SUPPLEMENTARY NOTES FOR MAT4520, SPRING 2010

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(Version: May 28, 2010)

1. Remark on integration in polar coordinates

Let $B^n = \{p \in \mathbb{R}^n \mid |p| \leqslant 1\}$*, and let* $f : B^n \to \mathbb{R}$ *be smooth. Then*

$$\int_{B^n} f = \int_{B^n} f\, dx_1 \wedge \cdots \wedge dx_n = \int_{S^{n-1}} g\sigma' ,$$

where $g(p) = \int_0^1 t^{n-1} f(tp)\, dt$*, and* σ' *is the restriction of the* $(n-1)$*–form* $\sigma = \Sigma_{i=1}^n (-1)^{i+1} x_i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$ *to* S^{n-1}*.*
(Corollary 8, p. 266 in Spivak.)

Proof. The n-form $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ on B^n is trivially closed, hence also exact, since B^n is contractible. (By the Poincaré lemma.) This means that there is a $(n-1)$–form θ such that $d\theta = \omega$, and Stokes' theorem says that

$$\int_{B^n} f = \int_{S^{n-1}} \theta.$$

In fact, the proof of the Poincaré lemma in chapter 7 (page 225) even gives an explicit formula for such a form θ, given a contracting homotopy H, namely $\theta = I(H^*\omega)$, where I is the algebraic homotopy operator defined on pp. 223-224.

The simplest contracting homotopy is given by $H(p,t) = tp$. Then we get

$$H^*\omega = H^*f\, H^*dx^1 \wedge \cdots \wedge H^*dx^n = (f \circ H)\, d(x^1 \circ H) \wedge \cdots \wedge d(x^n \circ H)$$

Here $f \circ H(p,t) = f(tp)$ and $x^i \circ H(p,t) = tp^i$ $(p = (p^1, \ldots, p^n))$, hence $d(x^i \circ H) = d(tx^i) = tdx^i + x^i dt$. Substituting in the formula above for $H^*\omega$ we get

$$\begin{aligned} H^*\omega_{(p,t)} &= f(tp)(t\, dx^1 \wedge \cdots \wedge t\, dx^n + \Sigma_{i=1}^n t\, dx^1 \wedge \cdots \wedge x^i dt \wedge \cdots \wedge t\, dx^n) \\ &= t^n f(tp)\, dx^1 \wedge \cdots \wedge dx^n + dt \wedge t^{n-1} f(tp)\, \sigma_p \end{aligned}$$

This has the form $\omega_1 + dt \wedge \eta$ required for the definition of the operator I, with $\eta_{(p,t)} = t^{n-1} f(tp)\, \sigma_p$. Since σ is independent of t, we get

$$\theta_p = I(H^*\omega)_p = (\int_0^1 t^{n-1} f(tp)\, dt)\, \sigma_p$$

□

2. de Rham cohomology of spheres. Applications.

We shall see that we can use the Mayer–Vietoris exact sequence (Spivak, p. 424) to compute all cohomology of the spheres S^n. For $n \geqslant 1$ the result is:

$$H^k(S^n) \cong \begin{cases} \mathbb{R} \text{ if } k = 0 \text{ or } k = n \\ 0 \quad \text{otherwise} \end{cases} \tag{1}$$

(The case $n = 0$ (two points) is trivial: again there are two $\mathbb{R}'$s, but they both lie in degree 0.)

Let $U_\pm = S^n - \{(0, \dots, \pm 1)\} \subset S^n$. Then $\{U_+, U_-\}$ is an open cover of S^n of manifolds diffeomorphic to R^n, and $U_+ \cap U_-$is diffeomorphic to $S^{n-1} \times \mathbb{R}$. By the homotopy invariance of cohomology, we then have $H^k(U_\epsilon) \cong H^k(\text{point})$ and $H^k(U_+ \cap U_-) \cong H^k(S^{n-1})$. We also recall that $H^0(S^n) \cong \mathbb{R}$ for $n > 0$, and $H^k(S^n) \cong 0$ for $k > n$.

We will prove (1) by induction on n, starting with $n = 1$. Then Mayer–Vietoris reduces to

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^2 \to H^1(S^1) \to 0$$

This can only be exact if $H^1(S^1) \cong \mathbb{R}$.

Now assume we have proved (1) for $n - 1$, $n \geqslant 2$. Then Mayer–Vietoris for S^n becomes

$$\begin{aligned} 0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to H^1(S^n) \to 0 \to \cdots \\ \to 0 \to H^{k-1}(S^{n-1}) \to H^k(S^n) \to 0 \cdots \end{aligned}$$

Hence we see that $H^1(S^n) \cong 0$ and $H^k(S^n) \cong H^{k-1}(S^{n-1})$ for $k > 1$. The induction step follows from this.

Note that by homotopy invariance it now also follows that we have computed the cohomology of $R^n - \{0\} \approx S^{n-1} \times \mathbb{R}$.

We give two classical applications of this result. The first is

Brouwer's fixed point theorem. *Let $f : B^n \to B^n$ be a smooth map. Then f has a fixed point, i. e. there is a point $x \in B^n$ such that $f(x) = x$.*

Proof. Suppose not, i. e. $f(x) \neq x$ for all x. Then there is a unique line ℓ_x through the two points x and $f(x)$, and ℓ_x must intersect the boundary sphere S^{n-1} of B^n in two points. Let $g(x)$ be the intersection point closer to x than to $f(x)$. To see that g is a smooth map $B^n \to S^{n-1}$, observe that $g(x) = x + t(x - f(x))$ for a unique $t \geqslant 0$, which can be found by solving the quadratic equation (in t) $|x + t(x - f(x)|^2 = 1$.

If $x \in S^{n-1}$ we get $g(x) = x$, hence the composition $S^{n-1} \overset{\iota}{\subset} B^n \xrightarrow{g} S^{n-1}$ is the identity map. Applying H^{n-1} we get that the composition

$$\mathbb{R} \cong H^{n-1}(S^{n-1}) \xrightarrow{g^*} H^{n-1}(B^n) \xrightarrow{\iota^*} H^{n-1}(S^{n-1}) \cong \mathbb{R}$$

is the identity map. But since $H^{n-1}(B^n) = 0$, this is impossible. □

Before stating the next result we make an important observation concerning the *antipodal map* $A : S^{n-1} \to S^{n-1}$ defined by $A(x) = -x$.

