Chapter 2:

Stochastic processes in event history analysis

SOLUTIONS TO EXERCISES

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EXERCISE 1

Throw a die several times. Let $Y_i$ be result in $i$th throw, and let $X_i = Y_1 + \ldots + Y_i$ be the sum of the $i$ first throws.

(a) Explain that

$$f(x_1, x_2) = P(X_1 = x_1, X_2 = x_2) = \frac{1}{36} I(1 \leq x_1 \leq 6, 1 + x_1 \leq x_2 \leq 6 + x_1)$$

(b) Find an expression for $f(x_2|x_1)$.

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$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f(x_1)} = \frac{1}{6} I(1 + x_1 \leq x_2 \leq 6 + x_1)$$

(c) Find an expression for $E(X_2|x_1)$ as a function of $x_1$.

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$$E(X_2|x_1) = x_1 + 7/2$$
(d) Find the expectation of the random variable $E(X_2|X_1)$. Find $E(X_2)$ from this. Find also $E(X_2)$ without using the rule of double expectation.

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$$E(X_2|X_1) = X_1 + 7/2$$

$$E(X_2) = E(E(X_2|X_1)) = E(X_1) + 7/2 = 7$$

or

$$E(X_2) = E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 7/2 + 7/2 = 7$$
EXERCISE 2

Let $Y_1, Y_2, \ldots$ be independent with $E(Y_n) = \mu$ for all $n$.

Let $X_n = Y_1 + \ldots + Y_n$.

Find $E(X_4|\mathcal{F}_2)$ and in general $E(X_n|\mathcal{F}_m)$ when $m \leq n$

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$$E(X_4|\mathcal{F}_2) = E(Y_1 + Y_2 + Y_3 + Y_4|Y_1, Y_2)$$
$$= Y_1 + Y_2 + E(Y_3) + E(Y_4)$$
$$= Y_1 + Y_2 + 2\mu = X_2 + 2\mu$$

$$E(X_n|\mathcal{F}_m) = X_m + (n - m)\mu$$
EXERCISE 3

Let $X_1, X_2, \ldots$ be independent with

$$P(X_i = 1) = P(X_i = -1) = 1/2$$

Think of $X_i$ as result of a game where one flips a coin and wins 1 unit if “heads” and loses 1 unit if “tails”.

Let $M_n = X_1 + \ldots + X_n$ be the gain after $n$ games, $n = 1, 2, \ldots$

Show that $M_n$ is a (mean zero) martingale.

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\[
E(M_n|\mathcal{F}_{n-1}) = E(X_1 + \ldots + X_n|X_1, \ldots, X_{n-1}) \\
= X_1 + \ldots + X_{n-1} + E(X_n) \\
= M_{n-1}
\]
EXERCISE 4

(a) Let $X_1, X_2, \ldots$ be independent with $E(X_n) = 0$ for all $n$.
Let $M_n = X_1 + \ldots + X_n$.
Show that $M_n$ is a (mean zero) martingale.

Same proof as for Exercise 3

(b) Let $X_1, X_2, \ldots$ be independent with $E(X_n) = \mu$ for all $n$.
Let $S_n = X_1 + \ldots + X_n$.
Is $\{S_n\}$ a martingale?
Can you find a simple transformation of $S_n$ which is a martingale?

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$E(S_n | \mathcal{F}_{n-1}) = S_{n-1} + \mu$

This is a martingale only if $\mu = 0$.
Easy to see that $M_n = S_n - n\mu$ is a martingale.
EXERCISE 5

Show that for a martingale \( \{M_n\} \) is

(a) \( E(M_n|F_m) = M_m \) for all \( m < n \).

This is essentially Exercise 2.1 in book.

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We use induction on \( k \) to prove that

\[
(*) \ E(M_n|F_{n-k}) = M_{n-k} \quad \text{for} \ k = 1, 2, \ldots, n - 1
\]

\( k = 1 \) is definition of martingale.

Suppose (*) holds for \( k \). Then

\[
E(M_n|F_{n-k-1}) = E\{E(M_n|F_{n-k})|F_{n-k-1}\} = E(M_{n-k}|F_{n-k-1}) = M_{n-k-1}
\]

Hence (*) holds for all the required \( k \).

(b) \( E(M_m|F_n) = M_m \) for all \( m < n \)

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This holds trivially since \( M_m \in F_n \)

(c) Give verbal (intuitive) interpretations of each of (a) and (b).
EXERCISE 6

Define the *martingale differences* by

\[ \Delta M_n = M_n - M_{n-1} \]

(a) Show that the definition of martingale, \( E(M_n|\mathcal{F}_{n-1}) = M_{n-1} \), is equivalent to

\[ E(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0, \text{ i.e. } E(\Delta M_n|\mathcal{F}_{n-1}) = 0 \]  

(Hint: Why is \( E(M_{n-1}|\mathcal{F}_{n-1}) = M_{n-1} \)?)

