# THE STRUCTURE OF $Simp_{<\infty}(A)$ FOR FINITELY GENERATED $k$-ALGEBRAS $A$.

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**Introduction.** Let $k$ be any field, most often assumed to be algebraically closed, and consider a finitely generated $k$-algebra $A$. Let

$$Simp_{<\infty}(A) = \bigcup_n Simp_n(A)$$

be the set of (iso-classes of) finite dimensional simple right $A$-modules. An n-dimensional simple $A$-module $V \in Simp_n(A)$ defines a surjectiv homomorphism of $k$-algebras, $\rho : A \to End_k(V)$, the kernel of which is a two-sided maximal ideal $\mathfrak{m}_V$, of $A$. Let $Max_{\leq\infty}$ be the set of all such maximal ideals of $A$, for $n \geq 1$. To exclude some strange and for our purposes non-interesting cases, we shall assume that $A$ has the following property:

$$Rad(A)^{\infty} := \bigcap_{\mathfrak{m} \in Max_{<\infty}(A), n \geq 0} \mathfrak{m}^n = 0$$

For want of a better name, we shall call such algebras *geometric*. It is easy to see that any finitely generated left (or right) Noetherian $k$-algebra $A$ is geometric. The condition above is actually satisfied for most finitely generated $k$-algebras that we have come across and, in particular, for the free $k$-algebra on $d$ symbols, $A = k < x_1, x_2, ..., x_d >$, see the example (4.19) of [La 1].

We shall be concerned with the structure of the individual $Simp_n(A)$, $n \geq 1$, and we shall construct natural completions $Simp_\Gamma(A)$, of the scheme $Simp_n(A)$, adding indecomposable modules. We shall also see that the scheme of indecomposable two-dimensional representations induces interesting correspondences for hypersurfaces, and in particular for plane curves. The study of $Ind_\Gamma(A) := Simp_\Gamma(A) - Simp_n(A)$ may also throw light on the classical McKay correspondence. As a tool for studying $Simp_\Gamma(A)$ we introduce the Jordan morphism, and corresponding generalizations of the Deligne-Simpson problem. Finally we shall discuss to what extent the the family $\{Simp_n(A)\}_{n\geq 1}$ of schemes determine the *globale* structure of $A$. In particular, are the K-groups (resp. the cyclic homology) of $A$ determined by the K-groups, (resp. the de Rham cohomology) of the different $Simp_n(A)$? Conversely, what can we learn about the de Rham cohomology of $Simp_n(A)$, knowing the cyclic cohomology of $A$?

This paper is meant as an introduction to a more comprehensive study of non-commutative plane curves, see [Jø-La-Sl].

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**Some general results.** In [La 1] we introduced non-commutative deformations of families of modules of non-commutative $k$-algebras, and the notion of *swarm* of right modules (or more generally of objects in a $k$-linear abelian category). Let $\underline{a}_r$ denote the category of $r$-pointed not necessarily commutative $k$-algebras $R$. The objects are the diagrams of k-algebras,

$$k^r \xrightarrow{\iota} R \xrightarrow{\rho} k^r$$

such that the composition of $\iota$ and $\rho$ is the identity. Any such r-pointed k-algebra $R$ is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of $R$ is the bilateral ideal $Rad(R) := ker\rho$, such that $R/Rad(R) \simeq k^r$. The dual k-vectorspace of $Rad(R)/Rad(R)^2$ is called the tangent space of $R$.

For $r = 1$, there is an obvious inclusion of categories

$$\underline{l} \subseteq \underline{a}_1$$

where $\underline{l}$, as usual, denotes the category of commutative local artinian $k$-algebras with residue field $k$.

Fix a not necessarily commutative $k$-algebra $A$ and consider a right $A$-module $M$. The ordinary deformation functor

$$Def_M : \underline{l} \to \underline{Sets}$$

is then defined. Assuming $Ext^i_A(M, M)$ has finite $k$-dimension for $i = 1, 2$, it is well known, see [Sch], or [La 0], that $Def_M$ has a noetherian prorepresenting hull $H$, *the formal moduli of* $M$. Moreover, the tangent space of $H$ is isomorphic to $Ext^1_A(M, M)$, and $H$ can be computed in terms of $Ext^i_A(M, M)$, $i = 1, 2$ and their *matric* Massey products, see [La 0].

In the general case, consider a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of right $A$-modules. Assume that,

$$dim_k Ext^1_A(V_i, V_j) < \infty.$$

Any such family of $A$-modules will be called a swarm. Define a deformation functor,

$$Def_{\mathcal{V}} : \underline{a}_r \to \underline{Sets}$$

generalizing the functor $Def_M$ above. Given an object $\rho : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the $k$-vectorspace and $R$-left module $(R_{i,j} \otimes_k V_j)$. $\rho$ defines a $k$-linear and left $R$-linear map,

$$\rho(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i,$$

inducing a homomorphism of $R$-endomorphism rings,

$$\tilde{\rho}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right $A$-module structure on the $V_i'$s is defined by a homomorphism of $k$-algebras, $\eta_0 : A \to \oplus_{i=1}^r End_k(V_i)$. Let

$$Def_{\mathcal{V}}(R) \in \underline{Sets}$$

be the isoclasses of homomorphisms of k-algebras,

$$\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

such that,

$$\tilde{\rho}(R) \circ \eta' = \eta_0,$$

where the equivalence relation is defined by inner automorphisms in the $k$-algebra $(R_{i,j} \otimes_k Hom_k(V_i, V_j))$. One easily proves that $Def_{\mathcal{V}}$ has the same properties as the ordinary deformation functor and we prove the following, see [La 1-2, (2.6)]:

**Theorem 1.** *The functor $Def_{\mathcal{V}}$ has a prorepresentable hull, i.e. an object $H$ of the category of pro-objects $\hat{\underline{a}}_r$ of $\underline{a}_r$, together with a versal family,*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathcal{V}}(H/\mathfrak{m}^n)$$

*such that the corresponding morphism of functors on $\underline{a}_r$,*

$$\rho : Mor(H, -) \to Def_{\mathcal{V}}$$

*is smooth, and an isomorphism on the tangent level. Moreover, $H$ is uniquely determined by a set of matric Massey products of the form*

$$Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{n-1}}, V_k) \cdots \to Ext^2(V_i, V_k).$$

The right action of $A$ on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j)),$$

and the k-algebra $O(\mathcal{V})$ acts on the family of $A$-modules $\mathcal{V} = \{V_i\}$, extending the action of $A$. If $dim_k V_i < \infty$, for all $i = 1, ..., r$, the operation of associating $(O(\mathcal{V}), \mathcal{V})$ to $(A, \mathcal{V})$ turns out to be a closure operation.

Moreover, we prove the crucial result,

**A generalized Burnside theorem.** *Let $A$ be a finite dimensional $k$-algebra, $k$ an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of simple $A$-modules, then*

$$\eta : A \longrightarrow O(\mathcal{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

*is an isomorphism.*

We also proved that there exists, in the noncommutative deformation theory, an obvious analogy to the notion of prorepresenting (modular) substratum $H_0$ of the formal moduli $H$. The tangent space of $H_0$ is determined by a family of subspaces

$$Ext^1_0(V_i, V_j) \subseteq Ext^1_A(V_i, V_j), \qquad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathcal{V}$, and by a subspace,

$$T_0(\Delta) \subseteq \prod_i Ext^1_A(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of $A$-modules generated by the family $\mathcal{V} = \{V_i\}_{i=1}^r$, see [La 1]. If $\mathcal{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some artinian $k$-algebra $A$, we show that the above notion of *almost split sequence* coincides with that of Auslander, see [R].

Using this we consider, in [La 2], the general problem of classification of iterated extensions of a family of modules $\mathcal{V} = \{V_i,\}_{i=1}^r$, and the corresponding classification of filtered modules with graded components in the family $\mathcal{V}$, and extension type given by a directed representation graph $\Gamma$, see under section Completion of $Simp_n(A)$. The main result is the following, see [La 2. (4.7)], and :

**Proposition 2.** *Let $A$ be any $k$-algebra, $\mathcal{V} = \{V_i\}_{i=1}^r$ any swarm of $A$-modules, i.e. such that,*

$$\dim_k Ext^1_A(V_i, V_j) < \infty \quad \textit{for all } i, j = 1, \dots, r.$$

*(i): Consider an iterated extension $E$ of $\mathcal{V}$, with representation graph $\Gamma$. Then there exists a morphism of $k$-algebras*

$$\phi : H(\mathcal{V}) \to k[\Gamma]$$

*such that*

$$E \simeq k[\Gamma] \otimes_\phi \tilde{V}$$

*in the above sense.*

*(ii): The set of equivalence classes of iterated extensions of $\mathcal{V}$ with representation graph $\Gamma$, is a quotient of the set of closed points of the affine algebraic scheme*

$$\underline{A}[\Gamma] = Mor(H(\mathcal{V}), k[\Gamma])$$

*(iii): There is a versal family $\tilde{V}[\Gamma]$ of $A$-modules defined on $A[\Gamma]$, containing as fibres all the isomorphism classes of iterated extensions of $\mathcal{V}$ with representation graph $\Gamma$.*

To any, not necessarily finite, swarm $\underline{c} \subset \underline{mod}(A)$ of right-$A$-modules, we have associated two associative $k$-algebras, see [La 1,3], $O(|\underline{c}|, \pi)$, and a sub-quotient $\mathcal{O}_\pi(\underline{c})$, together with natural $k$-algebra homomorphisms,

$$\eta(|\underline{c}|) : A \longrightarrow O(|\underline{c}|, \pi)$$

and,

$$\eta(\underline{c}) : A \longrightarrow \mathcal{O}_\pi(\underline{c})$$

with the property that the $A$-module structure on $\underline{c}$ is extended to an $\mathcal{O}$-module structure in an optimal way. We then defined an *affine non-commutative scheme* of right $A$-modules to be a swarm $\underline{c}$ of right $A$-modules, such that $\eta(\underline{c})$ is an isomorphism. In particular we considered, for finitely generated $k$-algebras, the swarm $Simp^*_{<\infty}(A)$ consisting of the finite dimensional simple $A$-modules, and the *generic* point $A$, together with all morphisms between them. The fact that this is a swarm, i.e. that for all objects $V_i, V_j \in Simp_{<\infty}$ we have $dim_k Ext^1_A(V_i, V_j) < \infty$, is easily proved. We have in [La 1] proved the following result, (see (5.20), loc.cit. and Lemma 2. above.)