Claim: The induced map $A^* : H^{n-1}(S^{n-1}) \to H^{n-1}(S^{n-1})$ *is multiplication by* $(-1)^n$.

To see this, recall that a generator of $H^{n-1}(S^{n-1})$ is represented by the form σ' defined by $\sigma'_p(v_1, \ldots, v_{n-1}) = \det \begin{pmatrix} p \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$, where p and the v_i's are considered as vectors in $\mathbb{R}^n$. The image of this generator under A^* is then represented by the form $A^*\sigma'$ given by

$$\begin{aligned}(A^*\sigma')_p(v_1, \ldots, v_{n-1}) &= \sigma'_{A(p)}(A_*v_1, \ldots, A_*v_n) = \det \begin{pmatrix} -p \\ -v_1 \\ \vdots \\ -v_{n-1} \end{pmatrix} \\ &= (-1)^n \sigma'_p(v_1, \ldots, v_{n-1})\end{aligned}$$

The claim follows.

We can now prove

Theorem. *The sphere* S^n *admits a vector field* v *such that* $v(p) \neq 0$ *for every* $p \in S^n$ *if and only if* n *is odd.*

(Cf. Spivak, p. 276)

Proof. Assume S^n does admit such a vector field v. By normalizing v at every point, we may assume that $|v(p)| = 1$. Hence we may think of v as a function $v : S^n \to S^n$ such that p and $v(p)$ are orthogonal for every p.

Now define a homotopy $F : S^n \times I \to S^n$ by

$$F(p, t) = \cos(\pi t)\, p + \sin(\pi t)\, v(p)\,.$$

Then $F(p, 0) = p$ and $F(p, 1) = -p$, which means that F is a smooth homotopy between the identity map on S^n and A. Since homotopic maps induce the same map on cohomology, it follows that $1 = (-1)^{n+1}$. Hence n must be odd.

Conversely, suppose $n = 2k - 1$. Then we can define a vector field v by the formula

$$v(p_1, \ldots, p_{2k}) = (-p_2, p_1, -p_4, p_3, \ldots, -p_{2k}, p_{2k-1})$$

Clearly, $v(p) \neq 0$ for all $p \in S^n$. ∎

3. More calculations: Projective spaces.

We start with the *real* projective spaces RP^n, which we think of as obtained from S^n by identifying antipodal points. Then the quotient map $\pi : S^n \to RP^n$ is a surjective local diffeomorphism, and such that $\pi(p) = \pi(q)$ if and only if $p = q$ or $p = -q = A(q)$.

Now we make use of the fact that $A^* : \Omega^k(S^n) \to \Omega^k(S^n)$ has order two — i.e. $(A^*)^2 = (A^2)^* = I$; the identity homomorphism. Here is an easy exercise in linear algebra:

Assume $P : V \to V$ is a linear map such that $P^2 = I$. Then the vector space V splits as a direct sum of the $+1$ and -1 eigenspaces V_+ and V_- of P. (Use the decomposition $v = \frac{1}{2}(v + Pv) + \frac{1}{2}(v - Pv)$.)

Applying this to A^* we see that $\Omega^k(S^n)$ splits as $\Omega^k(S^n)_+ \oplus \Omega^k(S^n)_-$, and it is also easy to see that the exterior derivative $d : \Omega^k(S^n) \to \Omega^{k+1}(S^n)$ respects this decomposition.

Taking cohomology, we get a similar decomposition of $H^k(S^n)$, and our calculations show that $H^n(S^n) \cong \mathbb{R}$ is equal to the $(-1)^{n+1}$–eigenspace of $A^* : H^n(S^n) \to H^n(S^n)$.

Now consider $\pi^* : \Omega^k(RP^n) \to \Omega^k(S^n)$. Since $\pi A = \pi$, we have $A^*\pi^* = \pi^*$, so $\pi^*(\Omega^k(RP^n)) \subseteq \Omega^k(S^n)_+$.

Claim: $\pi^* : \Omega^k(RP^n) \to \Omega^k(S^n)_+$ is an isomorphism.

Corollary. $H^k(RP^n) \cong \begin{cases} \mathbb{R} \text{ if } k = 0 \text{ or } k = n \text{ if } n \text{ is odd} \\ 0 \quad \text{otherwise} \end{cases}$

It remains to verify the *Claim.* Let $\eta \in \Omega^k(S^n)$ be a k–form. Recall that π^* is defined by

$$(\pi^*\omega)_p(v_1, \dots) = \omega_{\pi(p)}((\pi_*)_p\, v_1, \dots)\,.$$

Hence $\pi^*\omega = \eta$ if and only if

$$\omega_q(w_1, \dots) = \eta_p(v_1, \dots) \tag{2}$$

for every $p \in S^n$ such that $\pi(p) = q$ and $v_i \in S^n_p$ such that $(\pi_*)_p\, v_i = w_i$. But for every $q, w_1, \dots$ there are exactly two choices of $p, v_1, \dots$; if $p, (\pi_{*p})^{-1}w_1, \dots$ is one, then $A(p), A_*(\pi_{*p})^{-1}w_1, \dots$ is the other. Hence formula (2) determines ω uniquely if and only if

$$\eta_p(v_1, \dots) = \eta_{A(p)}(A_*v_1, \dots) = (A^*\eta)_p(v_1, \dots)$$

For all $p, v_1, \dots$ — i.e. if and only if $\eta \in \Omega^k_+(S^n)$.

We now turn to the *complex* projective spaces CP^n. We will compute the de Rham cohomology of CP^n by induction on n using the Mayer–Vietoris sequence.

Recall that $CP^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$, where the group $\mathbb{C}^* = \mathbb{C} - \{0\}$ acts on $\mathbb{C}^{n+1} - \{0\}$ by $\zeta \cdot (z_0, \dots, z_n) = (\zeta z_0, \dots, \zeta z_n)$. The orbit (equivalence class) of $(z_0, \dots, z_n)$ is denoted $[z_0 : \dots : z_n]$ (*homogeneous coordinates.*) Note that we can identify CP^{n-1} with the set of points in CP^n having the last homogeneous coordinate equal to zero.

