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Intuitive construction

- A hyperreal is a sequence of reals $\langle r_n \rangle$
Intuitive construction

- A hyperreal is a sequence of reals \( \langle r_n \rangle \)
  - If \( \lim_{n \to \infty} r_n = 0 \), \( \langle r_n \rangle \) is infinitely small.
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  - If $\lim_{n \to \infty} r_n = 0$, $\langle r_n \rangle$ is infinitely small.
  - If $\lim_{n \to \infty} r_n = \infty$, $\langle r_n \rangle$ is infinitely large.
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■ Elementwise addition and multiplication
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- Elementwise addition and multiplication
- Problem: This is not a field.

$$\langle 1, 0, 1, 0, \ldots \rangle \odot \langle 0, 1, 0, 1, \ldots \rangle = \langle 0, 0, 0, 0, \ldots \rangle$$
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  \langle 1, 0, 1, 0, \ldots \rangle \odot \langle 0, 1, 0, 1, \ldots \rangle = \langle 0, 0, 0, 0, \ldots \rangle
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- Solution: Introduce an equivalence relation.
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- Solution: Introduce an equivalence relation.
- Large subsets of $\mathbb{N}$. 
Ultrafilters

Definition

An ultrafilter on \( \mathbb{N}, \mathcal{F} \), is a set of subsets of \( \mathbb{N} \) such that:

- If \( X \in \mathcal{F} \) and \( X \subseteq Y \subseteq \mathbb{N} \), then \( Y \in \mathcal{F} \).
- If \( X \in \mathcal{F} \) and \( Y \in \mathcal{F} \), then \( X \cap Y \in \mathcal{F} \).
- \( \mathbb{N} \in \mathcal{F} \), but \( \emptyset \notin \mathcal{F} \).
- For any subset \( A \) of \( \mathbb{N} \), \( \mathcal{F} \) contains exactly one of \( A \) and \( \mathbb{N} \setminus A \).

We say that an ultrafilter is free if it contains no finite subsets of \( \mathbb{N} \).
Ultrafilters

Definition

An *ultrafilter* on $\mathbb{N}$, $\mathcal{F}$, is a set of subsets of $\mathbb{N}$ such that:

- If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq \mathbb{N}$, then $Y \in \mathcal{F}$.
- If $X \in \mathcal{F}$ and $Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.
- $\mathbb{N} \in \mathcal{F}$, but $\emptyset \notin \mathcal{F}$.
- For any subset $A$ of $\mathbb{N}$, $\mathcal{F}$ contains exactly one of $A$ and $\mathbb{N} \setminus A$.

We say that an ultrafilter is *free* if it contains no finite subsets of $\mathbb{N}$.

Theorem

*There exists a free ultrafilter on $\mathbb{N}$.*
Formal construction

- Fix a free ultrafilter $\mathcal{F}$ on $\mathbb{N}$. 

Equivalence relation:

$\langle r^n \rangle \equiv \langle s^n \rangle \iff \{ n \in \mathbb{N} | r^n = s^n \} \in \mathcal{F}$.

Defining the hyperreals $\mathbb{R}^*$ as

$\mathbb{R}^* = \{ [r] | r \in \mathbb{R} \} = \mathbb{R}_F = \mathbb{R} / \equiv$.

Addition and multiplication:

$[r] + [s] = [\langle r^n \rangle] + [\langle s^n \rangle] = [\langle r^n + s^n \rangle]$,

$[r] \cdot [s] = [\langle r^n \rangle] \cdot [\langle s^n \rangle] = [\langle r^n \cdot s^n \rangle]$.

Ordering relation:

$[r] < [s] \iff \{ n \in \mathbb{N} | r^n < s^n \} \in \mathcal{F}$. 

Formal construction

- Fix a free ultrafilter $\mathcal{F}$ on $\mathbb{N}$.
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Formal construction

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  \[
  \langle r_n \rangle \equiv \langle s_n \rangle \iff \{ n \in \mathbb{N} \mid r_n = s_n \} \in \mathcal{F}.
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- Defining the hyperreals ($^*\mathbb{R}$) as
  \[
  ^*\mathbb{R} = \{ [r] \mid r \in \mathbb{R}^\mathbb{N} \} = \mathbb{R}^\mathbb{N}/\equiv.
  \]
- Addition and multiplication:
  \[
  [r] + [s] = [\langle r_n \rangle] + [\langle s_n \rangle] = [\langle r_n + s_n \rangle]
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  [r] \cdot [s] = [\langle r_n \rangle] \cdot [\langle s_n \rangle] = [\langle r_n \cdot s_n \rangle].
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Formal construction

- Fix a free ultrafilter $\mathcal{F}$ on $\mathbb{N}$.
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  \[ [r] + [s] = [\langle r_n \rangle] + [\langle s_n \rangle] = [\langle r_n + s_n \rangle] \]
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- Ordering relation:

  \[ [r] < [s] \iff \{ n \in \mathbb{N} \mid r_n < s_n \} \in \mathcal{F}. \]
Infinitely small and large numbers

**Theorem**

There exist numbers $\varepsilon, \omega \in \mathbb{R}^*$ such that $0 < \varepsilon < r$ for any positive real number $r$ and $\omega > r$ for any real number $r$. 