**Proposition 3.** *Let $A$ be a geometric $k$-algebra, then the natural homomorphism,*

$$\eta(Simp^*(A)) : A \longrightarrow \mathcal{O}_\pi(Simp^*_{<\infty}(A))$$

*is an isomorphism, i.e. $Simp^*_{<\infty}(A)$ is a scheme for $A$.*

In particular, $Simp^*_{<\infty}(k < x_1, x_2, ..., x_d >)$, is a scheme for $k < x_1, x_2, ..., x_d >$. To analyze the local structure of $Simp_n(A)$, we need the following, see [La 2], §4:

**Lemma 4.** *Let $\mathcal{V} = \{V_i\}_{i=1,..,r}$ be a finite subset of $Simp_{<\infty}(A)$, then the morphisme of $k$-algebras,*

$$A \to O(\mathcal{V}) = (H_{i,j} \otimes_k Hom_k(V_i, V_j))$$

*is topologically surjective.*

*Proof.* Since the simple modules $V_i,\ i = 1, .., r$ are distinct, there is an obvious surjection, $\pi : A \to \prod_{i=1,..,r} End_k(V_i)$. Put $\mathfrak{r} = ker\pi$, and consider for $m \geq 2$ the finite-dimensional $k$-algebra, $B := A/\mathfrak{r}^m$. Clearly $Simp(B) = \mathcal{V}$, so that by the generalized Burnside theorem, see [La], §4, we find, $B \simeq O^B(\mathcal{V}) := (H^B_{i,j} \otimes_k Hom_k(V_i, V_j))$. Consider the commutative diagram,

$$\begin{array}{ccccc} A & \longrightarrow & (H^A_{i,j} \otimes_k Hom_k(V_i, V_j)) =: O^A(\mathcal{V}) & & \\ \downarrow & & \downarrow & \searrow & \\ B & \longrightarrow & (H^B_{i,j} \otimes_k Hom_k(V_i, V_j)) & \xrightarrow{\alpha} & O^A(\mathcal{V})/rad^m \end{array}$$

where all morphisms are natural. In particular $\alpha$ exists since $B = A/\mathfrak{r}^m$ maps into $O^A(\mathcal{V})/rad^m$, and therefore induces the morphism $\alpha$ commuting with the rest of the morphisms. Consequently $\alpha$ has to be surjective, and we have proved the contention.

□

**Localization and topology on** $Simp(A)$**.** Let $s \in A$, and consider the *open subset* $D(s) = \{V \in Simp(A) |\ \rho(s)$ invertible in $End_k(V)\}$. The Jacobson topology on $Simp(A)$ is the topology with basis $\{D(s)|\ s \in A\}$. It is clear that the natural morphism,

$$\eta : A \to \mathcal{O}_\pi(D(s))$$

maps $s$ into an invertible element of $O(D(s), \pi)$. Therefore we may define the localization $A_{\{s\}}$ of $A$, as the k-algebra generated in $O(D(s), \pi)$ by $\mathcal{O}_\pi(D(s))$ and the inverse of $\eta(s)$. This furnishes a general methode of localization with all the properties one would wish. And in this way we also find a canonical (pre)sheaf, $\mathcal{O}$ defined on $Simp(A)$.

**Definition 5.** *When the $k$-algebra $A$ is geometric, such that $Simp^*(A)$ is a scheme for $A$, we shall refer to the presheaf $\mathcal{O}$, defined above on the Jacobson topology, as the structure presheaf of the scheme $Simp(A)$.*

In the next § we shall see that the Jacobson topology on $Simp(A)$, restricted to each $Simp_n(A)$ is the Zariski topology for a classical scheme-structure on $Simp_n(A)$.

Notice that, working on non-commutative invariant theory, one is led to believe that the topology on $Simp(A)$ should be saturated with respect to infinitesimal incidence, i.e. should be such that $Ext^1_A(V,V') \neq 0$ implies $V'$ is in the closure of $V$. We shall come back to this later.

**The algebraic (scheme) structure on $Simp_n(A)$.** Recall that a standard n-commutator relation in a $k$-algebra $A$ is a relation of the type,

$$[a_1, a_2, ..., a_{2n}] := \sum_{\sigma \in \Sigma_{2n}} sign(\sigma) a_{\sigma(1)} a_{\sigma(2)} ... a_{\sigma(2n)} = 0$$

where $\{a_1, a_2, ..., a_{2n}\}$ is a subset of $A$. Let $I(n)$ be the two-sided ideal of $A$ generated by the subset,

$$\{[a_1, a_2, ..., a_{2n}] | \; \{a_1, a_2, ..., a_{2n}\} \subset A\}.$$

Consider the canonical homomorphism,

$$p_n : A \longrightarrow A/I(n) =: A(n).$$

It is well known that any homomorphism of $k$-algebras,

$$\rho : A \longrightarrow End_k(k^n) =: M_n(k)$$

factors through $p_n$, see e.g. [Formanek].

**Corollary 6.** *(i). Let $V_i, V_j \in Simp_{\leq n}(A)$ and put $\mathfrak{r} = \mathfrak{m}_{V_i} \cap \mathfrak{m}_{V_j}$. Then we have, for $m \geq 2$,*

$$Ext^1_A(V_i, V_j) \simeq Ext^1_{A/\mathfrak{r}^m}(V_i, V_j)$$

*(ii). Let $V \in Simp_n(A)$. Then,*

$$Ext^1_A(V, V) \simeq Ext^1_{A(n)}(V, V)$$

*Proof.* (i) follows directely from Lemma 2. To see (ii), notice that $Ext^1_A(V,V) = HH^1(A, End_k(V)) = Der_k(A, End_k(V))/Triv = Der_k(A(n), End_k(V))/Triv \simeq Ext^1_{A(n)}(V,V)$. The third equality follows from the fact that any derivation maps a standard n-commutator relation into a sum of standard n-commutator relations.

□

**Example 7.** Notice that, for distinct $V_i, V_j \in Simp_{\leq n}(A)$, we may well have,

$$Ext^1_A(V_i, V_j) \neq Ext^1_{A(n)}(V_i, V_j).$$

In fact, consider the matrix $k$-algebra,

$$A = \begin{pmatrix} k[x] & k[x] \\ 0 & k[x] \end{pmatrix},$$

and let $n = 1$. Then $A(1) = k[x] \oplus k[x]$. Put $V_i = k[x]/(x) \oplus (0)$, $V_j = (0) \oplus k[x]/(x)$, then it is easy to see that,

$$Ext^1_A(V_i, V_j) = k, \; Ext^1_{A(1)}(V_i, V_j) = 0.$$

**Lemma 8.** *Let $B$ be a $k$-algebra, and let $V$ be a vectorspace of dimension $n$, such that the $k$-algebra $B \otimes End_k(V)$ satisfies the standard $n$-commutator-relations, i.e. such that the ideal, $I(n) \subset B \otimes End_k(V)$ generated by the standard $n$-commutators $[x_1, x_2, .., x_{2n}],\ x_i \in B \otimes End_k(V)$, is zero. Then $B$ is commutative.*

*Proof.* In fact if $b_1, b_2 \in B$ is such that $[b_1, b_2] \neq 0$, then the obvious $n$-commutator,

$$b_1 e_{1,1} b_2 e_{1,1} e_{1,2} e_{2,2} ... e_{n-1,n} - b_2 e_{1,1} b_1 e_{1,1} e_{1,2} e_{2,2} ... e_{n-1,n}$$

is different from 0. Here $e_{i,j}$ is the $n \times n$ matrix with all elements equal to 0, except the one in the $(i,j)$ position, where the element is equal to 1.

□

**Lemma 9.** *If $A$ is a finite type $k$-algebra, then any $V \in Simp_n(A)$ is an $A(n) := A/I_n$-module, and the corresponding formal moduli, $H^{A(n)}(V)$ is isomorphic to $H^A(V)^{com}$, the commutativization of $H^A(V)$.*

*Proof.* Consider the natural diagram of homomorphisms of $k$-algebras,

$$\begin{array}{ccccc}
 & & A & \longrightarrow & O(Simp^*(A), \pi) \\
 & & \downarrow & & \downarrow \\
Z(A(n)) & \longrightarrow & A(n) & & O(Simp_n^*(A), \pi) \\
\downarrow & & \downarrow & & \downarrow \\
H(V)^{com} & \longrightarrow & H(V)^{com} \otimes_k End_k(V) & \longleftarrow & (H_{i,j} \otimes_k Hom_k(V_i, V_j))
\end{array}$$

where $Z(A(n))$ is the center of $A(n) := A/I_n$, $V_i, V_j \in Simp_n(A)$, and $H(V)^{com}$ is the commutativization of $H(V)$. Clearly there are natural morphisms of formal moduli,

$$H^A(V) \to H^{A(n)}(V) \to H^A(V)^{com} \to H^{A(n)}(V)^{com}.$$

Since moreover

$$A(n) \to H^{A(n)}(V) \otimes End_k(V)$$

is topologically surjective, we find using (Lemma 6), that $H^{A(n)}(V)$ is commutative. But then the composition,

$$H^{A(n)}(V) \to H^A(V)^{com} \to H^{A(n)}(V)^{com},$$

is an isomorphism. Since by Corollary 4. the tangent spaces of $H^{A(n)}(V)$ and $H^A(V)$ are isomorphic, the lemma is proved.

□

**Corollary 10.** *Let $A = k < x_1, .., x_d >$ be the free $k$-algebra on $d$ symbols, and let $V \in Simp_n(A)$. Then*

$$H^A(V)^{com} \simeq H^{A(n)}(V) \simeq k[[t_1, ..., t_{(d-1)n^2+1}]]$$

This should be compared with the results of [Procesi 1.], see also [Formanek]. There are further examples, some based upon the calculation of Tord Romstad, see [Romstad], showing that $H^A(V)$ is not commutative, even though $V \in Simp(A) = Simp_{\leq 2}(A)$.

In general the natural morphism,

$$\eta(n) : A(n) \to \prod_{V \in Simp_n(A)} H^{A(n)}(V) \otimes_k End_k(V)$$

is not an injection.

**Example 11.** *In fact, let*

$$A = \begin{pmatrix} k & k & k \\ k & k & k \\ o & 0 & k \end{pmatrix}.$$

*The ideal $I(2)$ is generated by $[e_{1,1}, e_{1,2}e_{2,2}e_{2,3}] = e_{1,3}$. So*

$$A(2) = \begin{pmatrix} k & k & k \\ k & k & k \\ o & k & k \end{pmatrix} / \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ o & 0 & 0 \end{pmatrix} = M_2(k) \oplus M_1(k).$$

*Hovever,*

$$\prod_{V \in Simp_2(A)} H^{A(2)}(V) \otimes_k End_k(V) = M_2(k),$$

*therefore $ker\ \eta(2) = M_1(k) = k$.*

Let $O(n)$, be the image of $A(n)$, then obviously,

$$O(n) \to \prod_{V \in Simp_n(A)} H^{O(n)}(V) \otimes_k End_k(V)$$

is injective and,

$$H^{O(n)}(V) \simeq H^{A(n)}(V).$$

for every $V \in Simp_n(A)$. Put $B = \prod_{V \in Simp_n(A)} H^{A(n)}(V)$. Let $x_i \in A, i = 1, ..., d$ be generators of $A$, and consider the images $(x^i_{p,q}) \in B \otimes_k End_k(k^n)$ of $x_i$ via the injective homomorphism of $k$-algebras,

$$O(n) \to B \otimes End_k(k^n),$$

obtained by choosing bases in all $V \in Simp_n(A)$. Now, $B$ is commutative, so the $k$-subalgebra $C(n) \subset B$ generated by the elements $\{x^i_{p,q}\}_{i=1,..,d;\ p,q=1,..,n}$ is commutative. We have an injection ,

$$O(n) \to C(n) \otimes_k End_k(k^n).$$

and for all $V \in Simp_n(A)$ there is a natural projection,

$$C(n) \otimes_k End_k(k^n) \to H^{A(n)}(V) \otimes_k End_k(V).$$

This defines a set theoretical map,

$$t : Simp_n(A) \longrightarrow Simp(C(n)).$$

Since $A(n) \to H^{A(n)}(V) \otimes_k End_k(V)$ is topologically surjective, $H^{A(n)}(V) \otimes_k End_k(V)$ is topologically generated by the images of $x_i$. It follows that we have a surjective homomorphism,

$$\hat{C}_{t(V)}(n) \to H^{A(n)}(V).$$

Categorical properties implies, as usual, that there is another natural morphism,

$$H^{A(n)}(V) \to \hat{C}_{t(V)}(n),$$

which composed with the former is an automorphism of $H^{A(n)}(V)$. Since

$$C(n) \otimes_k End_k(k^n) \subseteq \prod_{V \in Simp_n(A)} H^{O(n)}(V) \otimes_k End_k(V).$$

It follows that for $v \in Simp(C(n))$, corresponding to $V \in Simp_n(A)$, the finite dimensional $k$-algebra $C(n)/\underline{m}_v{}^2 \otimes_k End_k(k^n)$ sits in a finite dimensional quotient of,

$$\prod_{V \in V} H^{O(n)}(V) \otimes_k End_k(V).$$

where $V \subset Simp_n(A)$ is finite. However, by Lemma 4. the composition of the morphisms,

$$A \longrightarrow O(n) \longrightarrow \prod_{V \in V} H^{O(n)}(V) \otimes_k End_k(V)$$

is topologically surjectiv. Therefore the morphism,

$$A \longrightarrow C(n)/\underline{m}_v{}^2 \otimes_k End_k(k^n)$$

is surjectiv, implying that the map

$$H^{A(n)}(V) \to \hat{C}_{t(V)}(n),$$

is surjectiv, and consequently, $H^{A(n)}(V) \simeq \hat{C}(n)_v$.

