n-1} \psi_p \psi_{n-p},$$

and this is easily seen to be the boundary of the 1-cochain

$$\psi_{n+1} = (-)^{n+2} 1/((n+1)!) log^{n+1}.$$

Therefore all Massey products are zero. Of course, we have not yet proved that they could be different from zero, i.e. we have not computed the *obstruction*-group $Ext^2_A(V_0, V_0)$ and found it non-trivial! Now this is unnecessary.

Now, assume first $V_0 = \mathbf{C}[x, x^{-1}]$, then any,

$$\delta \in Ext^1_A(V_0, V_0) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_0, V_0))/Triv$$

is determined by the values of $\delta(x)$ and $\delta(\lambda),\ \lambda \in \mathbf{C}^*$. Since $Ext^1_{\mathbf{C}[x]}(V_0, V_1) = 0$, we may find a trivial derivation such that subtracting from δ we may assume $\delta(x) = 0$. But then the formula,

$$\delta(x \times \lambda) = \delta(\lambda \times (\lambda^{-1} x))$$

implies,

$$x\delta(\lambda) = \delta(\lambda)(\lambda^{-1} x),$$

from which it follows,

$$\delta(\lambda)(x^p) = (\lambda^{-1} x)^p \delta(\lambda)(1).$$

Now, since, $\lambda\mu = \mu\lambda$ in $\mathbf{C}^*$, we find,

$$(\lambda^{-1}\mu x)^p \delta(\lambda)(1)(\mu x). = (\lambda\mu^{-1} x)^p \delta(\lambda)(1)(\lambda x),$$

which should hold for any pair of $\mu, \lambda \in \mathbf{C}^*$, and any p. This obviously implies $\delta = 0$.

This argument shows not only that ,

$$Ext^1_A(V_1, V_1) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_1, V_1))/Triv = 0$$

when $V_1 = \mathbf{C}[x, x^{-1}]$ but also when $V_1 = \mathbf{C}[x]$. Finally we find that the formula above,

$$x\delta(\lambda) = \delta(\lambda)(\lambda^{-1} x),$$

shows that for,

$$\delta \in Ext^1_A(V_1, V_0) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_1, V_0))/Triv$$

$\delta(\lambda)(xx^p) = 0$ for all p. Therefore,

$$Ext^1_A(V_1, V_0) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_1, V_0))/Triv = 0$$

when $V_1 = \mathbf{C}[x, x^{-1}]$. However, when $V_1 = \mathbf{C}[x]$, we find that δ with, $\delta(\lambda)(1) \neq 0$ and with $\delta(\lambda)(x^p) = 0$ for $p \geq 1$, survives. These will, as above give rise to a logarithm of the real part of $\mathbf{C}^*$. Therefore, in this case,

$$Ext^1_A(V_1, V_0) = \mathbf{C}.$$

The miniversal families look like,

$$H = \begin{pmatrix} \mathbf{C}[[t]] & 0 \\ 0 & \mathbf{C} \end{pmatrix}$$

when $V_1 = \mathbf{C}[x, x^{-1}]$, and like

$$H = \begin{pmatrix} \mathbf{C}[[t]] & 0 \\ <\mathbf{C}> & \mathbf{C} \end{pmatrix}$$

when $V_1 = \mathbf{C}[x]$.

Localization and topology. Let $s \in A$, and consider the *open subset* $D(s) = \{V \in Simp(A) |\ \rho(s)$ invertible in $End_k(V)\}$. The Jacobson topology on $Simp(A)$ is the topology with basis $\{D(s)|\ s \in A\}$. It is clear that the natural morphism,

$$\eta : A \to O(D(s), \pi)$$

maps s into an invertible element of $O(D(s), \pi)$. Therefore we may define the localization $A_{\{s\}}$ of A, as the k-algebra generated in $O(D(s), \pi)$ by $im\,\eta$ and the inverse of $\eta(s)$. This furnishes a general method of localization with all the properties one would wish. And in this way we also find a canonical (pre)sheaf, $\mathcal{O}$ defined on $Simp(A)$.

