OPERATOR ALGEBRAS

1. Introduction

In the past decades the subject of operator algebras, which serves as a mathematical foundation of quantum physics, has seen an enormous expansion. Together with its algebraic-geometric offspring, noncommutative geometry, operator algebras is now a major mathematical discipline, which has attracted many eminent mathematicians, including the Fields medalists Connes and Jones, and several plenary speakers at ICMs: Haagerup and Popa at the last two. The subject now expands to such diverse fields as dynamical systems, ergodic theory, quantum groups, noncommutative topology, deformation theory, index theory, number theory, wavelet theory, and disciplines in mathematical physics like quantum field theory and quantum information theory.

The operator algebra group in Norway enjoys a strong scientific position, both nationally and internationally. According to ISI web of science, Bratteli has been cited in 2304 scientific papers to date; allegedly this is the highest number of citing papers in the period 1940 - 27 May 2008 for scientific work of any Norwegian mathematician (including Selberg). Størmer was the first chairman of the Abel Prize committee. Skau has done a tremendous job in popularizing the Abel Prize and the work of the Abel Prize winners to Norwegian and international media through interviews and TV programs.

The group's distinctions include:

- In the spring of 2007 the operator algebra group at UiO was named "toppforskningsmiljø" (top research environment).
- In 2001 Bratteli received Professor Ingerid Dal and Ulrikke Greve Dal's prize for Humanistic Research.
- In 2004 Bratteli received "The Fridtjof Nansen prize for excellence in research".
- In 2004 Bratteli and Størmer shared the "Prize for excellence in research" (The Møbius prize) from The Research Council of Norway, and the "Prize for excellence in research" from the Wedel-Jarlsberg Foundation.
- In 2008 Neshveyev received "The Mathematical Sciences of Paris Foundation" award.

1.1. Primary scientific objectives.

In this project our main goals are to:

(i) show for a large class of groups (conjecturally all amenable ones) that minimal actions are orbit equivalent to $\mathbb{Z}$-actions;

(ii) describe the structure of C^*-algebras arising in the study of actions of semigroups and Hecke-pairs, shift spaces and higher-rank graphs;

(iii) develop a theory of noncommutative geometry for quantum groups and their homogeneous spaces, including an algebraic index theorem;

(iv) identify data of arithmetic significance in quantum statistical dynamical systems from number fields, and explore links between number theory and ergodic theory;

(v) advance the theory of finite quantum systems by operator algebraic methods.

See Section 2 for a more elaborate description.

1.2. **Strategic objectives.**

The operator algebra group sees a challenge in maintaining its outstanding international profile given that two of its members have recently retired (Alfsen and Størmer), another will retire next year (Skau), and three more will reach the retirement age within five years. Of the 11 members, only two are aged between 40 and 60. One of them (Bédos) has pressing administrative duties (which he will fortunately be relieved of in 2009), while the second (Tuset) is employed at Oslo University College and has a considerable teaching load.

However, there is some cause for optimism: the University of Oslo has permanently employed two researchers under the age of forty in operator algebras (Larsen and Neshveyev), and NTNU has hired a permanent researcher in his early thirties (Carlsen, due to commence in August 2008).

Also at the postgraduate level there is some cause for optimism: there are currently two Ph.D. students in Oslo, and the Faculty of Mathematics has granted a Ph.D. stipend, starting in 2008 in connection with Larsen's position. Two new Ph.D. students will begin in 2008 at NTNU.

In general, recruitment of graduate students is made difficult by the challenging nature of the topic: one needs to master a wide breadth of knowledge from many mathematics courses before one can embark on a Ph.D. study.

The objectives of the project are to attract young new researchers, revitalize the scientific connection between Oslo and Trondheim, and maintain the fertile collaboration between Oslo University College and the University of Oslo, thus also bridging the generation gap, while at the same time retaining and extending the already extensive international network of collaborators.