Let $U = CP^n - CP^{n-1}$ — the open subset of points with the last homogeneous coordinate nonzero. Every such point has a unique representative of the form $(z_0, \dots, z_{n-1}, 1)$, and it follows that U is diffeomorphic to $\mathbb{C}^n$.

Let $V \subset CP^n$ be the complement of the point $[0 : \cdots : 0 : 1]$. Clearly $U \cup V = CP^n$ and $U \cap V \approx \mathbb{C}^n - \{0\} \approx \mathbb{R}^{2n} - \{0\}$, which has the same cohomology as S^{2n-1}.

A point with homogeneous coordinates $[z_0 : \cdots : z_n]$ lies in V if and only if $(z_0, \dots, z_{n-1}) \neq (0, \dots, 0)$. Therefore the formula

$$g([z_0 : \cdots : z_n]) = [z_0 : \cdots : z_{n-1} : 0]$$

defines a retraction of V to CP^{n-1}, i. e. a map $g : V \to CP^{n-1}$ such that $g \circ j = \mathrm{id}_{CP^{n-1}}$, where j is the inclusion $CP^{n-1} \subset V$. Moreover,

$$G([z_0 : \cdots : z_n], t) = [z_0 : \cdots : tz_n]$$

defines a homotopy $G : V \times I \to V$ such that $G(p, 1) = p$ and $G(p, 0) = jg(p)$. It follows that g and j induce inverse isomorphisms in cohomology, yielding isomorphisms $H^k(V) \cong H^k(CP^{n-1})$ for all k.

We are now ready to prove

$$H^k(CP^n) \cong \begin{cases} \mathbb{R} \text{ for } k \textit{ even} \text{ and } 0 \leqslant k \leqslant 2n \\ 0 \quad \text{otherwise} \end{cases}$$

Moreover, $j^* : H^k(CP^n) \to H^k(CP^{n-1})$ is an isomorphism for $k < 2n$.

Proof. We start with $n = 1$. $CP^1 \approx S^2$, and the result is a special case of the calculation for spheres.

Now assume we have proved the result for $n - 1$. We know already that $H^0(CP^n) \cong \mathbb{R}$, so let $k \geqslant 1$. Since then $H^k(U) \cong 0$, the Mayer–Vietoris sequence reduces to a sequence isomorphic to

$$\cdots \to H^{k-1}(S^{2n-1}) \to H^k(CP^n) \xrightarrow{j^*} H^k(CP^{n-1}) \to H^k(S^{2n-1}) \to \cdots$$

For $1 < k < 2n$ we see immediately that j^* is an isomorphism. (Note that $H^{2n-1}(CP^{n-1}) \cong 0$.) This is also true for $k = 1$, since e. g. the map $H^0(U) \to H^0(U \cap V)$ is surjective. Finally, for $k = 2n$ we get an isomorphism $H^{2n-1}(S^{2n-1}) \cong H^{2n}(CP^n)$. □

4. Cohomology with compact support

Notation. De Rham cohomology groups with compact support are defined as in Spivak, p. 268 and p. 430, except that we will use the notation $\Omega_c^k(M)$ for the vector space of k-forms, instead of Spivak's $C_c^k(M)$. If $f : M \to N$ is a smooth map which is *proper*, i. e. such $f^{-1}(K)$ is compact for every compact $K \subset N$, then it induces homomorphisms $\Omega_c^k(N) \to \Omega_c^k(M)$ and $H_c^k(N) \to H_c^k(M)$, both denoted f^* as before.

A new ingredient is the map induced by *inclusions* $j : U \subset V$ of open subsets of a manifold M: given $\omega \in \Omega_c^k(U)$ we can extend by 0 to the rest of V and obtain a form $j_*\omega \in \Omega_c^k(V)$. (This is denoted $j'(\omega)$ in Spivak (p. 430).) It commutes with the exterior differentials, $dj_* = j_*d$ — hence induces a homomorphism $j_* : H^k(U) \to H^k(V)$.

Easy observations.

- If M is compact, there is no differnce between $H^k_c(M)$ and $H^k(M)$. In general there is only a forgetful map $H^k_c(M) \to H^k(M)$. Note that if $j : U \subset V$ is as above, the diagram
$$\begin{array}{ccc} H^k_c(U) & \xrightarrow{j_*} & H^k_c(V) \\ \downarrow & & \downarrow \\ H^k(U) & \xleftarrow[j^*]{} & H^k(V) \end{array}$$
commutes.
- $H^0_c(M) = \ker(\Omega^0_c(M) \to \Omega^1_c(M))$ is the set of locally constant functions with compact support. In follows that if M is connected and non–compact (e. g. $M = \mathbb{R}^n$), then $H^0_c(M) = 0$.
- The usual proof of homotopy invariance works here, as well, provided the homotopy is proper as a map $M \times I \to N$. The crucial observation is that the operator I (p. 224) preserves forms of compact support.

Lemma *If M is oriented and of dimension n, then integration $\int_M$ induces a nontrivial homomorphism $H^n_c(M) \to R$.*

Proof. We first need to prove that if $\omega \in \Omega^n_c(M)$ is exact, then $\int_M \omega = 0$. So, let $\omega = d\eta$, where η has compact support.

Consider first the case $M = \mathbb{R}^n$. (Cf. p. 268.) Then there is a ball B_r of some radius r such that $\mathrm{Supp}\,(\omega) \subset \mathrm{int}\, B_r$, and Stokes' theorem gives
$$\int_{\mathbb{R}^n} \omega = \int_{B_r} d\eta = \int_{\partial B_r} \eta = 0$$

In the general case, cover the compact set $\mathrm{Supp}\,(\omega)$ with finitely many coordinate neighborhoods $U_1, \dots, U_l$ diffeomorphic to $\mathbb{R}^n$, and use a partition of unity argument to write η as a sum $\eta_1 + \cdots + \eta_l$ of forms such that η_i has compact support contained in U_i. Then
$$\int_M \omega = \int_M \Sigma_i d\eta_i = \Sigma_i \int_{U_i} d\eta_i = 0\,,$$
by the first case.