Definition 2.7. *When the k-algebra A is geometric, such that $Simp^*(A)$ is a scheme for A, we shall refer to the presheaf $\mathcal{O}$, defined above on the Jacobson topology, as the structure presheaf of the scheme $Simp(A)$.*

§3. The infinite phase space construction and Massey products.

Let, as above $\mathcal{V} = \{V_i\}_{i=1,\ldots,r}$ be a family of A-modules. To compute the relevant cohomology for the deformation theory, i.e. the $Ext^*_A(V_i, V_j)$, we may use the Leray spectral sequence of [La 0], together with the formulas,

$$\begin{aligned} Ext^n_A(V_i, V_j) =& HH^{n+1}(k, A; Hom_k(V_i, V_j)) \\ HH^{n+1}(k, A; W) =& \varprojlim_{\underline{Free}/A} {}^{(n)} Der_k(-, W),\ n > 0, \end{aligned}$$

where W is any A-bimodule. Choose a surjective morphism $\mu : F \to A$ of a free k-algebra F onto A, and put $I = ker\mu$, then,

$$Ext^2_A(V_i, V_j) = Hom_A(I/I^2, Hom_k(V_i, V_j))/Der,$$

where Der is the restrictions of the derivations, $Der_k(F, Hom_k(V_i, V_j))$, to I.

Now, let $\{\psi_{i,j}(l) \in Der_k(A; V_i, V_j)\}_{l=1,\ldots,d_{i,j}}$ represent a basis of $Ext^1_A(V_i, V_j)$, and let $E_{i,j} := \{t_{i,j}(l)\}_{l=1,\ldots,d_{i,j}}$ denote the dual basis. Consider, the free matrix k-algebra (quiver) $(T^1_{i,j})$, generated in slot (i,j) by the (formal) elements of $E_{i,j}$. There is a unique homomorphism,

$$\pi : T^1 := (T_{i,j}) \to \begin{pmatrix} k & 0 & .. & 0 \\ 0 & k & .. & 0 \\ . & . & .. & 0 \\ 0 & . & .. & k \end{pmatrix}.$$

Denote by the same letter the completion of T^1 with respect to the powers of the radical, $Rad(T^1) := ker\pi$. Then $T^1 \in \underline{\hat{a}}_r$. Consider the k-algebra, and the π-induced homomorphism,

$$\pi_1 : (T^1_{i,j} \otimes_k Hom_k(V_i, V_j)) \to (Hom_k(V_i, V_i)).$$

Clearly π_1 splits, and it is easy to see that,

$$\xi : A \to (T^1_{i,j} \otimes Hom_k(V_i, V_j))$$

defined by,

$$\xi = \sum_{i,j,l} t_{i,j}(l)\psi_{i,j}(l),$$

is a derivation, therefore inducing a unique homomorphism, $\tilde{\rho}_1$, making the following diagram commute,

$$\begin{array}{ccccc} A & \xrightarrow{id+d} & \mathcal{D}_1(A) & \longleftarrow & \mathcal{D}_2(A) \\ \downarrow{\scriptstyle \rho} & \searrow^{\tilde{\rho}_1} & \downarrow{\scriptstyle \rho_1} & & \\ (Hom_k(V_i, V_i)) & \longleftarrow & ((T^1/Rad^2(T^1))_{i,j} \otimes_k Hom_k(V_i, V_j)) & & \end{array}$$

Now we want to extend this diagram, completing it with commuting homomorphisms,

$$\begin{array}{ccc} A & \xrightarrow{id+d+1/2d^2} & \mathcal{D}_2(A) \\ \downarrow{\scriptstyle \tilde{\rho}_1} & & \downarrow{\scriptstyle \rho_2} \\ ((T^1/Rad^2(T^1))_{i,j} \otimes_k Hom_k(V_i, V_j)) & \longleftarrow & ((T^1/Rad^3(T^1))_{i,j} \otimes_k Hom_k(V_i, V_j)) \end{array}$$