Theorem

There exist numbers $\varepsilon, \omega \in \ast \mathbb{R}$ such that $0 < \varepsilon < r$ for any positive real number $r$ and $\omega > r$ for any real number $r$.

Proof of the first part.

- Real numbers in $\ast \mathbb{R}$: $r \mapsto *r = \langle r, r, r, \ldots \rangle$.
- Let $\varepsilon = [\langle 1, \frac{1}{2}, \ldots \rangle] = [\langle \frac{1}{n} \rangle]$. For any positive real $r$, the set $\{n \in \mathbb{N} \mid \frac{1}{n} > r\}$ is finite, and hence $\{n \in \mathbb{N} \mid \frac{1}{n} < r\}$ is cofinite, and so is in $\mathcal{F}$, which means that $\varepsilon < r$. Since $\{n \in \mathbb{N} \mid 0 < \frac{1}{n}\} = \mathbb{N} \in \mathcal{F}$, we also have that $0 < \varepsilon$. ■
Enlarging sets

- For a subset $A$ of $\mathbb{R}$, we create the enlarged subset $^*A$ of $^*\mathbb{R}$. 

\[ r \in ^*A \iff \{ n \in \mathbb{N} | r_n \in A \} \in F. \]

Examples:

- If $\omega = \langle 1, 2, 3, ... \rangle$, then $\{ n \in \mathbb{N} | \omega_n \in \mathbb{N} \} = \mathbb{N} \in F$, so $[\omega] \in ^*\mathbb{N}$.

$^*\{ \{ r \} \} = \{ ^*r \}$.

$^* (A \cup B) = ^*A \cup ^*B$.

$^* (0, 1) = \{ x \in ^*\mathbb{R} | 0 < x < 1 \}$. 

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Hyperreal Calculus
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Enlarging sets

- For a subset \( A \) of \( \mathbb{R} \), we create the enlarged subset \( \ast A \) of \( \ast \mathbb{R} \).
- Definition:

\[
[r] \in \ast A \iff \{ n \in \mathbb{N} \mid r_n \in A \} \in \mathcal{F}.
\]

Examples:

If \( \omega = \langle 1, 2, 3, \ldots \rangle \), then \( \{ n \in \mathbb{N} \mid \omega_n \in \mathbb{N} \} \in \mathcal{F} \), so \( \omega \in \ast \mathbb{N} \).

\( \ast \{ r \} = \{ \ast r \} \)

\( \ast (A \cup B) = \ast A \cup \ast B \)

\( \ast (0, 1) = \{ x \in \ast \mathbb{R} \mid 0 < x < 1 \} \)
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- For a subset $A$ of $\mathbb{R}$, we create the enlarged subset $\ast A$ of $\ast \mathbb{R}$.
- Definition:

  $$[r] \in \ast A \iff \{ n \in \mathbb{N} \mid r_n \in A \} \in \mathcal{F}.$$  

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Enlarging sets

- For a subset $A$ of $\mathbb{R}$, we create the enlarged subset $*A$ of $*\mathbb{R}$.
- Definition:
  \[ [r] \in *A \iff \{ n \in \mathbb{N} \mid r_n \in A \} \in \mathcal{F}. \]
- Examples:
  - If $\omega = \langle 1, 2, 3, \ldots \rangle$, then $\{ n \in \mathbb{N} \mid \omega_n \in \mathbb{N} \} = \mathbb{N} \in \mathcal{F}$, so $[\omega] \in *\mathbb{N}$. 