To show that the homomorphism is nontrivial, choose an orientation preserving diffeomorphism g from an open subset U of M to $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to [0, \infty)$ be a nonzero function with compact support. Let $j : U \subset M$ be the inclusion map. Then
$$\int_M j_* g^*(f\, dx^1 \wedge \cdots \wedge d\,x^n) = \int_{\mathbb{R}^n} f\, dx^1 \dots d\,x^n > 0$$
$\square$

Remark. If $j : U \subset V$ is an inclusion map of open sets in M with orientations induced from M, we have $\int_V j_*\omega = \int_U \omega$ for any $\omega \in \Omega^n_c(U)$.

Here is our first nontrivial complete calculation:

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} \text{ if } k = n \\ 0 \quad \text{otherwise} \end{cases}$$

Proof. For $a \in (-1, 1)$ let $U_a = \{(p^1, \ldots, p^{n+1}) \in S \mid p^{n+1} < a\} \subset S^n$ and $V_a = \{(p^1, \ldots, p^{n+1}) \in S \mid p^{n+1} > a\} \subset S^n$. Both subsets are diffeomorphic to $\mathbb{R}^n$. We know that $H_c^0(\mathbb{R}^n) = 0$ and $\mathbb{H}_c^n(\mathbb{R}^n) \neq 0$, hence it suffices to prove that the map

$$j_* : H_c^k(U_0) \to H_c^k(S^n) = H^k(S^n)$$

is *injective* for all k, where j is the inclusion map $U_0 \subset S^n$. This is trivial for $k = 0$, so we assume $k \geqslant 1$.

Let ω be a closed k–form on U_0 such that $j_*\omega = d\eta'$ for some $(k-1)$–form on S^n. η' may not have support in E_0, but we will arrange that by subtracting a suitable exact form $d\tau$.

Since ω has compact support contained in U_0, we can find a $b < 0$ such that $d\eta' = j_*\omega$ vanishes on V_b. Therefore we can find a $(k-2)$–form τ' on V_b such that $d\tau' = \eta'$ there. Let $\phi : S^n \to [0, 1]$ be a smooth function which is constant equal to 1 in a neighborhood of $\overline{V_0}$ and 0 on U_b. Define $\tau \in \Omega^{k-2}(S^n)$ to be equal to $\phi\tau'$ on V_b and 0 on $\overline{U_b}$. Then $\eta = \eta' - d\tau$ has support in U_0 and satisfies $d\eta = d\eta' = \omega$. □

5. The Mayer–Vietoris sequences

A diagram

$$\begin{array}{ccc} U \cap V & \xrightarrow{j_U} & U \\ {\scriptstyle j_V}\big\downarrow & & \big\downarrow{\scriptstyle i_U} \\ V & \xrightarrow{i_V} & U \cup V \end{array}$$

gives rise to two short exact sequences of de Rham complexes, and hence to two Mayer–Vietoris sequences. The sequence

$$0 \to \Omega^*(U \cup V) \xrightarrow{(i_U{}^*, i_V{}^*)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_U{}^* - j_V{}^*} \Omega^*(U \cap V) \to 0$$

yields

$$\cdots \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to H^{k+1}(U \cup V) \to \cdots$$

(Theorem 3, p. 424), and

$$0 \to \Omega_c^*(U \cap V) \xrightarrow{(j_{U*}, -j_{V*})} \Omega_c^*(U) \oplus \Omega_c^*(V) \xrightarrow{i_{U*} + i_{V*}} \Omega_c^*(U \cup V) \to 0$$

yields

$$\cdots \to H_c^k(U \cap V) \to H_c^k(U) \oplus H_c^k(V) \to H_c^k(U \cup V) \to H_c^{k+1}(U \cap V) \to \cdots$$

(Theorem 8, p 431). As an illustration of use of the second sequence, let us apply it to the same open cover $\{U_+, U_-\}$ of the sphere S^n as we used to compute $H^k(S^n)$. Then U_+ and U_- are both diffeomorphic to $\mathbb{R}^n$. Hence Mayer–Vietoris reduces to

$$0 \to H^0(S^n) \to H^1_c(U_+ \cap U_-) \to 0 \to \cdots \to 0 \to H^k_c(U_+ \cap U_-) \to 0$$
$$\cdots \to 0 \to H^n_c(U_+ \cap U_-) \to H^n_c(U_+) \oplus H^n_c(U_-) \to H^n(S^n) \to 0$$

But $U_+ \cap U_- \approx S^{n-1} \times \mathbb{R}$, so this means that

$$H^k_c(S^{n-1} \times \mathbb{R}) \cong \begin{cases} \mathbb{R} \text{ if } k = 1 \text{ or } k = n \\ 0 \quad \text{otherwise} \end{cases}$$

Note that this is isomorphic to the cohomology of S^{n-1}, only *shifted.*

6. Products and the Poincaré duality map

It is an easy exercise to show that wedge product of forms induces a bilinear pairing on de Rham cohomology:

$$\cup : H^k(M) \times H^l(M) \to H^{k+l}(M) \tag{3}$$

("Cup product".) In fact, the formula $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ immediately implies that the product of two cocycles is a cocycle, and if $d\alpha = 0$ and $d\beta = 0$ we have

$$(\alpha + d\gamma) \wedge (\beta + d\eta) = \alpha \wedge \beta + d(\gamma \wedge \beta + (-1)^k \alpha \wedge \eta + \gamma \wedge d\eta)\,.$$

Moreover, since $\operatorname{Supp}(\alpha \wedge \beta) \subset \operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$, the wedge product of two forms has compact support if at least one of them has compact support. Hence we also have a pairing

$$\cup : H^k(M) \times H^l_c(M) \to H^{k+l}_c(M) \tag{4}$$

The following properties of the cup product are easy to prove:

Lemma

(i) *The product in (3) gives $H^*(M) = \oplus_k H^k(M)$ the structure of a graded ring. A smooth map $f : M \to N$ induces a ring homomorphism $f^* : H^*(N) \to H^*(M)$.*

(ii) *(4) gives $H^*_c(M) = \oplus_k H^k_c(M)$ the structure of a graded module over the graded ring $H^*(M)$.*

(iii) *Let $j : U \subset V$ is the inclusion of open sets in M. If $x \in H^k(V)$ and $y \in H^l_c(U)$, then $j_*(j^*x \cup y) = x \cup j_*y$.*

Remark. Composition with j^* makes $H^*_c(U)$ a module also over $H^*(V)$, and property (iii) says that j_* is a homomorphism of $H^*(V)$–modules.