2. Scientific description

2.1. **Dynamical systems and crossed products.** The interplay between topological dynamical systems and ergodic theory on the one hand and operator algebras on the other, has proved fruitful. Members of the operator algebra group have been working actively on several aspects of this theory. One line of the group's research came to an end two years ago with the publication of a monograph on dynamical entropy in operator algebras [40], which is a comprehensive account of theory developed since the foundational work of Connes and Størmer [16]. Presently most of the group's research relates to C^*-dynamical systems and, in particular, to C^*-crossed products $C^*(X, G)$ associated to topological dynamical systems (X, G).

The focus of Skau's research has been on studying minimal topological dynamical systems (X, G) in terms of their orbit structure. For X the Cantor set, Skau and his collaborators have obtained remarkable results that are topological analogues of the famous results of Connes and Krieger in the measure-theoretic/von Neumann algebraic setting.

It turns out that to understand the orbit structure of (X, G), it is advantageous to study the associated equivalence relation R_G on X, appropriately topologized. The equivalence classes of R_G are the orbits of (X, G). In a celebrated paper by Connes, Feldman and Weiss (extending a result by Ornstein and Weiss) it was shown that countable, ergodic and amenable equivalence relations are hyperfinite, i.e. they can be approximated by finite equivalence relations. This can be shown to be the same as being orbit equivalent to an ergodic $\mathbb{Z}$-action. In three recent papers Skau and his collaborators Giordano, Matui and Putnam [23, 24, 25] have succeeded in proving that the Cantor minimal system (X, G) is topologically orbit equivalent to a minimal $\mathbb{Z}$-system if G is a finitely generated free abelian group.

An indispensable tool in their work is the concept of a Bratteli diagram, which provides a model for both the dynamics and the associated equivalence relation. Due to the novelty of the techniques developed, one is now in a good position to extend the results of Skau et al to countable amenable groups. A related project is to analyze the interplay between (X, G) and $C^*(X, G)$. For $G = \mathbb{Z}$ this has been done by Giordano, Putnam and Skau.

While noncommutative dynamical systems normally arise from an action of a group G on a C^*-algebra A, many outstanding examples of C^*-algebras do not exhibit this form of interaction, and one rather discovers an irreversible system involving an action of a semigroup or perhaps an action of a group G with a certain subgroup H forming a Hecke-pair. Landstad and his collaborators [29, 30] have introduced a new approach to the Hecke C^*-algebra of a Hecke group-subgroup pair (G, H) by looking at a Morita equivalent ideal in the group C^*-algebra of G. Landstad and Larsen have extended the methods of [29] to study the Hecke C^*-algebra associated to a Hecke-pair (G, H) and a finite-dimensional representation of H with finite range [34]. Landstad in collaboration with Kaliszewski and Quigg now continue their approach to also study Wiener-Hopf algebras, as defined by Nica in the context of C^*-algebras associated to highly non-amenable groups, by using crossed product algebras.

Bédos has made a significant contribution to the study of simplicity of group C^*-algebras [5]. It is also natural to look for primitive algebras. One knows that for a full group C^*-algebra a necessary condition for primitivity is that the group is ICC. This is also sufficient in the amenable case and also for certain families of non-amenable groups (as shown by Choi and Murphy). Bédos is investigating the primitivity of the full group C^*-algebra for other non-amenable ICC groups. In a somewhat related line of research, summation processes for Fourier series of elements in discrete reduced twisted C^*- crossed products are of interest when studying the ideal structure of such crossed products. In the case of reduced twisted group C^*-algebras, these processes have been studied in recent joint work of Bédos and Conti [6]. They now want to take the natural step forward and study the existence of summation processes in the more general setting of twisted crossed products.

Classification of crossed product algebras is closely related to classification of automorphic actions up to cocycle conjugacy. Since the pioneering work of Connes in the 70's, an important step towards such a classification has been to establish a Rohlin-type property. Bratteli, Kishimoto and Robinson have recently characterized the Rohlin property for quasi-free flows on Cuntz algebras [10], and have identified connections between approximately inner flows on C^*-algebras and various notions of convergence of the associated generators [11]. Bratteli is planning to continue this work on flows on operator algebras. An outstanding open problem is to decide whether a derivation which maps the algebra of smooth elements of a Lie group action into itself, is a generator [9]. There is a renewed interest in this problem in view of recent work of Connes [12] and Rosenberg [47].