There is an obstruction for this, which is easily computed, providing us with a nice way of computing cup-products $Ext^1_A(V_i, V_j) \otimes Ext^1_A(V_i, V_j) \to Ext^2_A(V_i, V_j)$. However it will be clear in the next construction, that the obvious continuation of this procedure does not work. In fact, the formalized higher differentials $\mathcal{D}(A)$ is not really the natural phase-space to work with for all purposes. In an obvious sense it is too homogenous. We are therefore led to the construction of a kind of *projective resolution* of A. Consider as above a surjective homomorphism, $\mu : F \to A$, with $F = k < x_1, x_2, ..., x_s >$ a free k-algebra, and $I = ker\mu$. Obviously $Ph^{(p)}(F)$ for $p \geq 1$, are also free, and $Ph^{(p+1)}(F)$ is a free $Ph^{(p)}(F)$-algebra. Let,

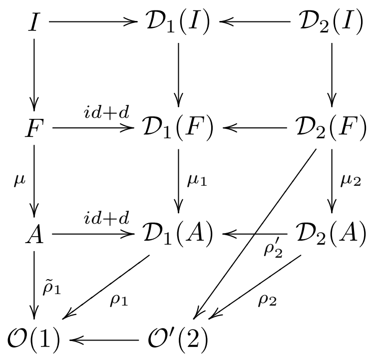
$exp(\delta) : F \to \mathcal{D}(F)$ be defined as in §1, $exp(\delta) = id + d + 1/2d^2 + ..$, and denote by $\eta_p : F \to \mathcal{D}_p(F)$ the induced homomorphism. Define,

$$\mathcal{H}_p := \mathcal{D}_p(F)/(i_0(I), \eta_p(I)).$$

Clearly, $\mathcal{H}_p = \mathcal{D}_p(A)$, for $p = 0, 1..$ For $p \geq 2$ there are only natural surjective homomorphisms,

$$\kappa_p : \mathcal{H}_p \to \mathcal{D}_p(A).$$

By functoriality, the diagram above induces another commutative diagram, which, may be completed to the commutative diagram, ρ_2 not yet included,



where we, in expectation of later constructions, have put,

$$\begin{aligned} H(1) =& T^1/Rad(T^1)^2 \\ H'(2) =& T^1/Rad(T^1)^3 \\ \mathcal{O}(n) :=& (H(n)_{i,j} \otimes_k Hom_k(V_i, V_j)), n \geq 1. \\ \mathcal{O}'(n) :=& (H'(n)_{i,j} \otimes_k Hom_k(V_i, V_j)), n \geq 2. \end{aligned}$$

Now the map $(id + d) \circ \mu_1 \circ \rho_1 : I \to (T^1_{i,j} \otimes_k Hom_k(V_i, V_j))$ is zero, and the map $\tilde{\rho}_1 : A \to (T^1/Rad(T^1)^2)_{i,j} \otimes_k Hom_k(V_i, V_j))$ is, as deformation of the family $\mathcal{V}$, the universal family at the tangent level. We want a homomorphism ρ_2' inducing a homomorphism,

$$\rho_2 : \mathcal{D}_2(A) \to (T^1/Rad(T^1)^3)_{i,j} \otimes_k Hom_k(V_i, V_j))$$

which produces a homomorphism,

$$\tilde{\rho}_2 := exp(d) \circ \rho_2 : A \to (T^1/Rad(T^1)^3)_{i,j} \otimes_k Hom_k(V_i, V_j)),$$

lifting $\tilde{\rho}_1$. This last condition implies that the image of $d \circ d \circ \rho_2'$ must sit in $(Rad(T^1)^2/Rad(T^1)^3)_{i,j} \otimes_k Hom_k(V_i, V_j))$. The map,

$$d \circ d \circ \rho_2' : I \to (Rad(T^1)^2/Rad(T^1)^3)_{i,j} \otimes_k Hom_k(V_i, V_j)),$$