Now assume that M is an *oriented* n–manifold. Then integration of n–forms defines a homomorphism $\int_U : H^n_c(U) \to \mathbb{R}$ for every open $U \subseteq M$, and we can define a new pairing

$$H^k(U) \times \mathbb{H}^{n-k}_c(U) \to \mathbb{R}$$

induced by $(\alpha, \beta) \mapsto \int_U \alpha \wedge \beta$. For each α this defines a map $\mathbb{H}^{n-k}_c(U) \to \mathbb{R}$, and bilinearity means that we get a linear map

$$\mu_U : H^k(U) \to H^{n-k}_c(U)^* = \operatorname{Hom}_{\mathbb{R}}(H^{n-k}_c(U), \mathbb{R})$$

Example. If $U = \mathbb{R}^n$, these groups are nontrivial only for $k = 0$. Then they are both isomorphic to $\mathbb{R}$, and $\mu_{\mathbb{R}^n} : H^0(\mathbb{R}^n) \to H_c^n(\mathbb{R}^n)^*$ is an isomorphism.

To see this, it is enough to verify that it is non–zero. But $H^0(\mathbb{R}^n)$ is generated by the class of the constant map $f(p) = 1$, hence the image under $\mu_{\mathbb{R}^n}$ is represented by the map $\beta \mapsto \int_{\mathbb{R}^n} \beta$, which we know takes non–zero values.

Our main result is a vast generalization of this:

Theorem ("Poincaré duality") *μ_M is an isomorphism for every oriented manifold M and every k.*

The proof will be based on a comparison of Mayer-Vietoris sequences: Given a diagram

$$\begin{array}{ccc} U \cap V & \xrightarrow{j_U} & U \\ {\scriptstyle j_V}\downarrow & & \downarrow{\scriptstyle i_U} \\ V & \xrightarrow{i_V} & U \cup V \end{array}$$

of open subsets of M, we consider the Mayer–Vietoris sequences for both H^* and H_c^*. If we dualize all vector spaces and maps in the compact support sequence we get a new long exact sequence, and since the spaces $U, V, U \cap V$ and $U \cup V$ are all n–manifolds with orientations coming from M, we can draw the following diagram:

(5)

$$\begin{array}{ccccccccc} \to H^k(U \cup V) & \to & H^k(U) \oplus H^k(V) & \to & H^k(U \cap V) & \xrightarrow{\delta} & H^{k+1}(U \cup V) \to \\ \downarrow{\scriptstyle \mu_{U\cup V}} & & \downarrow{\scriptstyle \mu_U \oplus \mu_V} & & \downarrow{\scriptstyle \mu_{U\cap V}} & & \downarrow{\scriptstyle \mu_{U\cup V}} \\ \to H_c^l(U \cup V)^* & \to & H_c^l(U)^* \oplus H_c^l(V)^* & \to & H_c^l(U \cap V)^* & \xrightarrow{\delta^*} & H_c^{l-1}(U \cup V)^* \to \end{array}$$

where $l = n - k$. (Note that then $l - 1 = n - (k + 1)$.)

Lemma *This diagram is commutative, provided the map labeled δ is replaced by $(-1)^k\delta$.*

Proof. Observe first that the lower sequence could also have been obtained from the short exact sequence of chain complexes gotten by dualizing

$$0 \to \Omega_c^*(U \cap V) \xrightarrow{(j_{U*}, -j_{V*})} \Omega_c^*(U) \oplus \Omega_c^*(V) \xrightarrow{i_{U*} + i_{V*}} \Omega_c^*(U \cup V) \to 0$$

The maps μ_U etc. are also defined on the level of forms. Hence we can construct a diagram of cochain complexes

(6)

$$\begin{array}{ccccc} 0 \to \Omega^*(U \cup V) & \xrightarrow{(i_U{}^*, i_V{}^*)} & \Omega^*(U) \oplus \Omega^*(V) & \xrightarrow{j_U{}^* - j_V{}^*} & \Omega^*(U \cap V) \to 0 \\ \downarrow{\scriptstyle \mu_{U\cup V}} & & \downarrow{\scriptstyle \mu_U \oplus \mu_V} & & \downarrow{\scriptstyle \mu_{U\cap V}} \\ 0 \to \Omega_c^*(U \cup V)^* & \xrightarrow[((i_{U*})^*, (i_{V*})^*)]{} & \Omega_c^*(U)^* \oplus \Omega_c^*(V)^* & \xrightarrow[(j_{U*})^* - (j_{V*})^*]{} & \Omega_c^*(U \cap V)^* \to 0 \end{array}$$

(Note that the dual of a linear map of the form $f+g : A \oplus B \to C$ is $(f^*, g^*) : C^* \to A^* \oplus B^*$, and the dual of $(f,g) : A \to B \oplus C$ is the map $f^* + g^* : B^* \oplus C^* \to A^*$.)

Observe that this is a three–dimensional diagram, with the third directions given by the exterior differential and its dual. We claim that it is commutative in each degree. This reduces to the following observation:

Let $j : W_1 \subseteq W_2$ be an inclusion of open subsets of M. Then the diagram

$$\begin{array}{ccc} \Omega^k(W_2) & \xrightarrow{j^*} & \Omega^k(W_1) \\ \downarrow{\scriptstyle \mu_{W_2}} & & \downarrow{\scriptstyle \mu_{W_1}} \\ \Omega_c^{n-k}(W_2)^* & \xrightarrow{{j_*}^*} & \Omega_c^{n-k}(W_1)^* \end{array}$$

is commutative. In fact, starting with $\omega \in \Omega^k(W_2)$ and going around the diagram in two ways results in the two linear maps $\Omega_c^{n-k}(W_1) \to \mathbb{R}$ given by $\eta \mapsto \int_{W_1} j^*\omega \wedge \eta$ and $\eta \mapsto \int_{W_2}(\omega \wedge j_*\eta)$. But these are equal, since $j_*(j^*\omega \wedge \eta) = \omega \wedge j_*\eta$ and this form has support inside W_1.