2.2. **Cuntz-Pimsner type algebras.** A successful path towards understanding the structure and the K-theory of large classes of C^*-algebras was opened by Pimsner's influential paper [45] on C^*-algebras associated to Hilbert bimodules. The classical examples fitting Pimsner's construction (Cuntz algebras, Cuntz-Krieger algebras, crossed products by a single automorphism or a single endomorphism) were followed by the large class of graph-algebras which later were shown to have higher rank analogues, see the monograph [46] for an excellent summary. C^*-algebras of higher rank graphs are good candidates for constructing coactions of discrete groups and are a good test case for the study of C^*-algebras associated to higher dimensional dynamical systems. Carlsen, in collaboration with Sims, are studying C^*-algebras of higher-rank graphs. Carlsen and Larsen aim to classify KMS-states of Cuntz-Pimsner and Toeplitz algebras according to their types as introduced by Exel and Laca. Here examples in the realm of graph-algebras bring useful insight. Larsen and Raeburn have recently constructed multi-resolution analyses in Hilbert bimodules over graph-algebras [37], and would like to employ their joint expertise on Nica's algebras and crossed products by semigroups to analyze structure and examples of C^*-algebras of higher dimensional systems.

Emerging evidence suggests that a combination of Nica's approach to C^*-algebras generated by isometries, Pimsner's approach, and work of Fowler [22], should provide new tools for studying various C^*-algebras associated to semigroups. Motivated by this Sims and Yeend [48] introduced Cuntz-Nica-Pimsner algebras for certain well-behaved product systems of Hilbert bimodules over semigroups. Carlsen, Larsen, Sims and Vittadello seek uniqueness results for such algebras in terms of coactions of the enveloping groups.

Carlsen and Eilers have for several years studied shift spaces via associated C^*-algebras. This collaboration will continue with a study of particular classes of shift spaces by means of the C^*-algebras associated to them, as done for shift spaces of primitive aperiodic substitutions in the papers [19], [20], [21] and with beta-shifts in the paper [4] with Bates. A good candidate for such a class is the shift space of a Toeplitz flow, because it possesses one of the properties which make the analysis of the C^*-algebra associated to the shift space of a primitive aperiodic substitution possible, but on the other hand has a more complex structure than the shift space of a substitution.

A successful trend in operator algebras has been to adapt various constructions of C^*-algebras to the purely algebraic case. This has, for example, been done for graph algebras. Carlsen and Ortega are working on adapting the construction of Cuntz-Pimsner algebras. This could potentially lead to new invariants and classification results for symbolic dynamical systems.

2.3. **Quantum groups and homogeneous spaces.** The Dirac operator plays a crucial role in the description of matter, both in the Standard model and in String theory. The extension by Atiyah and Singer of the Dirac operator to general spin manifolds, has also, thanks to their work on the index theorem, had a major impact on topology, geometry and representation theory of Lie groups.

With the recent breakthrough by Neshveyev and Tuset [43], where the Dirac operator has been extended to q-deformed compact quantum groups in a way compatible with the axioms of Connes, in that their quantum Dirac operator forms a summable spectral triple, all these quantum groups can now be regarded as noncommutative manifolds. Success in constructing a quantum Dirac operator was previously limited to the simplest case of quantum $SU_q(2)$. Quantum groups can thus be studied from the point of view of noncommutative geometry, and its arsenal of tools are made available in previous work of Neshveyev and Tuset, where they have set up the appropriate Hopf-equivariant framework within noncommutative geometry [41, 42].

The obvious goal is to obtain a local index formula for these generalized elliptic operators, that is, a formula for the equivariant Chern character of the corresponding spectral triples. However, the present formula by Connes and Moscovici [15] is not sufficient for computations. One needs a suitable language in the quantum case to formulate a computable index formula. The way the quantum Dirac operator is constructed offers a hint on how to proceed. The construction relies on a twist, which can be thought of as a 2-cochain on the dual discrete quantum group. The key properties of the quantum Dirac operator actually depend on the associator, that is, the coboundary of the twist. By a celebrated result of Drinfeld in the formal deformation setting, there exists a twist such that the corresponding associator is given by the monodromy of the KZ-equations satisfied by the m-point function in conformal field theory. The existence of a twist in the analytic setting is established in a recent paper by Neshveyev and Tuset [44] as a corollary of a lengthy and technical work of Kazhdan and Lusztig which shows that the analytic version of the Drinfeld category of a group is equivalent to the category of finite-dimensional representations of its q-deformation.