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Definition:

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- If $\omega = \langle 1, 2, 3, \ldots \rangle$, then $\{n \in \mathbb{N} \mid \omega_n \in \mathbb{N}\} = \mathbb{N} \in \mathcal{F}$, so $[\omega] \in ^*\mathbb{N}$.
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- **Definition:**

  
  $[r] \in *A \iff \{n \in \mathbb{N} | r_n \in A\} \in \mathcal{F}.$

- **Examples:**

  - If $\omega = \langle 1, 2, 3, \ldots \rangle$, then $\{n \in \mathbb{N} | \omega_n \in \mathbb{N}\} = \mathbb{N} \in \mathcal{F}$, so $[\omega] \in *\mathbb{N}$.
  - $*\{r\} = \{*r\}$
  - $*(A \cup B) = *A \cup *B$
Enlarging sets

- For a subset $A$ of $\mathbb{R}$, we create the enlarged subset $^*A$ of $^*\mathbb{R}$.

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- Examples:
  - If $\omega = \langle 1, 2, 3, \ldots \rangle$, then $\{n \in \mathbb{N} \mid \omega_n \in \mathbb{N}\} = \mathbb{N} \in \mathcal{F}$, so $[^*\omega] \in ^*\mathbb{N}$.
  - $[^*\{r\}] = \{^*r\}$
  - $^*(A \cup B) = ^*A \cup ^*B$
  - $^*(0, 1) = \{x \in ^*\mathbb{R} \mid 0 < x < 1\}$
Extending functions

- Take a function \( f : \mathbb{R} \to \mathbb{R} \), and extend it to \( *f : \ast \mathbb{R} \to \ast \mathbb{R} \).
Extending functions

- Take a function $f : \mathbb{R} \to \mathbb{R}$, and extend it to $\star f : \star \mathbb{R} \to \star \mathbb{R}$.
- Definition:

\[
\star f([\langle r_1, r_2, \ldots \rangle]) = [\langle f(r_1), f(r_2), \ldots \rangle].
\]
Extending functions

- Take a function $f : \mathbb{R} \to \mathbb{R}$, and extend it to $*f : *\mathbb{R} \to *\mathbb{R}$.
- Definition:

  
  
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  *f([\langle r_1, r_2, \ldots \rangle]) = [\langle f(r_1), f(r_2), \ldots \rangle].
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- Can also extend $f : A \to \mathbb{R}$ to $*f : *A \to *\mathbb{R}$ with the same idea, but with a small trick.
Extending functions

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- Definition:
  \[
  *f([\langle r_1, r_2, \ldots \rangle]) = [\langle f(r_1), f(r_2), \ldots \rangle].
  \]
- Can also extend \( f : A \to \mathbb{R} \) to \( *f : *A \to *\mathbb{R} \) with the same idea, but with a small trick.
- Note that \( *f(*r) = *(f(r)) \).
The transfer principle

- What is the transfer principle?
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- What is the transfer principle?
- $\mathcal{L}$-sentences
The transfer principle

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- $\mathcal{L}$-sentences
- Examples:

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n)$$

$$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = y)$$

$$(\forall x \in \mathbb{R})(x > 0 \rightarrow \neg(x < 0))$$

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n)$$

$$(\forall n \in \mathbb{N}^*)(\exists m \in \mathbb{N}^*)(m > n)$$

$$(\forall x \in \mathbb{R})(\sin(x) < 2)$$

$$(\forall x \in \mathbb{R}^*)(\sin(x)^* < 2^*)$$
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■ $\ast$-transforms

Theorem (Transfer principle)
An $\mathcal{L}$-sentence $\phi$ is true if and only if its $\ast$-transform $\ast\phi$ is true.
The transfer principle

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- Examples:
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- $\ast$-transforms
  - $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n)$ to $(\forall n \in \ast\mathbb{N})(\exists m \in \ast\mathbb{N})(m > n)$.
The transfer principle

What is the transfer principle?

$L$-sentences

Examples:

- $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n)$
- $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = y)$
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$\ast$-transforms

- $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m > n)$ to $(\forall n \in \ast\mathbb{N})(\exists m \in \ast\mathbb{N})(m \ast > n)$.
- $(\forall x \in \mathbb{R})(\sin(x) < 2)$ to $(\forall x \in \ast\mathbb{R})(\ast\sin(x) < \ast2)$. 
The transfer principle

- What is the transfer principle?
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- Examples:
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The transfer principle

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  - $\forall n \in \mathbb{N} (\exists m \in \mathbb{N}) (m > n)$
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Theorem (Transfer principle)

An $\mathcal{L}$-sentence $\varphi$ is true if and only if its $\ast$-transform $\ast\varphi$ is true.
Using the transfer principle

Theorem

The structure \( \langle \ast \mathbb{R}, +, \cdot, < \rangle \) is an ordered field with zero and unity.
Using the transfer principle

Theorem

The structure $\langle \ast \mathbb{R}, +, \cdot, \prec \rangle$ is an ordered field with zero and unity.

Proof.