is, of course not necessarily trivial. It is easily seen to be F-linear, both from left and right, and so it induces an element,

$$o_2 \in ((Rad(T^1)^2/Rad(T^1)^3)_{i,j} \otimes_k Ext^2_A(V_i, V_j))$$

independent upon the choice of extension ρ_2'. In fact if ρ_2'' is another extension of ρ_1, then since,

$$\begin{aligned}\rho_2''(d^2(fg)) &= \rho_2''(d^2fg + 2dfdg + fd^2g)\\ &= \rho_2''(d^2f)\rho_1(g) + 2\rho_1(df)\rho_1(dg) + \rho_1(f)\rho_2''(d^2g)\end{aligned}$$

and,

$$\begin{aligned}\rho_2'(d^2(fg)) &= \rho_2'(d^2fg + 2dfdg + fd^2g)\\ &= \rho_2'(d^2f)\rho_1(g) + 2\rho_1(df)\rho_1(dg) + \rho_1(f)\rho_2'(d^2g),\end{aligned}$$

the difference is a derivation,

$$d \circ d \circ (\rho_2'' - \rho_2') \in Der_k(F, ((Rad(T^1)^2/Rad(T^1)^3)_{i,j} \otimes_k Hom_k(V_i, V_j))).$$

Now

$$((Rad(T^1)^2/Rad(T^1)^3)_{i,j} \otimes_k Ext_A^2(V_i, V_j))$$

may be identified with,

$$Hom_k((Ext_A^2(V_i, V_j)^*), (Rad(T^1)^2/Rad(H)^3)_{i,j}))$$

which is a subspace of,

$$Mor_{\underline{a}_r}(k^r \oplus (Ext_A^2(V_i, V_j)^*), T^1/Rad(T^1)^3).$$

Denote by T^2 the free matrix algebra (quiver), in $\hat{\underline{a}}_r$, generated by $Ext_A^2(V_i, V_j)^*$, just like the construction of T^1 above, such that,

$$T^2/Rad(T^2)^2 = k^r \oplus (Ext_A^2(V_i, V_j)^*).$$

We may now state and prove the main result of this paper,

Theorem 3.1. *(i) For any finite family of (finite dimensional) A-modules,* $\mathcal{V} := \{V_i\}_{i=1,..,r}$, *there is a homomorphism* $\tilde{\rho}$, *making the following diagram commutative*

$$\begin{array}{ccc} A & \xrightarrow{exp(\delta)} & \mathcal{H} \\ \downarrow{\scriptstyle \rho_V} & & \downarrow{\scriptstyle \bar{\rho}} \\ End_k(V) & \longleftarrow & (H_{i,j} \otimes Hom_k(V_i, V_j)) \end{array}$$

such that the versal family, $\tilde{\rho} = exp(\delta) \circ \bar{\rho}$.

(ii) Moreover, $H = (H_{i,j})$ *may be constructed recursively, as a quotient of* $T^1 = (T^1_{i,j})$, *by annihilating a series of obstructions,* o_n, *defining a morphism, in* $\underline{a}_r$,

$$o : T^2 \to T^1,$$

such that

$$H \simeq T^1 \otimes_{T^2} k^r.$$

Proof. We have, above constructed an obstruction for lifting ρ_1 to a ρ_2. It is a unique element,

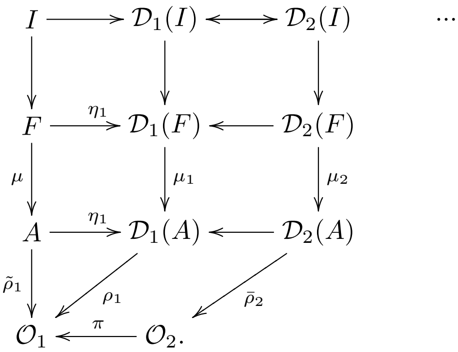
$$o_2 \in Mor_{\underline{a}_r}(k^r \oplus (Ext^2_A(V_i,V_j)^*), T^1/Rad(T^1)^3).$$

Obviously, the image $o_2((Ext^2_A(V_i,V_j)^*) \subset T^1/Rad(T^1)^3)$, generates an ideal of T^1, contained in $Rad(T^1)^2$.