Moreover $t$ is injective, so $Simp_n(A) \subset Simp(C(n))$. We have the following theorem, see Chapter VIII, §2, of the book of C. Procesi, [Procesi 2.], where part of this theorem is proved.

**Theorem 12.** *Let $V \in Simp_n(A)$, correspond to the point $v \in Spec(C(n))$. Then there exist a Zariski neighborhood $U_v$ of $v$ in $Spec(C(n))$ such that any $v' \in U$ corresponds to a point $V' \in Simp_n(A)$. Let $U(n)$ be the open subscheme of $Spec(C(n))$, the union of all $U_v$ for $V \in Simp_n(A)$. $O(n)$ defines a non-commutative structure sheaf $\mathcal{O}(n) := \mathcal{O}_{Simp_n(A)}$ of Azumaya algebras on the topological space $Simp_n(A)$*

*(Jacobson topology). The center $\mathcal{S}(n)$ of $\mathcal{O}(n)$, defines a scheme structure on $Simp_n(A)$. Moreover, there is a morphism of schemes,*

$$\kappa : U(n) \longrightarrow Simp_n(A),$$

*Such that for any $v \in U(n)$,*

$$\hat{\mathcal{S}}(n)_{\kappa(v)} \simeq H^{A(n)}(V)$$

*Proof.* Let $\rho : A \longrightarrow End_k(V)$ be the surjective homomorphism of $k$-algebras, defining $V \in Simp_n(A)$. Let, as above $e_{i,j} \in End_k(V)$ be the elementary matrices, and pick $y_{i,j} \in A$ such that $\rho(y_{i,j}) = e_{i,j}$. Let us denote by $\sigma$ the cyclical permutation of the integers $\{1, 2, ..., n\}$, and put,

$$s_k := [y_{\sigma^k(1),\sigma^k(2)}, y_{\sigma^k(2),\sigma^k(2)}, y_{\sigma^k(2),\sigma^k(3)} ... y_{\sigma^k(n),\sigma^k(n)}],\ s := \sum_{k=0,1,..,n-1} s_k \in A.$$

Clearly $s \in I(n-1)$. Since $[e_{\sigma^k(1),\sigma^k(2)}, e_{\sigma^k(2),\sigma^k(2)}, e_{\sigma^k(2),\sigma^k(3)} ... e_{\sigma^k(n),\sigma^k(n)}] = e_{\sigma^k(1),\sigma^k(n)} \in End_k(V)$, $\rho(s) := \sum_{k=0,1,..,n-1} \rho(s_k) \in End_k(V)$ is the matrix with non-zero elements, equal to 1, only in the $(\sigma^k(1), \sigma^k(n))$ position, so the determinant of $\rho(s)$ must be +1 or -1. The determinant $det(s) \in C(n)$ is therefore nonzero at the point $v \in Spec(C(n))$ corresponding to $V$. Put $U = D(det(s)) \subset Spec(C(n))$, and consider the localization $O(n)_{\{s\}} \subseteq C(n)_{\{det(s)\}} \otimes_k End_k(V)$, the inclusion following from general properties of the localization, see above. Now, any closed point $v' \in U$ corresponds to a $n$-dimensional representation of $A$, for which the element $s \in I(n-1)$ is invertible. But then this representation cannot have a $m < n$ dimensional quotient, so it must be simple.

Since $s \in I(n-1)$, the localized $k$-algebra $O(n)_{\{s\}}$ does not have any simple modules of dimension less than $n$, and no simple modules of dimension $> n$ . In fact, for any finite dimensional $O(n)_{\{s\}}$-module $V$, of dimension $m$, the image $\hat{s}$ of $s$ in $End_k(V)$ must be invertible. However, the inverse $\hat{s}^{-1}$ must be the image of a polynomial (of degree $m-1$) in s. Therefore, if $V$ is simple over $O(n)_{\{s\}}$, i.e. if the homomorphism $O(n)_{\{s\}} \to End_k(V)$ is surjective, $V$ must also be simple over $A$. Since now $s \in I(n-1)$, it follows that $m \geq n$. If $m > n$, we may construct, in the same way as above an element in $I(n)$ mapping into a nonzero element of $End_k(V)$. Since, by construction, $I(n) = 0$ in $A(n)$, and therefore also in $O(n)_{\{s\}}$, we have proved what we wanted. By a theorem of M.Artin, see [Artin], $O(n)_{\{s\}}$ must be an Azumya algebra over its center, $S(n)_{\{s\}} := Z(O(n)_{\{s\}})$. Therefore $O(n)$ defines a presheaf $\mathcal{O}(n)$ on $Simp_n(A)$, of Azumaya algebras over its center $\mathcal{S}(n) := Z(\mathcal{O}(n))$. Clearly, any $V \in Simp_n(A)$, corresponding to $v \in Spec(C(n))$ maps to a point $s := \kappa(v) \in Spec(\mathcal{O}(n))$. Since we know that,

$$H^{O(n)}(V) \simeq H^{A(n)}(V),$$

and since $O(n)$ is, locally Azumaya, it is clear that,

$$\hat{\mathcal{S}}(n)_s \simeq H^{O(n)}(V) \simeq H^{A(n)}(V).$$

The rest is clear.

□

Moreover, $Spec(C(n))$ is, in a sense, a compactification of $Simp_n(A)$, and we shall be able, using this embedding to study the degeneration processes that occur, at *infinity* in $Simp_n(A)$.

**Example 13.** *Let us check the case of* $A = k < x_1, x_2 >$, *the free non-commutative* $k$*-algebra on two symbols. First, let us compute* $Ext^1_A(V,V)$ *for* $V \in Simp_2(A)$, *and find a basis* $\{t_i^*,\}_{i=1}^5$, *represented by derivations* $\psi_i \in Der_k(A, End_k(V))$, *i=1,2,3,4,5. This is easy, since we have the exact sequence,*

$$\begin{aligned}0 \to Hom_A(V_1,V_2) \to Hom_k(V_1,V_2) \to Der_k(A, Hom_k(V_1,V_2))\\ \to Ext^1_A(V_1,V_2) \to 0\end{aligned}$$

*proving that,* $Ext^1_A(V_1,V_2) = Der_k(A, Hom_k(V_1,V_2))/Triv$, *where* $Triv$ *is the sub-vectorspace of trivial derivations. Pick* $V \in Simp_2(A)$ *defined by the homomorphism* $A \to M_2(k)$ *mapping the generators* $x_1, x_2$ *to the matrices*

$$X_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =: e_{1,2},\ X_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} =: e_{2,1}.$$

*Notice that*

$$X_1X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: e_{1,1} = e_1,\ X_2X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: e_{2,2} = e_2,$$

*and recall also that for any* $2\times 2$*-matrix* $(a_{p,q}) \in M_2(k)$, $e_i(a_{p,q})e_j = a_{i,j}e_{i,j}$. *The trivial derivations are generated by the derivations* $\{\delta_{p,q}\}_{p,q=1.2}$, *defined by,*

$$\delta_{p,q}(x_i) = x_i e_{p,q} - e_{p,q}x_i.$$

*Clearly* $\delta_{1,1} + \delta_{2,2} = 0$. *Now, compute and show that the derivations* $\psi_i$, $i = 1,2,3,4,5$, *defined by,*

$$\psi_i(x_p) = 0, \text{for } i = 1,2, p = 1,\ \psi_i(x_p) = 0, \text{for } i = 4,5, p = 2$$

*by,*

$$\psi_1(x_2) = e_{1,1}, \psi_2(x_2) = e_{1,2},\ \psi_3(x_1) = e_{1,2}, \psi_4(x_1) = e_{2,1}, \psi_5(x_1) = e_{2,1)}$$

*and by,*

$$\psi_3(x_2) = e_{2,1}$$

*form a basis for* $Ext^1_A(V,V) = Der_k(A, End_k(V))/Triv$. *Therefore* $H(V) = k[[t_1,t_2,t_3,t_4,t_5]]$, *and the formal versal family* $\tilde{V}$, *is defined by the actions of* $x_1, x_2$, *given by,*

$$X_1 := \begin{pmatrix} 0 & 1+t_3 \\ t_5 & t_4 \end{pmatrix},\ X_2 := \begin{pmatrix} t_1 & t_2 \\ 1+t_3 & 0 \end{pmatrix}.$$