All axioms for ordered fields can be expressed as $\mathcal{L}$-sentences. For example, that addition is commutative in $\mathbb{R}$ can be expressed as $(\forall x, y \in \mathbb{R})(x + y = y + x)$, which is true. Then, we conclude by transfer that $(\forall x, y \in \ast \mathbb{R})(x + y = y + x)$ is true, and hence addition is commutative in $\ast \mathbb{R}$ as well. Doing this for all the axioms proves the statement.
Terminology and notation

We say that a hyperreal number $b$ is:

- **limited** if $r < b < s$ for some $r, s \in \mathbb{R}$,
Terminology and notation

We say that a hyperreal number $b$ is:

- **limited** if $r < b < s$ for some $r, s \in \mathbb{R}$,
- **positive unlimited** if $r < b$ for all $r \in \mathbb{R}$,
- **negative unlimited** if $b < r$ for all $r \in \mathbb{R}$,
- **unlimited** if it is positive or negative unlimited,
- **positive infinitesimal** if $0 < b < r$ for all positive $r \in \mathbb{R}$,
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- **infinitesimal** if it is positive infinitesimal, negative infinitesimal or $0$,
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For a subset $X$ of $^*\mathbb{R}$, we define $X_\infty = \{ x \in X \mid x \text{ is unlimited} \}$ and $X^+ = \{ x \in X \mid x > 0 \}$. 

Arithmetic of hyperreals

If $\varepsilon$ and $\delta$ are infinitesimals, $b$ and $c$ are appreciable, $r$ and $s$ are limited and $H$ is unlimited, then:

- $\varepsilon + \delta$ is infinitesimal,
Arithmetic of hyperreals

If $\varepsilon$ and $\delta$ are infinitesimals, $b$ and $c$ are appreciable, $r$ and $s$ are limited and $H$ is unlimited, then:

- $\varepsilon + \delta$ is infinitesimal,
- $b + \varepsilon$ is appreciable,
Arithmetic of hyperreals

If $\varepsilon$ and $\delta$ are infinitesimals, $b$ and $c$ are appreciable, $r$ and $s$ are limited and $H$ is unlimited, then:

- $\varepsilon + \delta$ is infinitesimal,
- $b + \varepsilon$ is appreciable,
- $\varepsilon \cdot \delta$ is infinitesimal,
Arithmetic of hyperreals

If $\varepsilon$ and $\delta$ are infinitesimals, $b$ and $c$ are appreciable, $r$ and $s$ are limited and $H$ is unlimited, then:

- $\varepsilon + \delta$ is infinitesimal,
- $b + \varepsilon$ is appreciable,
- $\varepsilon \cdot \delta$ is infinitesimal,
- $\varepsilon \cdot b$ is infinitesimal,
Arithmetic of hyperreals

If $\varepsilon$ and $\delta$ are infinitesimals, $b$ and $c$ are appreciable, $r$ and $s$ are limited and $H$ is unlimited, then:

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- $\varepsilon \cdot b$ is infinitesimal,
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If $\varepsilon$ and $\delta$ are infinitesimals, $b$ and $c$ are appreciable, $r$ and $s$ are limited and $H$ is unlimited, then:

- $\varepsilon + \delta$ is infinitesimal,
- $b + \varepsilon$ is appreciable,
- $\varepsilon \cdot \delta$ is infinitesimal,
- $\varepsilon \cdot b$ is infinitesimal,
- $b \cdot c$ is appreciable,
- $\frac{b}{H}$ is infinitesimal,
Arithmetic of hyperreals

If $\varepsilon$ and $\delta$ are infinitesimals, $b$ and $c$ are appreciable, $r$ and $s$ are limited and $H$ is unlimited, then:

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- $\varepsilon \cdot b$ is infinitesimal,
- $b \cdot c$ is appreciable,
- $\frac{b}{H}$ is infinitesimal,
- $r + s$ is limited,
Arithmetic of hyperreals

If $\varepsilon$ and $\delta$ are infinitesimals, $b$ and $c$ are appreciable, $r$ and $s$ are limited and $H$ is unlimited, then:

- $\varepsilon + \delta$ is infinitesimal,
- $b + \varepsilon$ is appreciable,
- $\varepsilon \cdot \delta$ is infinitesimal,
- $\varepsilon \cdot b$ is infinitesimal,
- $b \cdot c$ is appreciable,
- $\frac{b}{H}$ is infinitesimal,
- $r + s$ is limited,
- $r \cdot s$ is limited.
Halos