Call it σ_2, and put,

$$H(2) = T^1/(Rad(T^1)^3 + \sigma_2).$$

Then there is a commutative diagram,



In fact, since we have divided out with the obstruction, we know that the morphism,

$$d \circ d \circ \rho_2' : I \to (Rad(T^1)^2/Rad(T^1)^3 + \sigma_2)_{i,j} \otimes_k Hom_k(V_i,V_j)),$$

is the restriction of a derivation,

$$\psi_2' : F \to (Rad(T^1)^2/Rad(T^1)^3 + \sigma_2)_{i,j} \otimes_k Hom_k(V_i,V_j)),$$

Now change the morphism, ρ_2', to ρ_2'' mapping d^2x_i to $\rho_2'(d^2x_i) - \psi_2(x_i)$. It is easily seen that for this new morphism, $d \circ d \circ \rho_2'$ is zero, and so $\eta_3 \circ \rho_2'' = 0$, proving the existence of $\bar{\rho}_2$. Recall that $\mathcal{D}_2(A) = \mathcal{H}_2$.

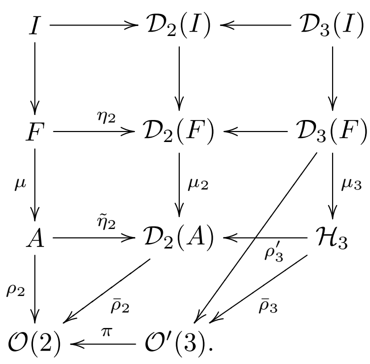
Now $\bar{\rho}_2$ defines $\tilde{\rho}_2 := \eta_2 \circ \rho_2$. Let σ_3' be the two-sided ideal in T^1 generated by

$$Rad(T^1)^4 + Rad(T^1)\sigma_2 + \sigma_2 Rad(T^1),$$

and let us put,

$$H'(3) := T^1/\sigma_3', \ \mathcal{O}'(3) := (H'(3) \otimes_k Hom_k(V_i,V_j))$$

The diagram above induces a commutative diagram, not yet including $\bar{\rho}_3$,



Consider now the map,

$$\eta_3 \circ \rho_3' : I \to \mathcal{O}'(3)$$

ending up in,

$$((Rad(T^1)^3 + \sigma_2)/\sigma_3)_{i,j} \otimes_k Hom_k(V_i, V_j)),$$

which, clearly is killed by $Rad(\mathcal{O}'(3))$, and therefore really is a matrix of vector spaces, as an $\mathcal{O}'(3)$)-module. As above, this map is easily seen to be a left and right linear map as F-modules, F acting on $\mathcal{O}(3)$ via $\tilde{\eta}_3 : F \to \mathcal{D}_3(F)$. Moreover, the induced element,

$$\begin{aligned} o_3 &\in ((Rad(T^1)^3 + \sigma_2)/\sigma_3')_{i,j} \otimes_k Ext^2_A(V_i, V_j)) \\ &= (Hom_k(Ext^2_A(V_i, V_j)^*, (Rad(T^1)^3 + \sigma_2)/\sigma_3')_{i,j}) \end{aligned}$$

is independent on the choice of ρ_3'. Now we define $H(3) := H'(3)/\sigma_3$, where σ_3 is defined by the image of o_3, and define $\kappa : \mathcal{O}'(3) \to \mathcal{O}(3)$ as above. Since by functoriality, the morphism,