The idea for the index formula is to use the explicit isomorphism between the cohomology groups of twist-isomorphic quasi-bialgebras [1]. This way one gets a cocycle living on the algebra of functions on the Lie group together with a nontrivial associator. The challenge is to find an explicit expression for this cocycle using some modified form of the Connes-Moscovici formula. Towards a solution of this problem it is instructive to consider the algebraic index theorem announced by Tamarkin and Tsygan [54]. There one starts with a deformation A_h of the algebra of functions on a Poisson manifold M, and expresses the trace density map from its periodic cyclic homology in terms of the principal symbol map $A_h \to C^\infty(M)$ and characteristic classes of M.

The construction of the Dirac operator suggests some concealed spin-structure in the categories of representations of (quantum) groups related to a spin KZ-equation and a spin monodromy. It would be extremely interesting if such a notion exists, especially in view of the close relation to string theory and to invariants of knots and 3-manifolds.

In another direction, Van Daele and Landstad have started a project based on their discovery that the existence of a compact, open subgroup in a group G is equivalent to the existence of a certain non-trivial pair; a convolution operator and a multiplication operator which commute [35, 36]. Other properties of the group G are shown to be equivalent to this property. For instance if G is abelian, it is equivalent to the existence of a compactly supported non-zero continuous function such that its Fourier transform also has compact support. It is desirable to generalize this to quantum groups. The first goal is then to find the proper analogue of a compact, open quantum subgroup.

2.4. **Number theory and quantum statistical mechanics.** The celebrated construction of Bost and Connes [7] was the first example of a quantum statistical mechanical system with implications in number theory. Since then the hope is that operator algebras can bring new insight into various number theoretical problems. One such tentative application is Manin's real multiplication program [38]. It is believed that a systematic analysis of analogies between elliptic curves and noncommutative tori will lead to a theory of real multiplication and, in particular, to a proof of Stark's conjectures for real quadratic fields [38].

One way of approaching this program, which is going to be pursued, is to analyze a specialization of the universal Bost-Connes type system for quadratic fields, the so-called GL_2-system of Connes and Marcolli [13], to a real quadratic field. The specialization of this system to an imaginary quadratic field K produces the right analogue of the Bost-Connes system for K and allows one to recast the theory of complex multiplication in operator algebraic terms [14]. The constructions of Connes and Marcolli have led Ha and Paugam [27] to suggest an analogue of the Bost-Connes system for arbitrary number fields. Recently Laca, Larsen and Neshveyev reinterpreted their systems in dynamical terms and gave a complete classification of KMS_β-states [33]. Furthermore, in [32] they solved a problem left open by Connes and Marcolli [13], and thus completed the analysis of the GL_2-system, by showing that for each β in the critical region $(1, 2]$ there exists a unique KMS_β-state. These advances put one in a good position to attack the case of real quadratic fields.

Another interesting problem is to compute the types of the von Neumann algebras defined by the KMS_β-states for small β and understand the relation of the structure of KMS_β-states with distribution results in number theory. For one-dimensional systems, that is, for the Bost-Connes systems for number fields, the KMS_β-states for $\beta \leq 1$ produce factors of type III_1, and the result is related to Landau's theorem on distribution of prime ideals (work in progress). In the higher dimensional case the factors are still amenable, despite the groups being nonamenable, but the computation of the type is much harder. The hope is to relate KMS-states to harmonic measures for Brownian motions and random walks on Bruhat-Tits buildings, and then possibly apply the technique of Bowen [8], [26].