- $r \simeq s$ iff $r - s$ is infinitesimal.
Halos

- $r \simeq s$ iff $r - s$ is infinitesimal.
- $\text{hal}(b) = \{ c \in \mathbb{R}^* \mid b \simeq c \}$
Halos

- $r \simeq s$ iff $r - s$ is infinitesimal.
- $\text{hal}(b) = \{ c \in \mathbb{R}^* \mid b \simeq c \}$

**Proposition**

*If two real numbers $b$ and $c$ are infinitely close, that is if $b \simeq c$, then $b = c$.***
Halos

- $r \simeq s$ iff $r - s$ is infinitesimal.
- $\text{hal}(b) = \{c \in \mathbb{R}^* | b \simeq c\}$

**Proposition**

*If two real numbers $b$ and $c$ are infinitely close, that is if $b \simeq c$, then $b = c$.***

**Proof.**

Suppose that $b \simeq c$ with $b$ and $c$ real, but that $b \neq c$. Then there is a non-zero real number $r$ such that $b - c = r$. But this contradicts the assumption that $b \simeq c$, since $r$ is not an infinitesimal. ■
Theorem (Existence of shadows)

Every limited hyperreal $b$ is infinitely close to one and only one real number $s$. This real number is called the shadow of $b$, which is denoted by $\text{sh}(b)$.
Shadows

Theorem (Existence of shadows)

Every limited hyperreal \( b \) is infinitely close to one and only one real number \( s \). This real number is called the shadow of \( b \), which is denoted by \( \text{sh}(b) \).

Proof sketch.

Let \( A = \{ r \in \mathbb{R} \mid r < b \} \). By the Dedekind completeness of \( \mathbb{R} \), \( A \) has a least upper bound in \( \mathbb{R} \). Call this real number \( s \). Then prove that \( b \simeq s \) using that \( s \) is a least upper bound of \( A \).
Continuity

Theorem

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ if and only if $f(x) \sim f(c)$ whenever $x \sim c$. 

Proof sketch. The definition of continuity can be expressed by the $L$-sentence $(\forall \varepsilon \in \mathbb{R}^+) (\exists \delta \in \mathbb{R}^+) (\forall x \in \mathbb{R}) (|x - c| < \delta \rightarrow |f(x) - f(c)| < \varepsilon)$. Then one can use the transfer principle, and some tricks, to prove the theorem. ■
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A function \( f : \mathbb{R} \to \mathbb{R} \) is continuous at \( c \in \mathbb{R} \) if and only if \( f(x) \simeq f(c) \) whenever \( x \simeq c \).

Proof sketch.

The definition of continuity can be expressed by the \( L \)-sentence \((\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in \mathbb{R})(|x - c| < \delta \to |f(x) - f(c)| < \varepsilon)\). Then one can use the transfer principle, and some tricks, to prove the theorem. ■
Continuity

**Theorem**

A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous at \( c \in \mathbb{R} \) if and only if \( f(x) \simeq f(c) \) whenever \( x \simeq c \).

**Proof sketch.**

The definition of continuity can be expressed by the \( \mathcal{L} \)-sentence \(( \forall \varepsilon \in \mathbb{R}^+) (\exists \delta \in \mathbb{R}^+) (\forall x \in \mathbb{R}) (|x - c| < \delta \rightarrow |f(x) - f(c)| < \varepsilon)\). Then one can use the transfer principle, and some tricks, to prove the theorem. □

**Theorem**

A function \( f : A \rightarrow \mathbb{R} \) is continuous at \( c \in A \) if and only if \( f(x) \simeq f(c) \) whenever \( x \simeq c \) and \( x \in *A \).
## A continuous function

**Proposition**

\[ \text{The function } f(x) = x^2 \text{ is continuous at any } a \in \mathbb{R}. \]
A continuous function

Proposition

The function $f(x) = x^2$ is continuous at any $a \in \mathbb{R}$.

Proof.

- We want $f(x) \simeq f(a)$ whenever $x \simeq a$. 
A continuous function

Proposition

The function $f(x) = x^2$ is continuous at any $a \in \mathbb{R}$.

Proof.

- We want $f(x) \approx f(a)$ whenever $x \approx a$.
- If $x \approx a$, then $x = a + \varepsilon$ for some infinitesimal $\varepsilon$. 

■
A continuous function

Proposition

The function \( f(x) = x^2 \) is continuous at any \( a \in \mathbb{R} \).

Proof.

- We want \( f(x) \approx f(a) \) whenever \( x \approx a \).
- If \( x \approx a \), then \( x = a + \varepsilon \) for some infinitesimal \( \varepsilon \).
- Now \( f(x) = f(a + \varepsilon) = a^2 + 2a\varepsilon + \varepsilon^2 \).
A continuous function

Proposition

The function $f(x) = x^2$ is continuous at any $a \in \mathbb{R}$.