However, the maps μ_W do not commute with exterior differentials, instead we have the formula

$$\mu_W(d\omega) = (-1)^{k+1} d^*(\mu_W(\omega)),$$

for $\omega \in \Omega^k(W)$. To see this, we evaluate both sides on a form $\eta \in \Omega_c^{n-k+1}(W)$:

$$\begin{aligned} \text{Lhs}: \quad & \mu_W(d\omega)(\eta) &= \int_W d\omega \wedge \eta \\ \text{Rhs}: \quad & d^*(\mu_W(\omega))(\eta) &= \int_W \omega \wedge d\eta \end{aligned}$$

But $\int_W d\omega \wedge \eta + (-1)^k \int_W \omega \wedge d\eta) = \int_W d(\omega \wedge \eta) = 0$.

Going back to diagram (6), we now see that if we change the sign of the exterior differentials in one of the exact sequences — e. g. if we replace $d^k : \Omega^k(W) \to \Omega^{k+1}(W)$ by $(-1)^{k+1} d^k$ — then all possible squares in the diagram will commute. This will not change any cohomology, and the maps induced by inclusions will remain the same as before. Therefore we will still get a Mayer–Vietoris sequence, and the only difference will be that the map $\delta : H^k(\mathcal{U} \cap V) \to H^{k+1}(U \cup V)$ will be changed by the same sign $(-1)^{k+1}$. But since diagram (6) now is completely commutative, it is easily seen that (5) also will be commutative. □

Using the 5–lemma, we now get the following easy, but important

Corollary *If Poincaré duality holds for U, V and $U \cap V$, it also holds for $U \cup V$.*

The following extension of this corollary provides one of our main tools for proving Poincaré duality (henceforth abbreviated to PD) for more and more complicated subsets of M.

Lemma A. *Suppose $\mathcal{O}$ is a set of open subsets which is closed under finite intersections. If PD holds for all sets in $\mathcal{O}$, then it holds for all finite unions of sets in $\mathcal{O}$.*

Proof. We prove that PD holds for sets of the form $U_1 \cup \cdots \cup U_k$, $U_i \in \mathcal{O}$, by induction on k, starting with $k = 1$ which holds by assumption.

So, assume PD holds for unions of $k - 1 \geqslant 1$ sets in $\mathcal{O}$. Then it holds for U_k and $U_1 \cup \cdots \cup U_{k-1}$, and by the assumption on $\mathcal{O}$ and the induction hypothesis it also holds for $U_k \cap (U_1 \cup \cdots \cup U_{k-1}) = (U_k \cap U_1) \cup \cdots \cup (U_k \cap U_{k-1})$. Therefore it also holds for $U_1 \cup \cdots \cup U_k$ by the Corollary above. □

Let us call a set of subsets an *fc–set* if it is closed under finite intersections. Observe that if $\mathcal{O}$ is an fc–set, then the set $\mathcal{O}_f$ of finite unions of sets in $\mathcal{O}$ is again fc. An example to keep in mind is the set of subsets of open boxes of $\mathbb{R}^n$ — the sets of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ for real numbers a_i, b_i. This set satisfies both of the conditions of Lemma A. Note that in this case $\mathcal{O}$ is in fact a *basis* for the topology, and every open subset of $\mathbb{R}^n$ can be written as a *countable* union of sets in $\mathcal{O}$. We would like to have a version pf Lemma A for infinite unions, but this can not be established the same way. However, one special case is easy:

Lemma B. *Let $U = \coprod_i U_i$ be the union of disjoint open sets such that PD holds for every U_i. Then PD holds for U.*

Proof. A form on U consists of a form on each U_i, and it follows easily that $H^k(U) \approx \prod_i H^k(U_i)$. However, a form with compact support is non-trivial only on finitely many U_i. Therefore $H_c^{n-k}(U) \approx \bigoplus_i H_c^{n-k}(U_i)$. But dualizing takes direct sums to direct products. □

It follows from Lemma A and Lemma B that if we start from a collection $\mathcal{O}$ of open subsets of M as in Lemma A, then PD will hold for all sets that can be obtained by finitely many times applying the processes of taking finite unions and arbitrary disjoint unions. The general Poincaré duality theorem follows if we can prove that M itself can be so obtained. The following Lemma is the crucial topological result we need:

Lemma C. *Let $\mathcal{O}$ be an fc–basis for the topology of a connected manifold M such that the closure of each element of $\mathcal{O}$ is compact. Then M can be written $M = V_1 \cup V_2$, where V_1 and V_2 are disjoint unions of countably many sets in $\mathcal{O}_f$.*

Before proving this, let us see how it leads to a

Proof of Poincaré duality. First, observe that PD follows for all open subsets of $\mathbb{R}^n$. Indeed, an open $U \in \mathbb{R}^n$ is a disjoint, countable union of connected open set, and if U is connected, it has a basis (e. g. the basis discussed above consisting of open boxes) satisfying the conditions of both Lemma A and Lemma B.

If M is an arbitrary connected manifold, it has a basis consisting of sets W such that W is diffeomorphic to an open subset of $\mathbb{R}^n$, and such that the closure of W is compact. This basis is easily seen to be fc, and we have just seen that all its elements satisfy Poincaré duality. □

It now only remains to prove Lemma C. We start by choosing an increasing sequence of open sets $U_1 \subset \cdots \subset U_i \subset U_{i+1} \subset \cdots$ such that

(1) $\bigcup_i U_i = M$

(2) Every U_i has compact closure $\overline{U}_i$, and $\overline{U}_i \subset U_{i+1}$.

(See e. g. the proof of Theorem 13, page 50 in Spivak.) We will define a new sequence of open sets W_i, $i = 1, 2, \ldots$ such that

(i) $\overline{U}_i \subset \bigcup_{j\leqslant i} W_j \subset U_{i+1}$ (hence $\bigcup_i W_i = M$)
(ii) Each $W_i \in \mathcal{O}_f$
(iii) $W_i \cap W_{i+2} = \emptyset$ for all i.

The sets W_i will be defined by induction on i. Since U_2 is open, it is a union of open sets from the basis $\mathcal{O}$. Choose a finite number of these sets which cover the compact set $\overline{U}_1$ and let W_1 be their union.