$$\eta_3 \circ \rho_3' \kappa : I \to \mathcal{O}(3),$$

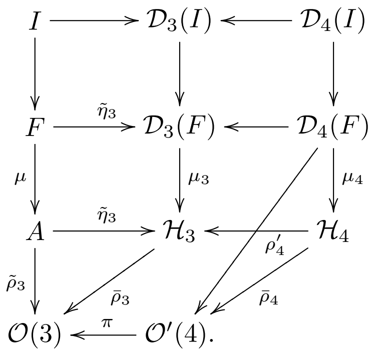
must induce the zero element in the corresponding

$$(Ext^1_A(V_i, V_j) \otimes ((Rad(T^1)^3 + \sigma_2)/\sigma_3')_{i,j})/imo_3)$$

it must be the restriction of a derivation,

$$\xi : F \to \mathcal{O}(3).$$

Now change ρ_3', by sending d^3x_i to $\rho_3'(d^3x_i) - \psi_3(x_i)$ leaving the other values of the parameters unchanged. Then a little calculation shows that the new ρ_3' maps each $\eta_3(f), f \in I$, to zero, inducing a morphism $\bar{\rho}_3 : \mathcal{H}_3 \to \mathcal{O}(3)$. We now have a new situation, given by a commutative diagram, not yet including ρ_4,



and it is clear how to proceed. This proves (i), and the rest is a consequence of the general theorem (4.2.4) of [La 0].

□

We cannot replace $\mathcal{H}$ with $\mathcal{D}$. This follows from the trivial example (ii) in (1.5). However, if we are in a graded situation, things are nicer.

Corollary 3.2. *Assume that A is a finitely generated, graded, in degree 1, k-algebra, and assume that $\mathcal{V}$ is a family of graded A-modules. Then there is a corresponding graded formal moduli $(H_{i,j})^{gr}$, and there is a commutative diagram,*

$$\begin{array}{ccc} A & \xrightarrow{exp(\delta)} & \mathcal{D} \\ {\scriptstyle \rho_V}\downarrow & & \downarrow{\scriptstyle \bar{\rho}^{gr}} \\ End_k(V) & \longleftarrow & (H_{i,j}^{gr} \otimes Hom_k(V_i, V_j)) \end{array}$$

such that the graded versal family, $\tilde{\rho}^{gr} = exp(\delta) \circ \bar{\rho}^{gr}$.

By definition, see [La 2], we also have,

Corollary 3.3. *Let A be a finitely generated k-algebra, and $\mathcal{V}$ a finite family of A-modules, with finite dimensional ext-spaces. Then the n-th. Massey products,*

$$[.,.,..,.] : Ext^1_A(V_i, V_j) \otimes Ext^1_A(V_j, V_k) \otimes ...Ext^1_A(V_l, V_m) \to Ext^2_A(V_i, V_m)$$

are given as follows: For n=2, it is the ordinary cup product, given by,

$$o_2 \in (Hom_k(Ext^2_A(V_i, V_j)^*, (Rad(T^1)^2)/Rad(T^1)^3)_{i,j}).$$

If, $\xi_{i,j} \in Ext^1_a(V_i, V_j)$, $\xi_{j,k} \in Ext^1_a(V_j, V_k)$, the pair $(\xi_{i,j}, \xi_{j,k})$ defines a linear map,

$$(\xi_{i,j}, \xi_{j,k}) : (Rad(T^1)^2)/Rad(T^1)^3)_{i,k} \to k,$$

and the image of o_2 is then $\xi_{i,j} \cup \xi_{j,k} \in Ext^2_A(V_i, V_k)$. In general, the element ,

$$o_n \in (Hom_k(Ext^2_A(V_{i_1}, V_{i_n})^*, (Rad(T^1)^n + \sigma_{n-1})/\sigma'_n)_{i,j})$$

defines a linear map,

$$D_n \to Ext^2_A(V_{i_1}, V_{i_n})$$

where D_n is defined as follows, $D_n = \{\Xi \in Ext^1_A(V_{i_1}, V_{i_2}) \otimes Ext^1_A(V_{i_2}, V_3) \otimes ...Ext^1_A(V_{i_{n-1}}, V_{i_n}) | \Xi$ killing $\sigma_{n-1}\}$.

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