Proof.

- We want $f(x) \approx f(a)$ whenever $x \approx a$.
- If $x \approx a$, then $x = a + \varepsilon$ for some infinitesimal $\varepsilon$.
- Now $f(x) = f(a + \varepsilon) = a^2 + 2a\varepsilon + \varepsilon^2$.
- Then $f(x) - f(a) = a^2 + 2a\varepsilon + \varepsilon^2 - a^2 = \varepsilon(2a + \varepsilon)$.
Proposition

The function \( f(x) = x^2 \) is continuous at any \( a \in \mathbb{R} \).

Proof.

- We want \( f(x) \approx f(a) \) whenever \( x \approx a \).
- If \( x \approx a \), then \( x = a + \varepsilon \) for some infinitesimal \( \varepsilon \).
- Now \( f(x) = f(a + \varepsilon) = a^2 + 2a\varepsilon + \varepsilon^2 \).
- Then \( f(x) - f(a) = a^2 + 2a\varepsilon + \varepsilon^2 - a^2 = \varepsilon(2a + \varepsilon) \).
- This is infinitesimal.
A continuous function

Proposition

The function \( f(x) = x^2 \) is continuous at any \( a \in \mathbb{R} \).

Proof.

- We want \( f(x) \approx f(a) \) whenever \( x \approx a \).
- If \( x \approx a \), then \( x = a + \varepsilon \) for some infinitesimal \( \varepsilon \).
- Now \( f(x) = f(a + \varepsilon) = a^2 + 2a\varepsilon + \varepsilon^2 \).
- Then \( f(x) - f(a) = a^2 + 2a\varepsilon + \varepsilon^2 - a^2 = \varepsilon(2a + \varepsilon) \).
- This is infinitesimal.
- Hence \( f(x) \approx f(a) \).
The Intermediate Value Theorem

**Theorem (The Intermediate Value Theorem)**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for every real number $d$ strictly between $f(a)$ and $f(b)$ there exists a real number $c \in (a, b)$ such that $f(c) = d$. 
Proof.

Assume that $f(a) < d < f(b)$.
Proof.

Assume that $f(a) < d < f(b)$.

- Partition $[a, b]$ into $n$ subintervals of equal length
Proof.

Assume that $f(a) < d < f(b)$.

- Partition $[a, b]$ into $n$ subintervals of equal length.
- These intervals have endpoints $p_k = a + k\frac{b-a}{n}$.
Proof.

Assume that $f(a) < d < f(b)$.

- Partition $[a, b]$ into $n$ subintervals of equal length
- These intervals have endpoints $p_k = a + k \frac{b-a}{n}$
- Let $s_n$ be the maximum of $\{p_k \mid f(p_k) < d\}$
Proof.

Assume that \( f(a) < d < f(b) \).

- Partition \([a, b]\) into \(n\) subintervals of equal length.
- These intervals have endpoints \( p_k = a + k \frac{b-a}{n} \).
- Let \( s_n \) be the maximum of \( \{p_k \mid f(p_k) < d\} \).
- We have \( a \leq s_n < b \) and \( f(s_n) < d \leq f(s_n + \frac{b-a}{n}) \) for all \( n \in \mathbb{N} \).

\[ \blacksquare \]
Proof.

Assume that $f(a) < d < f(b)$.

- Partition $[a, b]$ into $n$ subintervals of equal length
- These intervals have endpoints $p_k = a + k \frac{b-a}{n}$
- Let $s_n$ be the maximum of $\{p_k \mid f(p_k) < d\}$
- We have $a \leq s_n < b$ and $f(s_n) < d \leq f(s_n + \frac{b-a}{n})$ for all $n \in \mathbb{N}$.
- By transfer, this is true for all $n \in \mathbb{N}^*$, so pick an $N \in \mathbb{N}_\infty$.

$\blacksquare$
Proof.

Assume that $f(a) < d < f(b)$.

- Partition $[a, b]$ into $n$ subintervals of equal length.
- These intervals have endpoints $p_k = a + k \frac{b-a}{n}$.
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- We have $a \leq s_n < b$ and $f(s_n) < d \leq f(s_n + \frac{b-a}{n})$ for all $n \in \mathbb{N}$.
- By transfer, this is true for all $n \in {}^*\mathbb{N}$, so pick an $N \in {}^*\mathbb{N}_\infty$.
- $a \leq s_N < b$, so $s_N$ is limited, and has a shadow $c$.

\[\Box\]
Proof.