Now suppose W_j has been defined for $j < i$ and consider the inclusion of the compact set $\overline{U}_i - \bigcup_{j<i} W_j$ into the open set $U_{i+1} - \overline{U}_{i-1}$. The latter set is a union of sets from the basis $\mathcal{O}$, and a finite number of them cover $\overline{U}_i - \bigcup_{j\leqslant i-1} W_j$. Let W_i be the union of this finite number of basis sets.

Property (ii) is then fulfilled by definition. Property (i) follows since

$$\bigcup_{j\leqslant i} W_j = W_i \cup \bigcup_{j<i} W_j \supset (\overline{U}_i - \bigcup_{j<i} W_j) \cup \bigcup_{j<i} W_j = \overline{U}_i\,.$$

Finally, property (iii) is true because $W_i \subseteq U_{i+1} - \overline{U}_{i-1}$ and $W_{i+2} \subseteq U_{i+3} - \overline{U}_{i+1}$, and the two sets $U_{i+1} - \overline{U}_{i-1}$ and $U_{i+3} - \overline{U}_{i+1}$ are disjoint.

Now $V_1 = \bigcup_i W_{2i-1}$ and $V_2 = \bigcup_i W_{2i}$ are easily seen to be as in Lemma C. □

Poincaré duality is the most important result in de Rham cohomology. Here are some immediate consequences.

• Suppose M is a connected, orientable n–manifold. Then

$$H^n(M) \cong \begin{cases} \mathbb{R} \text{ if } M \text{ is compact} \\ 0 \text{ if } M \text{ is non–compact} \end{cases}$$

In contrast, for cohomology with compact support we get $H^n_c(M) \cong \mathbb{R}$ both in the compact and non–compact cases, and an isomorphim is given by $\omega \mapsto \int_M \omega$. Note that any nonzero element will be a generator. In particular it follows that we can represent a generator by a form with support in an arbitrarily small neighborhood of any point.

• The calculation of $H^n_c(M) \cong \mathbb{R}$ gives also rise to an important invariant of proper maps $f : M \to N$ between connected, oriented n–manifolds. In fact, the induced map $f^* : H^n_c(N) \to H^n_c(M)$ must be multiplication by a real number, which we call the *degree* of f — $\deg f$. If ω represents a generator of $H^n_c(N) \cong \mathbb{R}$, the degree of f can be calculated from the equation

$$\int_M f^*\omega = \deg f \int_N \omega\,.$$

For example, if f is a diffeomorphism, $\deg f = 1$ if f is orientation preserving and $\deg f = -1$ if f is orientation reversing. Also, if f is an n–fold, orientation preserving covering map, then $\deg f = n$. It is an amazing fact that the degree is always an integer! (See Theorem 12, page 275 in Spivak.)

• From the homotopy invariance of $H^k(M)$ we get $H^k(M) \cong H^k(M \times \mathbb{R})$ for all k. It follows that if M is orientable we have isomorphisms

$$H_c^k(M)^* \cong H^{n-k}(M) \cong H^{n-k}(M \times \mathbb{R}) \cong H_c^{k+1}(M \times \mathbb{R})^* , \quad \text{or}$$

$$H_c^k(M) \cong H_c^{k+1}(M \times \mathbb{R}) .$$

This generalizes the cases $M = S^n$ and $M = \mathbb{R}^n$ established earlier in terms of explicit calculations. It is a nice exercise to show that the isomorphism above can be represented by the map on the level of forms given by $\omega \mapsto \omega \wedge f\,dt$, where t is the coordinate on $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a function with compact support satisfying $\int_{\mathbb{R}} f\,dt = 1$.

This observation is vastly generalized in exercise 2.

• One can show that all smooth manifolds can be covered by open sets U_i diffeomorphic to R^n such that all finite intersections $U_{i_1} \cap \cdots \cap U_{i_l}$ are either diffeomorhic to $\mathbb{R}^n$ or empty. If M is compact, this simplifies the proof of Poincaré duality greatly, since we in this case only need the induction procedure of Lemma A. But then it also follows that the cohomology ($H^k = H_c^k$ in this case) is *finitely dimensional*. Hence, if M is a compact, orientable n–manifold there are isomorphisms

$$H^k(M) \cong H^{n-k}(M)$$

for all k.

• Recall that $H^k(CP^n)$ is isomorphic to $\mathbb{R}$ if k is even $\leqslant 2n$ and 0 otherwise. We will now use Poincaré duality to determine the *ring structure* on $H^*(CP^n) = \bigoplus_k H^k(CP^n)$.

Claim: $H^*(CP^n) \cong \mathbb{R}[u]/(u^{n+1})$, where u corresponds to an element in degree 2.

What this says is that if u is a generator (i. e. a non–zero element) of $H^2(CP^n)$, then u^k is non–zero for all $k \leqslant n$. We prove this by induction on n, starting trivially with $n = 1$.

Assume that $H^*(CP^{n-1}) \cong \mathbb{R}[v]/(v^n)$. By the computation earlier we know that the inclusion $\iota : CP^{n-1} \subset CP^n$ induces an isomorphism on H^j for $j < 2n$. Hence there is an element $u \in H^2(CP^n)$ such that $\iota^*(u) = v$, and since $\iota^*(u^k) = (\iota^*(u))^k = v^k \neq 0$ for $k < n$, it follows that $u^k \neq 0$ for $k < n$. Now use that the Poincaré duality isomorphism $H^{2(n-1)}(CP^n) \to H^2(CP^n)^*$ is given by $\mu_{CP^n}(w)(u) = w\,u$. This must be non–zero for all $w \neq 0$, in particular for $w = u^{n-1}$. Thus $u^n \neq 0$.