*One checks that there are polynomials of* $X_1, X_2$ *which are equal to* $t_i e_{p,q}$, *modulo the ideal* $(t_1,..,t_5)^2 \subset H(V)$, *for all* $i,p,q = 1,2$. *This proves that* $\hat{C}(2)_v \simeq H(V)$, *and that the composition,*

$$A \longrightarrow A(2) \longrightarrow M_2(C(2)) \subset M_2(H(V)))$$

*is topologically surjective.*

**Completions of $Simp_n(A)$.** In the example above it is easy to see that elements of the complement of $Simp_n(A)$ in the affine subscheme $Spec(C(n))$ may not be represented by simple, nor indecomposable, representations. A decomposable representation $W$ will not, however, in general be deformable into a simple representation, since good deformations should conserve $End_A(W)$. Therefore, even though we have termed $Spec(C(n))$ a compactification of $Simp_n(A)$, it is a bad *completion*. The missing points *at infinity* of $Simp_n(A)$, should be represented as indecomposable representations, with $End_A(W) = k$. Any such is an iterated extension of simple representations $\{V_i\}_{i=1,2,..s}$, with representation graph $\Gamma$ (corresponding to an *extension type*, see [La 2]), and $\sum_{i=1}^{s} dim(V_i) = n$. To simplify the notations we shall write, $|\Gamma| := \{V_i\}_{i=1,2,..s}$. In [La 2] we treat the problem of classifying all such, up to isomorphisms. Assume now that this problem is solved, i.e. that we have identified the *non-commutative* scheme of indecomposable $\Gamma$-representation, call it $Ind_\Gamma(A)$. Put $Simp_\Gamma(A) := Simp_n(A) \cup Ind_\Gamma(A)$. Now, repeat the basics of the construction of $Spec(C(n))$ above. Consider for every open affine subscheme $D(s) \subset Simp_\Gamma(A)$, the natural morphism,

$$A \to \varprojlim_{\underline{c} \subset D(s)} O(\underline{c}, \pi)$$

$\underline{c}$ running through all finite subsets of $D(s)$, and consider, in particular, its projection,

$$A \to A(n) \to \prod_{V \in D(s)} H^{A(n)}(V)^{com} \otimes_k End_k(V).$$

Put $B_s(\Gamma) := \prod_{V \in D(s)} H^{A(n)}(V)^{com}$. Let $x_i \in A, i = 1, ..., d$ be generators of $A$, and consider the images $(x^i_{p,q}) \in B_s(n) \otimes_k End_k(k^n)$ of $x_i$ via the homomorphism of $k$-algebras,

$$A \to B_s(\Gamma) \otimes M_n(k),$$

obtained by choosing bases in all $V \in Simp_\Gamma(A)$. Notice that since $V$ no longer is (necessarily) simple, we do not know that this map is topologically surjectiv. Now, $B_s(\Gamma)$ is commutative, so the $k$-subalgebra $C_s(\Gamma) \subset B_s(\Gamma)$ generated by the elements $\{x^i_{p,q}\}_{i=1,..,d;\ p,q=1,..,n}$ is commutative. We have a morphism,

$$I_s(\Gamma) : A \to C_s(\Gamma) \otimes_k M_n(k) = M_n(C_s(\Gamma)).$$

Moreover, these $C_s(\Gamma)$ define a presheaf, $\mathcal{C}(\Gamma)$, on the Jacobson topology of $Simp_\Gamma(A)$. The rank $n$ free $C_s(\Gamma)$-modules with the $A$-actions given by $I_s(\Gamma)$, glue together to form a locally free $\mathcal{C}(\Gamma)$-Module $\mathcal{E}(\Gamma)$ on $Simp_\Gamma(A)$, and the morphisms $I_s(n)$ induce a morphism of sheaves of algebras,

$$I(\Gamma) : A \to End_{\mathcal{C}(\Gamma)}(\mathcal{E}(\Gamma)).$$

As for every $V \in Simp_\Gamma(A)$, $End_A(V) = k$, the commutator of $A$ in $H^A(V)^{com} \otimes_k End_k(V)$ is $H^A(V)^{com}$. The morphism,

$$\zeta(V) : H^A(V)^{com} \to HH^0(A, H^A(V)^{com} \otimes_k End_k(V))$$

is therefore an isomorphism, and we may assume that the corresponding morphism,

$$\zeta : \mathcal{C}(\Gamma) \to HH^0(A, End_{\mathcal{C}(\Gamma)}(\mathcal{E}(\Gamma)))$$

is an isomorphism of sheaves. For all $V \in D(s) \subset Simp_\Gamma(A)$ there is a natural projection,

$$\kappa(\Gamma) : \mathcal{C}_s(\Gamma) \otimes_k M_n(k) \to H^{A(n)}(V)^{com} \otimes_k End_k(V),$$

which, composed with $I_s(\Gamma)$ is the natural homomorphism,

$$A \longrightarrow H^{A(n)}(V)^{com} \otimes_k End_k(V)$$

$\kappa$ defines a set theoretical map,

$$t : Simp_\Gamma(A) \longrightarrow Spec(\mathcal{C}(\Gamma)),$$

and a natural surjectiv homomorphism,

$$\hat{\mathcal{C}}(\Gamma)_{t(V)} \to H^{A(n)}(V)^{com}.$$

Categorical properties implies, as usual, that there is another natural morphism,

$$\iota : H^{A(n)}(V) \to \hat{\mathcal{C}}(\Gamma)_{t(V)},$$

which composed with the former is the obvious surjection, and such that the induced composition,

$$A \longrightarrow H^{A(n)}(V)^{com} \otimes_k End_k(V) \to \hat{\mathcal{C}}(\Gamma)_{t(V)} \otimes_k End_k(V),$$

is $I(\Gamma)$ formalized at $t(V)$. From this, and from the definition of $C(\Gamma)$, it follows that $\iota$ is surjective, such that for every $V \in Simp_\Gamma(A)$ there is an isomorphism $H^{A(n)}(V)^{com} \simeq \hat{\mathcal{C}}(\Gamma)_{t(V)}$. For $V \in Simp_\Gamma(A)$ there is also a natural commutative diagram,

$$\begin{array}{ccc}
ZA(n) & \longrightarrow & \mathcal{C}(\Gamma) \\
\downarrow & & \downarrow \\
A(n) & \longrightarrow & End_{\mathcal{C}(\Gamma)}(\mathcal{E}(\Gamma)) \\
\downarrow & & \downarrow \\
H^{A(n)}(V) \otimes_k End_k(V) & \longrightarrow & \hat{\mathcal{C}}(\Gamma)_{t(V)}(n) \otimes_k End_k(V)
\end{array}$$

Formally at a point $V \in Simp_\Gamma(A)$, we have therefore proved that the local, commutative structure of $\mathcal{S}imp_\Gamma(A)$ (as $A$ or $A(n)$-module), and the corresponding local structure of $Spec(\mathcal{C}(\Gamma))$ at $V$, coincide. We have actually proved the following,

**Theorem 14.** *The topological space $Simp_\Gamma(A)$, with the Jacobson topology, together with the sheaf of commutative $k$-algebras $\mathcal{C}(\Gamma)$ defines a scheme structure on $\mathcal{S}imp_\Gamma(A)$, containing an open subscheme, etale over $Simp_n(A)$. Moreover, there is a morphism,*

$$\pi(\Gamma) : \mathcal{S}imp_\Gamma(A) \to Spec(ZA(n)),$$

*extending the natural morphism,*

$$\pi_0 : Simp_n(A) \to Spec(ZA(n)).$$

*Proof.* As in Theorem 12. we prove that if $v = t(V)$, $V \in Simp_\Gamma(A)$, then there exists an open subscheme of $Spec(\mathcal{C}(\Gamma))$ containing only indecomposables with $End_A(V) = k$. The rest is clear.

□

These morphisms $\pi(\Gamma)$ are our candidates for the possibly different completions of $Simp_n(A)$. Notice that for $W \in Spec(C(n)) - Simp_n(A)$, the formal moduli $H^A(W)$ is not always prorepresenting, since $End_A(W) \neq k$ when $W$ is semisimple, but not simple. The corresponding modular substratum will, locally, correspond to the semisimple deformations of $W$, thus to a closed subscheme of $Spec(C(n)) - Simp_n(A) \subset Spec(C(n))$.

**The McKay correspondence.** Let us consider a special case, where a finite group $G$ acts on a finite dimensional $k$-vectorspace, $U$. Put, $A_0 := Sym_k(U^*)$, and let $A := O(Simp^*(A_0 - G))$ be the $k$-algebra of observables of the $A - G$-swarm of orbits of the $G$-action. Recall, see [La 3], §8, that $ZA = A_0^G$, and that the classical quotient scheme $U/G$ (exist and) is isomorphic to $Spec(A_0^G)$. Let $\{V_i\}_{i=1}^r$ be the finite family of irreducible (simple) $G$-representations. Let $\Gamma$ be a representation graph (defining an extension type) of dimension $n$, i.e. such that $|\Gamma| = \{V_{i_p}\}_{p=1}^s$, $\sum_{p=1}^s dim_k V_{i_p} = n$, and use Theorem 14. It says that there exist a scheme $\mathcal{S}imp_\Gamma(A)$ and a morphism,

$$\pi : \mathcal{S}imp_\Gamma(A) \to U/G = Spec(A_0^G),$$

extending the natural morphism,

$$\pi_0 : Simp_n(A) \to Spec(A_0^G).$$

If $n \geq ordG + 1$ the scheme $Simp_n(A)$ has to be empty, since any $V \in Simp_n(A)$ with support outside the origin in $Spec(A_0)$, correspond to a reduced orbit, and so necessarily have length less or equal to the order of $G$, and any $V$ with support in $\{\underline{0}\}$ is a simple $G$-representation with trivial $A$-action, so $dim_k V \leq |G|$. Now, suppose G acts freely on an open subset of $Spec(A_0)$, and let $\Gamma$ be a representation graph (corresponding to the extension type) of the regular representation of $G$. Under which conditions is the morphism,

$$\pi : \mathcal{S}imp_\Gamma(A) \to Spec(A_0^G),$$

a desingularization of the affine scheme $Spec(A_0^G)$? If it is, is the representation graph uniquely determined? We shall come back to these well known problems in a later paper. However, to see how we may compute the morphism $\pi$ let us here consider two very simple examples:

1. Consider the group $G = Z/(2)$, generated by $\tau$, acting on $U = k^2$ by $\tau = -id$. In this case $A_0 = k[x, y]$, and $\tau(x) = -x,\ \tau(y) = -y$, and $A_0^G = k[x^2, y^2, xy]$ is the well known singularity. Clearly $G$ has two simple (irreducible) representations of dimension 1, $V_i,\ i = 0, 1$, where $\tau$ acts as $(-1)^i$, respectively, and the regular representation, is the sum of these. The orbits of $G$ in $Spec(A_0) = \mathbf{A}^2$, are either of length 2, corresponding to a simple $A$-module of dimension 2, or is reduced to the origin. Therefore the indecomposable $A$-modules of dimension 2, must all have support at the origin. They must therefore be given by the indecomposables of representation graph,

$$V_0 \bullet \longrightarrow \bullet\ V_1.$$

Now all such are given in terms of the following actions of $x, y, \tau$ on the vectorspace $k^2$.

$$V_t : X = \begin{pmatrix} 0, 0 \\ 1, 0 \end{pmatrix}, Y = \begin{pmatrix} 0, 0 \\ t, 0 \end{pmatrix}, \tau = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix},\ t \in k$$

or

$$V_\infty : X = \begin{pmatrix} 0, 0 \\ 0, 0 \end{pmatrix}, Y = \begin{pmatrix} 0, 0 \\ 1, 0 \end{pmatrix}, \tau = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}$$

Compute,

$$Ext^1_A(V_t, V_t) = HH^1(A, End_k(V_t)) = Der_k(A, End_k(V_t))/Triv.$$

It is easy to see that $Ext^1_A(V_t, V_t) = k^2$, generated by the derivations, acting as follows:

$$\delta(x) = \begin{pmatrix} 0, w \\ 0, 0 \end{pmatrix}, \delta(y) = \begin{pmatrix} 0, tw \\ v,\ 0 \end{pmatrix}, \delta(\tau) = \begin{pmatrix} 0, 0 \\ 0, 0 \end{pmatrix}$$

parametrized by $v, w$. The corresponding formal moduli, and formal miniversal family are given by,

$$H(V_t)^{com} = k[[v, w]], \tilde{x} = \begin{pmatrix} 0, w \\ 1, 0 \end{pmatrix}, \tilde{y} = \begin{pmatrix} 0, tw + vw \\ v + t, 0 \end{pmatrix}, \tilde{\tau} = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}.$$

This is easily seen by checking the relations, $xy = yx, x\tau = -\tau x, y\tau = -\tau y$ in $A$.