Assume that $f(a) < d < f(b)$.

- Partition $[a, b]$ into $n$ subintervals of equal length
- These intervals have endpoints $p_k = a + k \frac{b-a}{n}$
- Let $s_n$ be the maximum of $\{p_k \mid f(p_k) < d\}$
- We have $a \leq s_n < b$ and $f(s_n) < d \leq f(s_n + \frac{b-a}{n})$ for all $n \in \mathbb{N}$.
- By transfer, this is true for all $n \in {}^*\mathbb{N}$, so pick an $N \in {}^*\mathbb{N}_\infty$.
- $a \leq s_N < b$, so $s_N$ is limited, and has a shadow $c$.
- $\frac{b-a}{N}$ is infinitesimal, and so $s_N \sim c$ and $s_N + \frac{b-a}{N} \sim c$. 

\[\square\]
Proof.

Assume that \( f(a) < d < f(b) \).

- Partition \([a, b]\) into \( n\) subintervals of equal length
- These intervals have endpoints \( p_k = a + k \frac{b-a}{n} \)
- Let \( s_n \) be the maximum of \( \{ p_k \mid f(p_k) < d \} \)
- We have \( a \leq s_n < b \) and \( f(s_n) < d \leq f(s_n + \frac{b-a}{n}) \) for all \( n \in \mathbb{N} \).
- By transfer, this is true for all \( n \in \mathbb{N}^\ast \), so pick an \( N \in \mathbb{N}^\infty \).
- \( a \leq s_N < b \), so \( s_N \) is limited, and has a shadow \( c \).
- \( \frac{b-a}{N} \) is infinitesimal, and so \( s_N \simeq c \) and \( s_N + \frac{b-a}{N} \simeq c \).
- By continuity, \( f(s_N) \simeq f(c) \) and \( f(s_N + \frac{b-a}{N}) \simeq f(c) \).

\( \blacksquare \)
Proof.

Assume that $f(a) < d < f(b)$.

- Partition $[a, b]$ into $n$ subintervals of equal length.
- These intervals have endpoints $p_k = a + k \frac{b-a}{n}$.
- Let $s_n$ be the maximum of $\{p_k | f(p_k) < d\}$.
- We have $a \leq s_n < b$ and $f(s_n) < d \leq f(s_n + \frac{b-a}{n})$ for all $n \in \mathbb{N}$.
- By transfer, this is true for all $n \in *\mathbb{N}$, so pick an $N \in *\mathbb{N}_\infty$.
- $a \leq s_N < b$, so $s_N$ is limited, and has a shadow $c$.
- $\frac{b-a}{N}$ is infinitesimal, and so $s_N \simeq c$ and $s_N + \frac{b-a}{N} \simeq c$.
- By continuity, $f(s_N) \simeq f(c)$ and $f(s_N + \frac{b-a}{N}) \simeq f(c)$.
- Therefore $f(c) \simeq f(s_N) < d \leq f(s_N + \frac{b-a}{N}) \simeq f(c)$.

$\blacksquare$
Proof.

Assume that \( f(a) < d < f(b) \).

- Partition \([a, b]\) into \( n\) subintervals of equal length
- These intervals have endpoints \( p_k = a + k \frac{b-a}{n} \)
- Let \( s_n \) be the maximum of \( \{p_k \mid f(p_k) < d\} \)
- We have \( a \leq s_n < b \) and \( f(s_n) < d \leq f(s_n + \frac{b-a}{n}) \) for all \( n \in \mathbb{N} \).
- By transfer, this is true for all \( n \in \mathbb{N}^\ast \), so pick an \( N \in \mathbb{N}^\infty \).
- \( a \leq s_N < b \), so \( s_N \) is limited, and has a shadow \( c \).
- \( \frac{b-a}{N} \) is infinitesimal, and so \( s_N \sim c \) and \( s_N + \frac{b-a}{N} \sim c \).
- By continuity, \( f(s_N) \sim f(c) \) and \( f(s_N + \frac{b-a}{N}) \sim f(c) \).
- Therefore \( f(c) \sim f(s_N) < d \leq f(s_N + \frac{b-a}{N}) \sim f(c) \).
- So \( f(c) \sim d \), but both are real, and so \( f(c) = d \).
Theorem

If \( f \) is defined at \( x \in \mathbb{R} \), then \( L \in \mathbb{R} \) is the derivative of \( f \) at \( x \) if and only if for every nonzero infinitesimal \( \varepsilon \), \( f(x + \varepsilon) \) is defined, and

\[
\frac{f(x + \varepsilon) - f(x)}{\varepsilon} \simeq L.
\]
Differentiation

Theorem

If $f$ is defined at $x \in \mathbb{R}$, then $L \in \mathbb{R}$ is the derivative of $f$ at $x$ if and only if for every nonzero infinitesimal $\varepsilon$, $f(x + \varepsilon)$ is defined, and

$$\frac{f(x + \varepsilon) - f(x)}{\varepsilon} \simeq L.$$ 

Remark

If $f$ is differentiable, we can find the derivative as $f'(x) = \text{sh} \left( \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \right)$ for any non-zero infinitesimal $\varepsilon$. 