We end with some remarks on *non–orientable manifolds.* If M is non-orientable, we can construct a double covering space $\pi : \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable, in a natural way. Let

$$\widetilde{M} = \{(p, o_p) \,|\, p \in M \text{ and } o_p \text{ an orientation of the tangent space } M_p\} ,$$

and define $\pi : \widetilde{M} \to M$ by $\pi(p, o_p) = p$. Then π is a surjective map such that $\pi^{-1}(p)$ consists of two points — the two orientations of M_p. If $(p, o_p) \in \widetilde{M}$, let $x : U \to \mathbb{R}^n$ be a local coordinate map such that $p \in U$. We can clearly choose x such that it is orientation preserving at p with respect to the given orientation o_p and the standard orientation on $\mathbb{R}^n$. Requiring that x be orientation preserving in all of U defines an orientation o_q of M_q for every

$q \in U$, hence a bijection between U and the subset $\{(q, o_q) | q \in U\}$. This defines a differentiable atlas on $\widetilde{M}$ such that π becomes a smooth covering map. Moreover, π_{*q} is an isomorphism $\widetilde{M}_{(q,o_q)} \cong M_p$. Giving $\widetilde{M}_{(q,o_q)}$ the orientation corresponding to o_q via this isomorphism defines an orientation of $\widetilde{M}$. $\pi : \widetilde{M} \to M$ is called the *orientation covering of* M.

Example. If M is an even–dimensional real projective space RP^{2m}, the orientation covering can be identified with the covering $S^{2m} \to RP^{2m}$ which we used to compute the cohomology of RP^{2m}. In that computation a crucial rôle was played by the antipodal map, which we now generalize to an arbitrary orientation covering.

Let $A : \widetilde{M} \to \widetilde{M}$ be the map that interchanges the two points of $\pi^{-1}(p)$ for every p. Then A shares the following formal properties with the antipodal map on even spheres:

– A is an orientation reversing diffeomorphism
– $\pi \circ A = \pi$
– $A \circ A = \mathrm{id}_{\widetilde{M}}$.

Just as in the sphere case we now get splittings of the vector spaces of forms (with and without compact support) into the $+1$ and -1 eigenspaces of A_*, and these splittings induce the corresponding splittings in cohomology. Moreover, just as before we can prove that we have isomorphisms $\Omega^k(M) \cong \Omega^k(\widetilde{M})_+$ and $\Omega^k_c(M) \cong \Omega^k_c(\widetilde{M})_+$. In particular we obtain isomorphisms

$$H^n(M) \cong H^n(\widetilde{M})_+ \quad \text{and} \quad H^n_c(M) \cong H^n_c(\widetilde{M})_+$$

But these groups both vanish! This is trivial in the case H^n for M non–compact, since then $H^n(\widetilde{M}) = 0$. The remaining cases follow from the fact that A is orientation reversing, since then $A_* : H^n_c(\widetilde{M}) \to H^n_c(\widetilde{M})$ is multiplication by -1.

Hence we have proved that $H^n(M) = H^n_c(M) = 0$ for all connected non–orientable manifolds M. Taken in combination with our earlier computations, we obtain the following characterization of orientable manifolds:

Corollary. *A connected n–manifold M is orientable if and only if $H^n_c(M)$ is non–vanishing.*

If M is orientable, a choice of generator of $H^n_c(M)$ defines an orientation. Two generators g_1 and g_2 determine the same orientation if and only if $g_2 = ag_2$ with $a > 0$.

It follows that if M is oriented, a generator represented by a form ω such that $\int_M \omega = 1$ is uniquely determined. This generator is called the *orientation class* of M.

Here are a couple of slightly challenging exercises for the ambitious student:

Exercise 1. Poincaré duality as discussed here breaks down in the non–orientable case, but there are still interesting relations between cohomology with and without compact support. As an example, show that

$$H^k(M) \oplus H^{n-k}_c(M)^* \cong H^k(\widetilde{M}) \,.$$

(Hint: show first that the diagram

$$\begin{array}{ccc} H^k(\widetilde{M}) & \xrightarrow{A^*} & H^k(\widetilde{M}) \\ \downarrow \mu_{\widetilde{M}} & & \downarrow \mu_{\widetilde{M}} \\ H_c^{n-k}(\widetilde{M})^* & \xleftarrow{(A^*)^*} & H_c^{n-k}(\widetilde{M})^* \end{array}$$

commutes up to multiplication by -1.)

Exercise 2. This is an exercise to test if you really understand the techniques used to prove Poincaré duality.

Let M and N be two manifolds, and let π_M, π_N be the two projections from $M \times N$ to M and N. Then we can define maps

$$\times : \Omega^i(M) \times \Omega^j(N) \to \Omega^{i+j}(M \times N)$$

by $(\omega, \eta) \mapsto \omega \times \eta = \pi_M^* \omega \wedge \pi_N^* \eta$

(a) Show that $\times$ also restricts to a bilinear map

$$\times : \Omega_c^i(M) \times \Omega_c^j(N) \to \Omega_c^{i+j}(M \times N),$$

which can be used to define a linear map

$$H_c^i(M) \otimes H_c^j(N) \to H_c^{i+j}(M \times N).$$

Doing this for all i and j such that $i + j = k$ defines a linear map

$$\kappa_{M,N} : \bigoplus_j H_c^j(M) \otimes H_c^{k-j}(N) \to H_c^k(M \times N)$$

(b) Show that $\kappa_{M,N}$ is an isomorphism for all M and N and all k. (*The Künneth theorem.*)

(Hint: Fix N and consider the source and target of $\kappa_{M,N}$ as functors of M and open inclusions. Start by showing that these functors have Mayer–Vietoris sequences.)

(c) What goes wrong if we try to prove the Künneth theorem for H^k? Prove that this theorem also holds if either M or N has finite dimensional cohomology in every dimension.

(d) Prove that $S^2 \times S^4$ and CP^3 are two manifolds which have isomorphic cohomology groups in every dimension but are not smoothly homotopy equivalent.

Remark. The tensor product of the graded rings $H_c^*(M)$ and $H_c^*(N)$ can be organized into a graded ring by setting

$$(H_c^*(M) \otimes H_c^*(N))^k = \bigoplus_j H_c^j(M) \otimes H_c^{k-j}(N)$$

and defining $(a \otimes b)(c \otimes d) = (-1)^{rs} ac \otimes bd$ if $b \in H_c^r(N)$ and $c \in H_c^s(M)$. Then $\kappa_{M,N} : H_c^*(M) \otimes H_c^*(N) \to H_c^*(M \times N)$ becomes an isomorphism of graded rings.

Similar remarks apply to H^*.