Notice that the formal miniversal family is algebraic, and that for $w = 0$ this gives us indecomposable $A$-modules, while for $w \neq 0$ the corresponding $A$-module is simple. Moreover, the map,

$$A_0^G = k[x^2, y^2, xy] \subset k[v, w]$$

is given by,

$$x^2 = w,\ y^2 = (t + v)^2 w,\ xy = (v + t)w,$$

which proves that,

$$\pi(\Gamma) : \mathcal{S}imp_\Gamma(A) \to Spec(A_0^G),$$

is a desingularization of the affine scheme $Spec(A_0^G)$. In fact it is just the ordinary desingularization of the $A_1$-singularity $A_0^G = k[x^2, y^2, xy]$, and $\Gamma$ is just the corresponding Dynkin diagram. The exceptional fibre of $\pi$ is obviously $\mathbf{P}^1$, given by $w = 0$, and $V_\infty$, see above.

2. Consider now the group $G = Z/(2)$, generated by $\tau$, acting on $U = k^2$ by $\tau = \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix}$. In this case $A_0 = k[x, y]$, and $\tau(x) = x,\ \tau(y) = -y$, and $A_0^G = k[x, y^2]$ is non-singular. $G$ has the two simple (irreducible) representations of dimension 1, $V_i,\ i = 0, 1$, where $\tau$ acts as $(-1)^i$, respectively, and the regular representation, is the sum of these. The orbits of $G$ in $Spec(A_0) = \mathbf{A}^2$, are either of length 2, corresponding to a simple $A$-module of dimension 2, or is supported by the $x$-axis. Therefore the non-simple indecomposable $A$-modules of dimension 2, must all have support at the $x$-axis. They must be given by the indecomposables with representation graph,

$$V_0 \bullet \longrightarrow \bullet\, V_1.$$

Now all such are easily seen to be given by $k[x, y]/(x - t, y^2)$, identified with the $x$-axis. A similar computation as above shows that,

$$\pi(\Gamma) : \mathcal{S}imp_\Gamma(A) \to Spec(A_0^G),$$

is an isomorphism.

The general problem posed above seems not to be very easy, although the story is well known in case $G \subset Sl_2(k)$, and there is a long list of papers on the subject, see [B-K-R].

Now, consider for $s_2 \le s_1 \le n$, $V_1 \in Simp_{s_1}(A),\ V_2 \in Simp_{s_2}(A)$, the commutative diagram,

$$\begin{array}{ccccc} Z(n) & \longrightarrow & A(n) & & \\ \downarrow{\scriptstyle \rho_1} & & \downarrow & & \\ Z(s_1) & \longrightarrow & A(s_1) & \longrightarrow & End_k(V_1) \\ \downarrow{\scriptstyle \rho} & & \downarrow & & \\ Z(s_2) & \longrightarrow & A(s_2) & \longrightarrow & End_k(V_2). \end{array}$$

Put $\rho_2 := \rho\rho_1$, and let $t(V_i) \in Simp(Z(s_i))$ be the points corresponding to the simple modules $V_i$.

**Lemma 15.** *In the situation above, if* $Ext^1_{A(n)}(V_i, V_j) \neq 0$ *then*

$$\rho_i : Simp(Z(s_i)) \to Simp(Z(n)),\ i = 1, 2.$$

*maps* $t(V_i)$ *to the same point.*

*Proof.* If $\rho_1(t(V_1)) \neq \rho_2(t(V_2))$, the two corresponding maximal ideals $\mathfrak{m}_i,\ i = 1, 2$, of $Z(n)$ will be distinct, the sum $\mathfrak{m}_1 + \mathfrak{m}_2$ is then $Z(n)$. However, $\mathfrak{m}_i$ annihilates $V_i$, therfore the sum will annihilate $Ext^1_A(V_i, V_j)$, which therefore must be zero.

□

**Quantum correspondences of plane curves.**

Let $f \in k < x_1, x_2 >$, and put $A = k < x_1, x_2 > /(f)$. Consider the algebraic plane curve,

$$C := Simp_1(A) = Spec(k[x_1, x_2]/(f)).$$

Put

$$\Gamma = \bullet \longrightarrow \bullet$$

then there are two natural morphisms

$$pr_i : Ind_\Gamma(A) \longrightarrow Simp_1(A),\ i = 1, 2,$$

defining a correspondence,

$$\Phi = pr_1 pr_2^{-1} : C \dashrightarrow C.$$

We shall be interested in computing $\Phi$, in general, or rather, we shall be concerned with the domain of definition of $\Phi$, and its degree.

Clearly $p_2 \in \Phi(p_1)$ if and only if

$$E((p_1), k(p_2)) := Ext^1_A(k(p_1), k(p_2)) \neq 0$$

since then

$$\begin{pmatrix} k(p_1) & E((p_1), k(p_2))^* \\ 0, & k(p_2) \end{pmatrix}$$

will be an indecomposable $A$-module of dimension 2. Here $k(p_1), k(p_2)$ are, of course the two simple one-dimensional $A$-modules, corresponding to the points $p_1, p_2 \in C$. Now, we have an exact sequence of Hochschild cohomology,

$$Hom_k(k(p_1), k(p_2)) \rightarrow^{\phi} Der_k(A, Hom_k(k(p_1), k(p_2)) \rightarrow Ext^1_A(k(p_1), k(p_2)) \rightarrow 0.$$

The kernel of $\phi$ is $Hom_A(k(p_1), k(p_2))$, which is zero if $p_1 \neq p_2$, so $\phi$ must be injective, and therefore,

$$Ext^1_A(k(p_1), k(p_2)) = Der_k(A, Hom_k(k(p_1), k(p_2)))/k.$$

This implies that $Ext^1_A(k(p_1), k(p_2)) \neq 0$ if and only if,

$$J_x(f : p_1; p_2) = 0,\ J_y(f : p_1; p_2) = 0$$

Here $J_{x_i}(f : (x_1, x_2); (u_1, u_2)),\ i = 1, 2.$ are polynomials in two sets of non-commuting variables, $(x_1, x_2)$ and $(u_1, u_2)$, linear functions in $f$, and defined on monomials $m_1 m_2$ such that

$$J_{x_i}(m_1 m_2) = J_{x_i}(m_1) m_2(u_1, u_2) + m_1(x_1, x_2) J_{x_i}(m_2),\ J_{x_i}(x_j) = \delta_{i,j}.$$

In particular, $J_{x_1}([x_1, x_2]) = u_2 + x_2,\ J_{x_2}([x_1, x_2]) = x_1 + u_1$. Assume the two equations,

$$J_{x_i}(f : (x_1, x_2); (u_1, u_2)) = 0,\ i = 1, 2.$$

admits solutions, $x_i = x_i(u_1, u_2),\ i = 1, 2$, then put,

$$\tilde{f} := f(x_1(u_1, u_2), x_2(u_1, u_2)).$$

Clearly, the condition for the correspondence $\Phi$ to be defined on an open subscheme of the curve $C$, is that $\tilde{f}(u_1, u_2) = 0$ on an open subscheme of $C = Z(f(u_1, u_2))$.

The remarkable fact is that for any $f \in k < x_1, x_2 >$, we have the following result,

**Proposition 16.** *Put* $f_q = f + q[x_1, x_2]$. *Then, for generic* $q$, $\tilde{f}_q$ *vanish on an open subscheme of* $C$.

This is equivalent to saying that for generic $q$, the morphisms

$$pr_i : Ind_\Gamma(A) \longrightarrow Simp_1(A),\ i = 1, 2,$$

are dominant and finite. We notice that if they are finite, they must be of degree $\leq$(deg(f)-1)$^2$

**Non-commutative Maclaurin series..**

Before we prove the Proposition, let us take a second look at the Maclaurin expansion in classical calculus.

**Definition 17.** *Let* $f \in k < x_1, x_2 >$ *then, for any sequence* $I_r = \{i_1, i_2, ..., i_r\}$ *with* $i_l \in \{1, 2\}$ *we define inductively,*

$$J_{x_{i_1}, x_{i_2}, ..., x_{i_r}}(f : (x_1, x_2); (u_1, u_2)) =$$
$$J_{x_{i_r}}(J_{x_{i_1}, x_{i_2}, ..., x_{i_{r-1}}}(f : (x_1, x_2); (u_1, u_2)) : (x_1, x_2); (u_1, u_2))$$

*We shall call* $J_{x_{i_1}, x_{i_2}, ..., x_{i_r}}(f : (x_1, x_2); (u_1, u_2))$ *the non-commutative r'th derivative (Jacobian) of* $f$ *with respect to* $x_{i_1}, x_{i_2}, ..., x_{i_r}$.

Now these derivatives are really very nice, in fact they have the properties of divided powers,

**Lemma 18.** *Let* $S(I_r)$ *be the group of permutations of the sequence* $I_r$, *then in* $k[u_1, u_2]$

$$\sum_{S(I_r)} J_{x_{i_1}, x_{i_2}, ..., x_{i_r}}(f : (u_1, u_2); (u_1, u_2))$$
$$= 1/r_1!r_2!(\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_{r-1}}..\frac{\partial}{\partial x_1} f)(u_1, u_2),$$

*where* $r_1, r_2$ *are the numbers of, respectively 1 and 2's in the sequence* $\{i_1, i_2, ..., i_r\}$

*Proof.* The formula is true for $r = 1$, by definition. Assume that it is true for all monomials $f$ of degree $\leq n - 1$, and consider $f = m.x_i$, then, putting $x_l := x_{i_l}$ to save space,

$$J_{x_j}(m.x_i : (x_1, x_2); (u_1, u_2)) = J_{x_j}(m : (x_1, x_2); (u_1, u_2))u_i + m.\delta_{i,1}$$

Therefore,

$$\sum_{S(I_r)} J_{x_{i_1}, x_{i_2}, ..., x_{i_r}}(m.x_i : (x_1, x_2); (u_1, u_2)) =$$
$$\sum_{S(I_r)} J_{x_{i_1}, x_{i_2}, ..., x_{i_r}}(m : (x_1, x_2); (u_1, u_2))u_i +$$
$$\sum_{S(I_r)} J_{x_{i_2}, x_{i_3}, ..., x_{i_r}}(m : (x_1, x_2); (u_1, u_2))\delta_{i,1}$$

By induction, this is equal to,

$$1/r_1!r_2!(\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_{r-1}}..\frac{\partial}{\partial x_1}m)(u_1,u_2)u_i+1/(r_1-1)!r_2!(\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_{r-1}}..\frac{\partial}{\partial x_2}m)(u_1,u_2)\delta_{i,1}$$

which is easily seen to be equal to

$$1/r_1!r_2!(\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_{r-1}}..\frac{\partial}{\partial x_1}m.x_i)(u_1,u_2)$$

proving the theorem.