Arne Tobias

Hyperreal Calculus

September 2, 2016
Increments

- Notation: $\Delta f = f(x + \Delta x) - f(x) (= \Delta f(x, \Delta x))$
Increments

- Notation: \( \Delta f = f(x + \Delta x) - f(x) (= \Delta f(x, \Delta x)) \)
- \( f'(x) = \text{sh} \left( \frac{\Delta f}{\Delta x} \right) \), if \( f'(x) \) is defined
Increments

- Notation: \( \Delta f = f(x + \Delta x) - f(x) \) (= \( \Delta f(x, \Delta x) \))
- \( f'(x) = \text{sh} \left( \frac{\Delta f}{\Delta x} \right) \), if \( f'(x) \) is defined

**Lemma (Incremental Equation)**

If \( f'(x) \) exists at real \( x \) and \( \Delta x \) is infinitesimal, then there exists an infinitesimal \( \varepsilon \), dependent on \( x \) and \( \Delta x \), such that \( \Delta f = f'(x)\Delta x + \varepsilon \Delta x \).
Increments

- Notation: \( \Delta f = f(x + \Delta x) - f(x) (= \Delta f(x, \Delta x)) \)
- \( f'(x) = \operatorname{sh} \left( \frac{\Delta f}{\Delta x} \right) \), if \( f'(x) \) is defined

Lemma (Incremental Equation)

If \( f'(x) \) exists at real \( x \) and \( \Delta x \) is infinitesimal, then there exists an infinitesimal \( \varepsilon \), dependent on \( x \) and \( \Delta x \), such that \( \Delta f = f'(x)\Delta x + \varepsilon \Delta x \).

Proof.

Since \( f'(x) \) exists, we have that \( f'(x) \sim \frac{\Delta f}{\Delta x} \), and hence that \( f'(x) - \frac{\Delta f}{\Delta x} = \varepsilon \) for some infinitesimal \( \varepsilon \). Multiplying through by \( \Delta x \) and rearranging, we get that \( \Delta f = f'(x)\Delta x + \varepsilon \Delta x \), which is what we wanted. \( \blacksquare \)
Theorem (Product rule)

If \( f \) and \( g \) are differentiable at \( x \), so is \( fg \), and we have that \( (fg)'(x) = f(x)g'(x) + g(x)f'(x) \).
Theorem (Product rule)

If $f$ and $g$ are differentiable at $x$, so is $fg$, and we have that $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$.

Proof.

We get that $\Delta(fg) = f(x)\Delta g + g(x)\Delta f + \Delta f\Delta g$ which yields that

$$
\frac{\Delta(fg)}{\Delta x} = f(x)\frac{\Delta g}{\Delta x} + g(x)\frac{\Delta f}{\Delta x} + \frac{\Delta f}{\Delta x}\Delta g
\approx f(x)g'(x) + g(x)f'(x) + 0,
$$

from which our theorem follows.
Theorem (Chain Rule)

If $f$ is differentiable at $x \in \mathbb{R}$, and $g$ is differentiable at $f(x)$, then $g \circ f$ is differentiable at $x$ with derivative $g'(f(x))f'(x)$. 
Theorem (Chain Rule)

If $f$ is differentiable at $x \in \mathbb{R}$, and $g$ is differentiable at $f(x)$, then $g \circ f$ is differentiable at $x$ with derivative $g'(f(x))f'(x)$.

Proof.

We have that $\Delta(g \circ f)(x, \Delta x) = \Delta g(f(x), \Delta f)$. By the incremental equation, we have that $\Delta(g \circ f) = g'(f(x))\Delta f + \varepsilon \Delta f$ for some infinitesimal $\varepsilon$, and hence that

$$\frac{\Delta(g \circ f)}{\Delta x} = g'(f(x)) \frac{\Delta f}{\Delta x} + \varepsilon \frac{\Delta f}{\Delta x} \approx g'(f(x))f'(x) + 0$$

which establishes our claim.