The following are clearly equivalent by 'Hint' which is obvious:

\[
\begin{align*}
E(M_n|\mathcal{F}_{n-1}) &= M_{n-1} \\
E(M_n|\mathcal{F}_{n-1}) - M_{n-1} &= 0 \\
E(M_n|\mathcal{F}_{n-1}) - E(M_{n-1}|\mathcal{F}_{n-1}) &= 0 \\
E(M_n - M_{n-1}|\mathcal{F}_{n-1}) &= 0
\end{align*}
\]
(b) Show that for a martingale we have

\[ \text{Cov}(M_{n-1} - M_{n-2}, M_n - M_{n-1}) = 0 \text{ for all } n \quad (2) \]

i.e.

\[ \text{Cov}(\Delta M_{n-1}, \Delta M_n) = 0 \]

Explain in word what this means.

\[ \text{Cov}(M_{n-1} - M_{n-2}, M_n - M_{n-1}) = E\{(M_{n-1} - M_{n-2})(M_n - M_{n-1})\} \]

\[ = E[E\{(M_{n-1} - M_{n-2})(M_n - M_{n-1})|\mathcal{F}_{n-1}\}] \]

\[ = E[(M_{n-1} - M_{n-2})E\{(M_n - M_{n-1})|\mathcal{F}_{n-1}\}] \]

\[ = 0 \]

(c) Show that (1) and (2) automatically hold when \( M_n = X_1 + \ldots + X_n \)

for independent \( X_1, X_2, \ldots \).

Note that in this case the differences \( \Delta M_n = M_n - M_{n-1} = X_n \) are independent.

Thus (2) shows that martingale differences correspond to a weakening of the independent increments property.
EXERCISE 7

Do Exercise 2.6 in book:

Show that the stopped process $M^T$ is a martingale. (*Hint: Find a predictable process $H$ such that $M^T = H \cdot M$.*

We have

$$M^T_n = M_{n \wedge T} \equiv \begin{cases} M_n & \text{if } n \leq T \\ M_T & \text{if } n > T \end{cases}$$

Want to write

$$(*) \quad M^T_n = H_1(M_1 - M_0) + H_2(M_2 - M_1) + \ldots + H_n(M_n - M_{n-1})$$

Let

$$H_i = \begin{cases} 1 & \text{if } T \geq i \\ 0 & \text{otherwise} \end{cases}$$

Then $H_i$ is known at time $i - 1$ and we may check that $(*)$ does the job.
EXERCISE 8

Suppose $X_n = U_1 + \ldots + U_n$ where $U_1, \ldots, U_n$ are independent with $E(U_i) = \mu$.

Find the Doob decomposition of the process $X$ and identify the predictable part and the innovation part.

Recall the Doob decomposition:

$$X_n = E(X_n|\mathcal{F}_{n-1}) + (X_n - E(X_n|\mathcal{F}_{n-1})) = \text{predictable} + \text{innovation}$$

Here the predictable part is:

$$E(X_n|\mathcal{F}_{n-1}) = E(X_{n-1} + U_n|\mathcal{F}_{n-1}) = X_{n-1} + \mu$$

and the innovation is hence

$$X_n - (X_{n-1} + \mu) = X_n - X_{n-1} - \mu = U_n - \mu$$
EXERCISE 9

Prove the results on the covariances (see Exercise 2.2 in book)

Let \( 0 \leq s < t < u < v \leq \tau \)

\[
\text{Cov}(M(t) - M(s), M(v) - M(u)) = E\{(M(t) - M(s))(M(v) - M(u))\}
\]

\[
= E[E\{(M(t) - M(s))(M(v) - M(u))|\mathcal{F}_u\}]
\]

\[
= E[(M(t) - M(s))E\{(M(v) - M(u))|\mathcal{F}_u\}]
\]

\[
= 0
\]
POISSON PROCESSES

\( N(t) = \# \) events in \([0, t]\)

Characterizing properties:

- \( N(t) - N(s) \) is Poisson-distributed with parameter \( \lambda(t - s) \)

- \( N(t) \) has independent increments, i.e. number of events in disjoint intervals are independent.

EXERCISE 10

Show that

\[ M(t) = N(t) - \lambda t \]

is a martingale. Identify the compensator of \( N(t) \).

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The compensator must be \( \lambda t \) by Doob-Meyer (and \( \lambda t \) is increasing, predictable).
\[ E(M(t)|\mathcal{F}_s) = E\{N(t) - \lambda t|\mathcal{F}_s\} \]
\[ = E\{N(t)|\mathcal{F}_s\} - \lambda t \]
\[ = E\{N(t) - N(s) + N(s)|\mathcal{F}_s\} - \lambda t \]
\[ = E\{N(t) - N(s)|\mathcal{F}_s\} + N(s) - \lambda t \]
\[ = E\{N(t) - N(s)\} + N(s) - \lambda t \]
\[ = \lambda t - \lambda s + N(s) - \lambda t \]
\[ = N(s) - \lambda s = M(s) \]
EXERCISE 11

This is Exercise 2.3 in book, plus a new question (d). (a) was done as our Exercise 3(a)

Thus let $X_1, X_2, \ldots$ be independent with $E(X_n) = 0$ and $\text{Var}(X_n) = \sigma^2$ for all $n$. Let $M_n = X_1 + \ldots + X_n$.