□

Therefore we have, formally, the following result,

**Proposition 19.** *The Maclaurin ( or Taylor) series expansion in* $k[u_1,u_2,x_1,x_2]$ *of* $f\in k<x_1,x_2>$ *is the following formula:*

$$f(x_1,x_2)=f(u_1,u_2)+\sum_{i_1,i_2,\dots,i_r}J_{x_{i_1},x_{i_2},\dots,x_{i_r}}(f:(u_1,u_2);(u_1,u_2))(x_{i_1}-u_{i_1})(x_{i_2}-u_{i_2})...(x_{i_r}-u_{i_r})$$

It is easy to see that this may be extended to a Taylor-series expansion in the non-commutative polynomial k-algebra. In fact, introduce the following notation:

**Definition 20.** *Let* $f\in k<x_1,x_2>$*, and let* $\{v_1,v_2\}$ *be new non-commuting variables. Denote by,*

$$J_{x_i}(f:\underline{x};\underline{v},\underline{u})\in k<\underline{x},\underline{v},\underline{u}>$$

*the linear function in* $f$*, defined for* $f=x_i$*, resp for* $f=mx_j$*, by:*

$$J_{x_i}(x_j:\underline{x};\underline{v},\underline{u})=\delta_{i,j}v_i$$
$$J_{x_i}(mx_j:\underline{x};\underline{v},\underline{u})=J_{x_i}(m:\underline{x};\underline{v},\underline{u})u_j+\delta_{i,j}mv_i$$

**Proposition 21.** *For* $f\in k<x_1,x_2>$*, and for some non-commuting variables* $\{v_1,v_2\}$ *we have, in* $k<\underline{u},\underline{v}>$*, the following identity,*

$$f(u_1+v_1,u_2+v_2)=f(u_1,u_2)+\sum_{i_1,i_2,\dots,i_r}J_{x_{i_1},x_{i_2},\dots,x_{i_r}}(f:(u_1,u_2);(v_1,v_2);(u_1,u_2))$$

Now, let us prove Proposition 16. For $f=f_1+q[x_1,x_2]$ the equations,

$$J_{x_i}(f:(x_1,x_2);(u_1,u_2))=0,\ i=1,2.$$

admits solutions, $x_i=x_i(u_1,u_2),\ i=1,2,$ in $k[u_1,u_2]$. Use the Maclaurin series expansion of,

$$J_{x_i}(f:(x_1,x_2);(u_1,u_2))\ i=1,2,$$

in k[[u$_1,u_2$]].

Then vi get,

$$J_{x_{i_1}}(f:(x_1,x_2);(u_1,u_2)) = J_{x_{i_1}}(f:(u_1,u_2);(u_1,u_2)) + \sum_{i_1,i_2,\ldots,i_r} J_{x_{i_1},x_{i_2},\ldots,x_{i_r}}(f:(u_1,u_2);(u_1,u_2))(x_{i_2}-u_{i_2})\ldots(x_{i_r}-u_{i_r})$$

Since

$$J_{x_i}(f:(x_1(u_1,u_2),x_2(u_1,u_2));(u_1,u_2)) = 0,\ i=1,2.$$

we find

$$J_{x_{i_1}}(f:(u_1,u_2);(u_1,u_2))(x_{i_1}-u_{i_1}) = -\sum_{i_1,i_2,\ldots,i_r} J_{x_{i_1},x_{i_2},\ldots,x_{i_r}}(f:(u_1,u_2);(u_1,u_2))(x_{i_1}-u_{i_1})(x_{i_2}-u_{i_2})\ldots(x_{i_r}-u_{i_r})$$

Using the Maclaurin series in the above Proposition, we obtain,

$$\tilde{f} := f(x_1(u_1,u_2),x_2(u_1,u_2)) = f(u_1,u_2),$$

in $k[[u_1,u_2]]$.

It is easy to see that the above can be extended to any hypersurface, and so to schemes in general. In fact, what we obtain is a kind of Abels addition theorem. See forthcoming preprint, Oslo University.

**The smooth locus of an affine non-commutative scheme.**

Recall from [La] that a point $V \in Simp_n(A)$ is called smooth (regular would probably have been better), if the natural $k$-linear map,

$$\kappa : Der_k(A,A) \longrightarrow Ext^1_A(V,V)$$

is surjective.

**Definition 22.** *Let $V \in Simp_n(A)$, then $V$ is called formally smooth if,*

$$HH^2(A, End_k(V)) = 0$$

Problem: Does

$$HH^2(A,A) = 0$$

imply that all $V \in Simp_n(A)$ are (formally) smooth?

Let $V \in Simp_n(A)$, and let $v \in Simp(Z(A))$ be the point corresponding to $V$. Denote by $\mathfrak{m}_v$ the corresponding maximal ideal of $Z(A)$. Clearly $Z(A)$ operate naturally on the Hochschild cohomology, $HH^1(A,A)$, and the map $\kappa$ factors through, $HH^1(A,A)/\mathfrak{m}_v HH^1(A,A)$, so that if $V$ is smooth, we obtain a surjectiv $k$-linear map,

$$\kappa_0 : HH^1(A,A)/\mathfrak{m}_v HH^1(A,A) \longrightarrow Ext^1_A(V,V).$$

It follows that $max_{V\in Simp(A)}\{dim_k HH^1(A,A)/\mathfrak{m}_v HH^1(A,A)\}$ is an upper bound for the dimensions of the smooth locus of $Simp_n(A)$ for all $n \geq 1$.

Clearly the definition of (formal) smoothness also works for any representation $V$.

**Proposition 23.** *If $V \in Simp_n(A)$ is smooth or formally smooth, then the corresponding point $v \in Spec(C(n))$ is also smooth.*

*Proof.* Assume that $V \in Simp_n(A)$ is formally smooth, then obviously the completion of the local ring of $Simp_n(A)$ at $V$ is $H(V)^{com}$, which since $H(V)$ has no obstructions and therefore must be the completion of the free non-commutative $k$-algebra, is a formal power series algebra, and thus $V$ is a smooth point of $Simp_n(A)$.

Now, assume $V$ is smooth, and consider the natural commutative diagram,

$$\begin{array}{ccc}
Der_k(A,A) & & \\
\downarrow \rho & & \\
Der_k(A(n),A(n)) & \overset{\gamma}{\searrow} & \\
\downarrow \kappa & & \\
Der_k(O(n),O(n)) & \xrightarrow{\delta} & Ext^1_A(V,V) \\
\downarrow \lambda & \overset{\epsilon}{\nearrow} & \uparrow \beta \\
Der_k(O(n)_{\{s\}},O(n)_{\{s\}}) & & \\
\uparrow \alpha & & \\
Der_k(S(n),S(n)) & \longrightarrow & Der_k(S(n),k(v)).
\end{array}$$

Notice that $\beta$ is an isomorphism. This has been proved above. That $\rho$ exists is easily seen, since for any derivation $\delta \in Der_k(A)$, and for any standard commutator $[x_1, x_2, ..., x_{2n}] \in I(n)$, we must have $\delta([x_1, x_2, ..., x_{2n}]) \in I(n)$. Notice that the kernel of the homomorphism, $A(n) \to O(n)$ is the image in $A(n)$ of

$$\mathfrak{n} = \bigcap_{\mathfrak{m} \in Max_n(A), m \geq 1} \mathfrak{m}^m.$$

Clearly any derivation will map an element of $\mathfrak{n}$ into $\mathfrak{n}$, proving the existence of $\kappa$. $\lambda$ is defined by localization at the point $v \in Spec(C(n))$, as in the proof of Theorem 9. We may assume $O(n)_{\{s\}}$ is a matrix algebra $M_n(S(n))$, and use the fact that any derivation of a matrix algebra is given by a derivation of the centre and an inner derivation, ($HH^1$ is Morita invariant). The inner derivation will map to zero in $Ext^1_A(V,V)$, and so the composition of $\alpha$ and $\epsilon$ is surjective.

□

The converse is not true.

**Some examples.**

1. Let $S$ be any commutative algebra, and denote by $\mathfrak{b} \subseteq \mathfrak{a} \subset S$ two ideals of $S$. Consider the $k$-algebra,

$$A := \left\{ \begin{pmatrix} a_{1,1}, & a_{1,2} \\ a_{2,1}, & a_{2,2} \end{pmatrix} \mid a_{i,j} \in S,\ a_{1,1} - a_{2,2} \in \mathfrak{a}, a_{1,2}, a_{2,1} \in \mathfrak{b} \right\}.$$

Clearly the centre of $A = A(2) = O(2)$, is $S(2) = C(2) = S$ and a simple calculation shows that,

$$A(1) = \left\{ \begin{pmatrix} \tilde{a}_{1,1}, & \tilde{a}_{1,2} \\ \tilde{a}_{2,1}, & \tilde{a}_{2,2} \end{pmatrix} \mid \tilde{a}_{i,j} \in \mathfrak{b}/\mathfrak{ab},\ i \neq j,\ \tilde{a}_{1,1}, \tilde{a}_{2,2} \in S/\mathfrak{b}^2, \tilde{a}_{1,1} - \tilde{a}_{2,2} \in \mathfrak{a}/\mathfrak{b}^2 \right\}.$$

Then $A(1)$ is the commutative $k$-algebra expressed by Nagata rings, i.e.

$$A(1) = ((S/\mathfrak{b}^2)[(\mathfrak{a}/\mathfrak{b}^2)])[(\mathfrak{b}/\mathfrak{ab})^2].$$

Consider the subschemes $V(\mathfrak{a}) \subset V(\mathfrak{b}) \subset Spec(S)$. Then, $Simp_2(A) = Spec(S) - V(\mathfrak{b})$ and a simple calculation shows that $Simp_1(A) = Spec(A(1))$ is a thickening of the affine scheme $Spec((S/\mathfrak{b}^2)[(\mathfrak{a}/\mathfrak{b}^2)])$. In the special case,

$$S = k[t_1, t_2],\ \mathfrak{a} = (f, g), \mathfrak{b} = (f)$$

where $f, g \in S$, correspond to two curves, $V(f), V(g)$ that intersect in a finite set U, one finds that $Simp_2(A)$ is an open affine subscheme of $Spec(S)$, and that $Simp_1(A) = Spec(A(1))$ is the disjoint union of the curve $V(f)$ with itself, amalgamated at the points of $U$. If both $V(f)$ and $V(g)$ are smooth, and intersect normally at the points of $U$, then the embedding-dimension of $Simp_1(A) = Spec(A(1))$ at a point not in $U$, is 2, and at the points of $U$, 6!

2. Let in the above example, $\mathfrak{b} = \mathfrak{a} = (t_1, t_2)$, then $Simp_2(A) = Spec(s) - \{(0, 0)\}$, therefore not affine, and $Simp_1(A) = Spec(A(1))$ is a thick point situated at the origin of the affine 2-space $Spec(S)$.

3.Let us compute the $Simp_2(A)$ for the non-commutative cusp, i.e. for the $k$-algebra,

$$A = k < x, y > /(x^3 - y^2).$$

We first notice that the center $Z(A) \subset A$ is the subalgebra of $A$ generated by $t := x^3 = y^2$. Put

$$u_1 = x^2 y,\ v_1 = yx^2.$$

Then there is a surjective morphism,

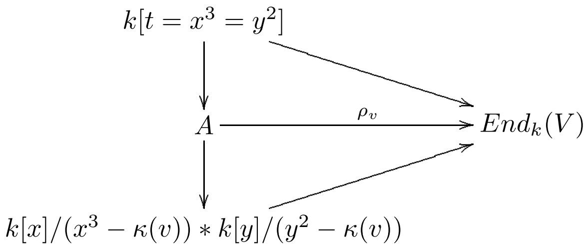
$$k[t, t^{-1}] < u, v > /(uvu - vuv) \longrightarrow A(t^{-1})$$

mapping $u$ to $u_1$ and $v$ to $v_1$. In fact, $u_1 v_1 = t^2 x$ and $v_1 u_1 v_1 = t^3 y$, and finally $u_1 v_1 u_1 = t^3 y = v_1 u_1 v_1$. (The relations with the equation of Yang-Baxter, if any, will have to be discovered.)

Now let us compute the $Simp_n(A)$. It is clear that any surjectiv homomorphism of $k$-algebras,

$$\rho_v : A \longrightarrow End_k(V)$$

will map $Z(A) = k[t]$ into $Z(End_k(V)) = k$, inducing a point $v \in Simp(k[t]) = \mathbf{A}^1$. This means that $Simp_n(A)$ is fibred over the affine line $Spec(k[t]) = \mathbf{A}^1$. Let $\rho_v(x)^3 = \rho_v(y)^2 = \kappa(v)\mathbf{1}$, where $\mathbf{1}$ is the identity matrix, and where $\kappa(v)$ is a parameter of the cusp. Then either $v = origin =: \underline{o}$ or we may assume $\kappa(v) \neq 0$. Consider now the diagram:




Clearly, if $\kappa(v) \neq 0$ the simple representations of $A$ are fibered on the cusp with fibres being the simple representations of $U := k[x]/(x^3 - \kappa(v)) * k[y]/(y^2 - \kappa(v))$, isomorphic to the group algebra of the modular group $Sl_2(\mathbf{Z})$. Since the representation theory of $Sl_2(\mathbf{Z})$ is known, this shows, in principle, how to go about describing the open subscheme of $Simp_n(A)$ corresponding to $\kappa(v) \neq 0$, for all $n \geq 0$.

We shall however have to work a little to find the fibre of $Simp_n(A)$ corresponding to the singular point of the cusp. When $n = 2$ it is clear that we have no choice, but to fix the Jordan form of $\rho_v(y)$ equal to the Jordan form of

$$\rho_v(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $I(\rho_v(x))$ be the isotropy subgroup of the action of $Gl_n(k)$ on $M_n(k)$, at $\rho_v(x)$. Set theoretically, the fiber is then the double quotient,

$$I(\rho_v(x)) \backslash Gl_n(k) / I(\rho_v(x))$$

To find the scheme structure we may compute the formal moduli of the simple module given by,

$$\rho_v(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \rho_v(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We compute and find the following,

**Example 24.** *Let $A$ be the non-commutative cusp. Then*

*(i) $Simp_1(A) = Spec(k[x,y]/(x^3 - y^2))$*

*(ii) $Simp_2(A)$ is fibered on the cusp minus the origin, with fiber $E(\underline{t}) = U_2/T^2$ where $U_2$ is an open subscheme of the 3-dimensional scheme of all pairs of 2-vectors, with vector product equal 1, and $T^2$ is a two dimensional torus, acting naturally on $U_2$.*

*(iii) $S(2) = k[t^2, t^3, u]$.*

*(iv) The fiber $E(\underline{o}))$ over $\underline{o}$ is given by,*

$$\tilde{\rho}(x) = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}, \ \tilde{\rho}(y) = \begin{pmatrix} u & 0 \\ 1+v & -u \end{pmatrix}$$

*parametrized by the $k$-algebra $k[t,u,v]/(t^2, u^2, (1+v)t)$, i.e. it is the open subscheme of the double line parametrized by $v$, with the point $v = -1$ removed.*

*(v) In particular we find that $E(\underline{o}))$ is a component of $Simp_2(A)$.*

**The Jordan correspondence.** As we have seen in the above example, the computation of the structure of the different $Simp_n(A)$ for a given $k$-algebra $A$, is naturally related to the problem of finding the possible Jordan forms for the action of the generators $\{x_i\}_{i=1}^d$ of $A$ on a vector space of dimension $n$.

Notice that when $A$ is the group algebra of the homotopy group of the p-pointed Poincaré sphere this problem is, in some quarters, called the Deligne-Simpson problem, and is related to classical problems in monodromy theory, see e.g. [Katz], [Kostov] and [Simpson].

We shall now see how this can be formulated in non-commutative algebraic geometry, using the existence of a non-commutative moduli space for iso-classes of endomorphisms, developed in [La 1], § 8. Let $End_k(k^n) = Spec(k[x_{i,j}])$, and let $B := k[x_{i,j}]$ and $G := Gl_n(k)$. For each *formal normal Jordan form* of dimension $n$, there is an orbite, such that the affine ring of its closure is a $B-G$-representation $\rho_i : B \to V_i$. Corresponding to a family $\mathcal{V} = \{V_i\}_i$ of $B-G$-modules, there is a deformation functor and a versal family of $B-G$-modules, $\tilde{\mathcal{V}}$, together with a homomorphism of $B$-modules,

$$\tilde{\rho} : B \to \tilde{\mathcal{V}} = (H_{i,j} \otimes V_j).$$

In all cases known to us, there is an algebraic $k$-algebra $H' \subset (H_{i,j})$, and a universal family defined on $H'$, inducing the formal one above. This $H'$, from now on called $End(n)$, is simply $O(\mathcal{V}^*, \pi)$, the affine $k$-algebra of the non-commutative moduli scheme $\mathbf{End}(n)$ of iso-classes of endomorphisms, see [La 2]. Here $\mathcal{V}^*$ is the $A-G$-swarm defined by the morphisms, $\rho_i : B \to V_i$. There is a homomorphism of $k$-algebras,

$$\eta : B \to O(\mathcal{V}, \pi) = (H_{i,j} \otimes Hom_k(V_i, V_j)).$$

inducing a homomorphism of $k$-algebras,

$$\eta : B^G \to End(n).$$

In [La 1] we have computed $End(n)$ for $n = 2$ and in a forthcoming paper, see [Siq 2], Arvid Siqveland has computed $End(n)$ for $n = 3$. There is, however, a problem with this set up; the lack of *an algebraic structure* on the map,

$$M_n(k) := End_k(k^n) \longrightarrow \mathbf{End}(n).$$

To overcome this, let us go back to the general theory for a while. Let $A$ be given, as above, and consider a swarm, $\underline{c} \subset A-mod$. Let $V_i, V_j \in |\underline{c}|$. We shall say that $V_i$ is *above* $V_j$, and write it $V_i > V_j$ if $Ext^1_A(V_i, V_j) \neq 0$. Given a point $V \in |\underline{c}|$ we shall call the subset $\{V' \in |\underline{c}|| \ V' > V\}$ *the focal swarm* of $V$, and the subset $\{V' \in |\underline{c}|| \ V > V'\}$ will be called the *local swarm* of $V$. In the case of the swarm $Simp(B-G)$, if $V$ is an object, i.e. the affine algebra of the closure of an orbit, there is a finite focal swarm $\mathcal{V}_V$ of $V$, corresponding to the orbits $Simp(V_i)$ containing $Simp(V)$ in their closure,i.e. to the set of points $V_i$ for which there is a $B-G$-module homomorphism of $V_i$ onto $V$.

Now consider the left $End(n)$ and right $B$-module $\tilde{\mathcal{V}}$, and fix an element $q \in Simp(B)$. Then there exists a unique closed orbit $Simp(V(q))$ containing $q$, such that $q \in Simp(V) - \cup_{V_j < V} Simp(V_j)$. Let $\mathcal{V}_q := \mathcal{V}_{V(q)}$, and consider the commutative diagram,

$$\begin{array}{ccc} \tilde{\mathcal{V}} & \longrightarrow & \prod_{\mathcal{V} \subset Simp(B-G)} (H_{i,j}(\mathcal{V}) \otimes V_j) \\ \downarrow & & \downarrow \\ \tilde{\mathcal{V}} \otimes_B k(q) & \longrightarrow & \prod_{\mathcal{V} \subset Simp(B-G)} (H_{i,j}(\mathcal{V}) \otimes V_j \otimes_B k(q)) \\ \downarrow & & \downarrow \\ \tilde{H}(\mathcal{V}_q) & \longrightarrow & H(\mathcal{V}_q) \end{array}$$

Here $\mathcal{V}$ runs through all finite subsets of $Simp(B - G)$, and $k(q)$ is the residue field of the point $q \in Simp(B)$. This induces an $End(n)$-module homomorphism,

$$\tilde{q} : \tilde{\mathcal{V}} \longrightarrow H(\mathcal{V}_q)$$

Notice that the points, i.e. simple quotient modules, of the $End(n)$-module $H(\mathcal{V}_q)$ correspond precisely to the local swarm $\mathcal{V}_q$. Moreover, this defines a unique, *algebraic*, morphism, the Jordan morphism,

$$J : End_k(k^n) \longrightarrow \{\mathcal{V} \subset \mathbf{E}nd(n) | \ \mathcal{V} \text{ local swarm}\}.$$

Notice also that $\tilde{H}(\mathcal{V}_q)$ is a left $End(n)$ and a right $B$-module. Fixing $q$, any element $c \in Simp(End(n))$, i.e. any Jordan form, therefore defines a simple $B$-module, an element $c(q) \in Simp(B)$. In this way we obtain a local section of the $Gl_n(k)$-orbit stratification of $M_n(k)$ parametrized by $Simp(End(n))$.

Now, assume given a $k$-algebra $A$, generated by the elements $\{x_i\}_{i=1}^p$, and a simple $n$-dimensional representation $V \in Simp_n(A)$. Recall again the commutative diagram,

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow & & \\ S(n) & \longrightarrow & O(n) & \longrightarrow & C(n) \otimes_k End_k(V) \\ & & \downarrow & & \downarrow \\ & & H^O(V) \otimes_k End_k(V) & \simeq & \hat{C}_{t(V)} \otimes_k End_k(V). \end{array}$$

Clearly, the element $x_i \in A$ induces a homomorphism,

$$x_i : B \to C(n),$$

therefore a natural map,

$$X_i : Simp_n(A) \to Simp(B) = M_n(k).$$

Together we have proved the following,

**Theorem 25.** *There exists a natural algebraic correspondence,*

$$J(x_1, x_2, ..., x_r) : Simp_n(A) \longrightarrow \{\mathcal{V} \subset \mathbf{E}nd(n)|\ \mathcal{V}\ local\ swarm\}^r$$