(a) Show that $M_n$ is a (mean zero) martingale.

(b) Compute $\langle M \rangle_n$

\[ \langle M \rangle_n = \sum_{i=1}^{n} \text{Var}(\Delta M_i | \mathcal{F}_{i-1}) \]

Here $\Delta M_i = M_i - M_{i-1} = X_i$, so we get

\[ \langle M \rangle_n = \sum_{i=1}^{n} \text{Var}(\Delta M_i | \mathcal{F}_{i-1}) = \sum_{i=1}^{n} \text{Var}(X_i | \mathcal{F}_{i-1}) = \sum_{i=1}^{n} \text{Var}(X_i) = n\sigma^2 \]

since $X_i$ is independent of $\mathcal{F}_{i-1}$. (Why?)
(c) Compute $[M]_n$

$[M]_n = \sum_{i=1}^{n} (\Delta M_i)^2 = \sum_{i=1}^{n} X_i^2$

(d) Compute $E \langle M \rangle_n$, $E [M]_n$ and $\text{Var}(M_n)$.

$E \langle M \rangle_n = E [M]_n = \text{Var}(M_n) = n\sigma^2$

(and these are equal in general, see next exercise)
EXERCISE 12

Consider a general discrete time martingale $M$

(a) $M_n^2 - \langle M \rangle_n$ is a mean zero martingale - Exercise 2.4 in book

Write

$$M_n^2 = (M_{n-1} + M_n - M_{n-1})^2$$

$$= M_{n-1}^2 + 2M_{n-1}(M_n - M_{n-1}) + (M_n - M_{n-1})^2$$

Thus

$$E\{M_n^2 - \langle M \rangle_n | F_{n-1}\}$$

$$= E\{M_{n-1}^2 + 2M_{n-1}(M_n - M_{n-1}) + (M_n - M_{n-1})^2 - \langle M \rangle_n | F_{n-1}\}$$

$$= M_{n-1}^2 + 2M_{n-1}E\{(M_n - M_{n-1})|F_{n-1}\} + E\{(M_n - M_{n-1})^2|F_{n-1}\}$$

$$- \sum_{i=1}^{n} E\{(M_i - M_{i-1})^2|F_{i-1}\}$$

$$= M_{n-1}^2 + 2 \cdot 0 - \sum_{i=1}^{n-1} E\{(M_i - M_{i-1})^2|F_{i-1}\}$$

$$= M_{n-1}^2 - \langle M \rangle_{n-1}$$
(b) $M_n^2 - [M]_n$ is a mean zero martingale - See book p. 45

(c) Use (a) and (b) to prove that

$$Var(M_n) = E\langle M\rangle_n = E[M]_n$$

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$$Var(M_n) = E(M_n^2) = E\langle M\rangle_n = E[M]_n$$
EXERCISE 13

Consider again the Poisson process, where we have shown that

\[ M(t) = N(t) - \lambda t \]

is a martingale.

Prove now that:

\[ M^2(t) - \lambda t \]

is a martingale (Exercise 2.9 in book)

Then prove that \( \langle M \rangle (t) = \lambda t \)

What is \([M](t)\)?
\[ E(M^2(t) - \lambda t | \mathcal{F}_s) \]
\[ = E\{(N(t) - \lambda t)^2 - \lambda t | \mathcal{F}_s\} \]
\[ = E\{(N(t) - N(s) - \lambda(t-s) + N(s) - \lambda s)^2 | \mathcal{F}_s\} - \lambda t \]
\[ = E\{(N(t) - N(s) - \lambda(t-s)) \mathcal{F}_s\} \]
\[ + 2E\{(N(t) - N(s) - \lambda(t-s))(N(s) - \lambda s) | \mathcal{F}_s\} \]
\[ + E\{(N(s) - \lambda s)^2 | \mathcal{F}_s\} - \lambda t \]
\[ = E\{(N(t) - N(s) - \lambda(t-s))^2\} \]
\[ + 2(N(s) - \lambda s)E\{N(t) - N(s) - \lambda(t-s)\} \]
\[ + (N(s) - \lambda s)^2 - \lambda t \]
\[ = \lambda(t - s) + 0 + (N(s) - \lambda s)^2 - \lambda t \]
\[ = (N(s) - \lambda s)^2 - \lambda s \]
\[ = M^2(s) - \lambda s \]