Let us compute $J$ in the first non-trivial case, i.e. for $n = 2$. For this we first need to compute the versal family, $\tilde{\mathcal{V}}$, i.e. the action of $B$ on $\tilde{\mathcal{V}} = H \otimes \mathcal{V}$. This is easily done by using the $k$-linear and $Gl(2)$-invariant section of the morphism $B \to V_1 = B/(s_1, s_2)$, induced by fixing a $k$-basis for $V_1$,

$$\{x_{1,1}^{n_0} x_{1,2}^{n_1} x_{2,1}^{n_2} =: x_{1,1}^{n_0} v_0\}_{0 \le n_0 \le 1, 0 \le n_1, n_2}$$

mapping, multiplicatively, $x_{1,1}$ to $1/2(x_{1,1} - x_{2,2})$, and $x_{i,j}, i \neq j$ to $x_{i,j}$, see §10 of [La 1]. We obtain,

$$\tilde{\mathcal{V}} = (H(\{V_i\})_{i,j} \otimes V_j) = \begin{pmatrix} k[s_1, s_2] \otimes V_1 & H_{1,2} \otimes V_2 \\ 0 & k[s] \otimes V_2 \end{pmatrix}$$

where $V_2 = k$, subject to the relation in $H_{1,2} = k[s_1, s_2] < t_1, t_2 > k[s]$,

$$t_1 s^2 - s_2 t_1 - 2 \cdot t_2 s + s_1 t_2 = 0,$$

with the $k[x_{i,j}]$-action given by,

$$\begin{pmatrix} 1 \otimes v_1 & 0 \\ 0 & 1 \otimes v_2 \end{pmatrix} x_{i,j} = \begin{pmatrix} 1 \otimes v_1 x_{i,j} & 0 \\ 0 & 0 \end{pmatrix}$$

if $i \neq j$, and,

$$\begin{pmatrix} 1 \otimes v_0 & 0 \\ 0 & 1 \otimes v_2 \end{pmatrix} x_{1,1} = \begin{pmatrix} 1 \otimes v_0 x_{1,1} - 1/2 s_1 \otimes v_0 & -1/2 t_1 \otimes \overline{v}_0 \\ 0 & -s \otimes v_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \otimes v_0 & 0 \\ 0 & 1 \otimes v_2 \end{pmatrix} x_{2,2} = \begin{pmatrix} -1 \otimes v_0 x_{1,1} - 1/2 s_1 \otimes v_0 & -1/2 t_1 \otimes \overline{v}_0 \\ 0 & -s \otimes v_2 \end{pmatrix}$$

Moreover, the (1,1)-term of the matrix,

$$\begin{pmatrix} 1 \otimes v_1 & 0 \\ 0 & 1 \otimes v_2 \end{pmatrix} x_{1,1}$$

for $v_1 = v_0 x_{1,1}$, looks like,

$$-1 \otimes v_0 x_{1,2} x_{2,1} - 1/2 s_1 \otimes v_0 x_{1,1} + s_2 \otimes v_0$$

and the (1,2)-term has the form,

$$t_2 \otimes v_0' - 1/2 t_1 s \otimes v_0' - (s_1/2)^2/(1 - (s_1/2)) t_1 \otimes v_0',$$

The (1,1)-term of the matrix,x,

$$\begin{pmatrix} 1 \otimes v_1 & 0 \\ 0 & 1 \otimes v_2 \end{pmatrix} x_{2,2}$$

has the form,

$$1 \otimes v_0 x_{1,2} x_{2,1} - 1/2 s_1 \otimes v_0 x_{1,1} - s_2 \otimes v_0$$

and the (1,2)-term looks like,

$$-t_2 \otimes v_0' + 1/2 t_1 s \otimes v_0' + (s_1/2)^2/(1 - (s_1/2)) t_1 \otimes v_0'.$$

Here $v_0'$ is the image of $v_0$ in $V_2$. Notice that for $s_1 = 2$ these formulas are undefined. Assume $s_1 \neq 2$, then $J$ is defined, and in particular,

$$J(\underline{0}) = ((0, 0), 0).$$

The (generalized Deligne-Simpson) problem we encountered above, is now the following:

**Problem 26.** *Given a $k$-algebra $A$, finitely generated by the elements $\{x_i\}_{i=1}^r$, characterize the image of the morphism,*

$$J(x_1, x_2, ..., x_r) : Simp_n(A) \to \mathbf{End}(n)^p.$$

In the case of the cusp above, it is easy to compute the image of $J$, when $n = 1, 2$, and not so easy when $n \geq 3$.

**A structure theorem for geometric $k$-algebras.** Let $A$ be a geometric algebra, and assume moreover that $I(n) = 0$ thus, $A \simeq A(n)$, so that $A$ does not have any simple modules of dimension greater than $n$. Now, for any $m \leq n$, consider the natural morphism,

$$A \to \prod_{\mathcal{V} \subset Simp_m(A)} O^A(\mathcal{V})$$

where $\mathcal{V}$ runs through all finite subsets of $Simp_m(A)$. Call the image $D(m)$. Clearly there is a natural surjectiv homomorphism,

$$D(m) \longrightarrow O(m) \subset \prod_{V \in Simp_m(A)} H^{A(m)} \otimes End_k(V),$$

see Proposition 11. Let $\mathcal{D}(m)$, $\mathcal{O}(m)$, be corresponding (non-commutative) sheaves on $Simp_m(A)$. Consider the diagram,

$$\begin{array}{ccccc}
K(n) & \longrightarrow & A(n) & \longrightarrow & \mathcal{D}(n) \\
\downarrow{\scriptstyle \rho_1} & & \downarrow & & \\
K(n-1) & \longrightarrow & A(n-1) & \longrightarrow & \mathcal{D}(n-1) \\
\downarrow{\scriptstyle \rho} & & \downarrow & & \\
0 & \longrightarrow & A(1) & \longrightarrow & \mathcal{D}(1).
\end{array}$$

where $K(m)$ is the kernel of the morphism $A(m) \to D(m)$. Clearly $K(1) = 0$.

**Theorem 27.** *For any geometric $k$-algebra with $I(n) = 0$, there is a sheaf of matrix algebras $\mathcal{D}$, defined on $Simp_n(A)$, and an injectiv homomorphism of $k$-algebras,*

$$A \longrightarrow \mathcal{D},$$

*where $\mathcal{D}$ is generated by matrices of the type,*

$$\begin{pmatrix} \mathcal{D}(n) & * & ... & * \\ * & \mathcal{D}(n-1) & .. & \\ * & * & . & \mathcal{D}(1) \end{pmatrix},$$

*such that $Simp_m(A) = Simp(\mathcal{D}(m))$.*

*Proof.* This is now just another way of stating Proposition 1., i.e. saying that $A \simeq O(Simp^*(A))$, since clearly $O(Simp^*(A)) \subseteq \mathcal{D}$.

□

The following simple consequence of the $O$-construction, is going to be rather useful,

**Corollary 28.** *Suppose the geometric $k$-algebra $A$ satisfies the following conditions,*

(1) *I(n)=0*
(2) $Ext^1_A(V,V')=0,\ if\ dimV<dimV'(resp.\ if\ dimV>dimV')$

*Then $\mathcal{D}$ is a sheaf of upper triangular (resp. lower triangular) matrices of the form,*

$$\mathcal{D}=\begin{pmatrix}\mathcal{D}(n) & * & ... & * \\ 0 & \mathcal{D}(n-1) & .. & \\ 0 & 0 & . & \mathcal{D}(1)\end{pmatrix}.$$

**Remark.** *The above condition (2) is very often satisfied, and in particular, it is satisfied for the the coordinate $k$-algebras of affine subschemes of (non-commutative) orbit spaces of the action of a (finite dimensional) reductive Lie group. In fact, if the Lie group $G$ acts on the affine scheme $X=Spec(B)$ such that the (non-commutative) orbit space, see* [La ?] *is an affine (non-commutative) $k$-algebra $A$, then, for any local swarm $\mathcal{V}=\{V_1,V_2,...,V_r\}$, of $B-G$-modules, corresponding to closed orbits $Spec(V_1)\supset Spec(V_2)\supset ...\supset Spec(V_r)$, then*

$$Ext^1_{A-G}(V_i,V_j)=0, for\ all\ j<i. \tag{3}$$

*This implies that the corresponding formal moduli of $\mathcal{V}=\{V_1,V_2,...,V_r\}$ has the form,*

$$H(V)=\begin{pmatrix}H_{1,1} & * & ... & * \\ 0 & H_{2,2} & .. & \\ 0 & 0 & . & H_{r,r}\end{pmatrix}.$$

*This again will imply that $A$ will have the form,*

$$A=\begin{pmatrix}\mathcal{S}(n) & * & ... & * \\ 0 & \mathcal{S}(n-1) & .. & \\ 0 & 0 & . & \mathcal{S}(0)\end{pmatrix}.$$

*Here $Simp(\mathcal{S}(p))$ is the (possibly non-commutative) subscheme of $Simp(A)$ corresponding to the $p$-dimensional orbits. Let us prove (3) above. There are two spectral sequences converging to $Ext^*_{A-G}(V_i,V_j)$, one given by*

$$E_2^{p,q}=H^p(G,Ext^q_B(V_i,V_j)),$$

*the other with,*

$$E_2^{p,q}=HH^p(B,H^q(G,Hom_k(V_i,V_j))).$$

*If $p+q=1$, then the last one will be reduced to,*

$$E_2^{0,1}=HH^0(B,H^1(G,Hom_k(V_i,V_j)))=0,$$

*since $G$ is reductive, and*

$$E_2^{1,0}=HH^1(B,H^0(G,Hom_k(V_i,V_j)))=0,$$

*since, obviously, $H^0(G,Hom_k(V_i,V_j))=Hom_G(V_i,V_j)=0$ for $j<i$.*

**A spectral sequence.** Let the finitely generated $k$-algebra $A$ be such that $A \simeq A(n)$. Then $Simp_m(A) = \emptyset$, for $m \geq n$. To what extent will the globale scheme structures of the $Simp_p(A)$ determine the *globale* structure of $A$, and vice versa? In particular, is the cyclic homology of $A$ determined by the de Rham cohomology of the different $Simp_p(A)$, and conversely, what can we learn about the de Rham cohomology of $Simp_p(A)$ knowing the cyclic cohomology of $A$? The first result in this direction is the following trivial observation,

**Lemma 29.** *Suppose, in the above situation, that the ideals $J(m-1) := I(m-1)/I(m) \subset A(m),\ m \geq 1$, are H-unital, then there existe a spectral sequence with,*

$$E^1_{p,m} = HC_p(J(m-1)),$$

*converging to (abutting at) $HC_*(A)$.*

*Proof.* See, e.g. [Loday]

□

**Theorem 30.** *Let $A$ satisfy the following conditions,*

(1) *I(n)=0*
(2) $Ext^1_A(V, V') = 0,\ if\ dimV < dimV'(resp.\ if\ dimV > dimV').$
(3) $Simp_m(A) = Spec(C(n))$ *is affine for* $m \geq 1$.

*Then,*

$$A \simeq \mathcal{D}$$

*and there is a spectral sequence with,*

$$E^1_{p,m} = HC_p(C(m)),$$

*converging to (abutting at) $HC_*(A)$. Moreover, if all $Simp_m(A)$ are smooth affine schemes, then*

$$HC_p(C(m)) = \oplus_{l\geq 1} H^{p-2l}_{d.R}(Simp_m(A)) \oplus (\Omega^p_{Simp_m(A)}/d\Omega^{p-1}_{Simp_m(A)})$$

*Proof.* Use the Lemma 23. If $Simp_m(A)$ is affine for $m \geq 1$, it follows that the map $A(m) \to C(m) \otimes M_m$ is surjectiv. The problem is to show that $I(m-1)$ maps surjectively onto $C(m) \otimes M_m$. However, the image of $I^A(m-1)$ in $C(m) \otimes M_m$ is $I^{C(m)\otimes M_m}(m)$ which, obviously, is $C(m) \otimes M_m$, since $C(m) \otimes M_m$ has no modules of dimension strictly less then $m$. But then $A \simeq \mathcal{D}$. Now, $A \simeq \mathcal{D}$ is triangular, and the ideals $J(m-1) := I(m-1)/I(m) \subset A(m)$ are obviously H-unital. Since cyclic homology is Morita invariant, the result follows from, e.g. [Loday], see 2.2.12, and Chapter 3.

□

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