2.5. **Operator algebras in quantum physics.** The question of quantum theory over non-Archimedean space time arose out of a seminal work of Volovich, [56] and [57], which is based on the fact that when general relativity and quantum theory operate together, a new scale emerges, the Planck scale, with the property that in domains of dimensions less than this scale, no measurements are possible.

During the past few years, Digernes, Weisbart, and Varadarajan have extended various aspects of quantum theory to p-adic systems, see [17] and [55]. Their goal is to explore the ground of Volovich's hypothesis from the point of view of quantum measurement. Concretely, they are working on extending the results of Digernes, Varadarajan, Varadhan (recipient of the Abel Prize 2007) and Weisbart in [17] and [55] to p-adic manifolds and matrix potentials, see [18] for a promising progress, and they are exploring the structure of the Feynman–Kac formula in the p-adic setting. Another objective is that of extending the results in [31] to quantum field theory over more general compact p-adic manifolds. This work relates p-adic quantum theory to statistical mechanics, but works only on 4-dimensional p-adic space time. The hope is to extend the results to higher dimensions and to other manifolds, and this will require the use of Hamiltonians which are not quadratic.

Quantum computation and quantum information theory (QIT) are tremendously active fields of research, and lately much effort has been devoted by mathematicians and mathematical physicists to the understanding of conceptually and mathematically difficult problems of quantum entanglement. At the other extreme, a "non-entangled" state of a quantum system is called "separable".

An important problem in the analysis of the state spaces of composed quantum systems is that of finding a satisfactory characterization of the convex set of separable states. Alfsen has had a year-long collaboration with Shultz on the general theory of state spaces, which culminated in the monograph [2]. They are currently studying state spaces of composed quantum systems, a topic which is also related to recent work of Arveson and Størmer.

Most of the theory on QIT is developed for finite dimensions and has been almost disjoint from operator algebra theory. Størmer has played a key role in tying the theories closer together by using operator algebra techniques and drawing on expertise in positive maps. In two recent papers, one jointly with Arveson, [3, 51], Størmer studies multiplicative and asymptotic properties of positive maps. His recent work is more directly related to QIT. In the theory there are several candidates for entropy of systems, one of which is Holevo's capacity. Størmer has proved a reduction theorem for capacity based on the decomposition of a positive map with respect to its definite set [50]. There is a duality between positive maps and linear functional on tensor products of operator systems and the trace class operators. This duality was studied by Størmer more than 20 years ago [49] and has been very fruitful in studying states and their properties such as separability [52, 53]. A central result in the theory is the Horodecki theorem [28], which characterizes separable states on tensor algebras of matrix algebras in terms of their action on all positive maps between the algebras. Størmer has used duality arguments to obtain extensions of this result, in particular to nuclear C^*-algebras, and is presently looking for smaller classes of maps which can replace all maps in the Horodecki theorem.

3. Perspectives

3.1. **Other programs.** The operator algebra group is connected to an EU Research training network in noncommutative geometry, see http://www.cf.ac.uk/maths/opalg/qsng.html. Neshveyev is the coordinator of the Norwegian node.

There will be a special session on operator algebras and noncommutative geometry, organized by Larsen, Neshveyev and Nest, at the 25th Nordic and 1st British-Nordic congress of Mathematicians in 2009.

Størmer is in the organizing committee for a project on QIT at the Mittag-Leffler Institute in the fall of 2010.

3.2. **Impact on society.** In areas like computer science and nano technology, where quantum effects are significant, a deeper understanding of the mathematical structures behind quantum physics is important. This project with its focus on operator algebras, should therefore contribute towards this.

Of course the recruitment of young researchers in mathematics is essential for mathematical education, and this project meets the challenge of maintaining a sufficiently high mathematical standard in society.

4. Project management

4.1. **Planned activities.** We plan a cooperation at the level of Ph.D. education by offering advanced and concentrated series of lectures. In addition, we plan an international summer school and hope to attract many students from the large operator algebra groups in, for example, Denmark (Copenhagen and Odense), Germany (Münster) and Belgium (Leuven). It is our experience and belief that participation in such events is very inspiring and also creates important contacts for future research.

As part of the national network activity, we are going to organize several joint seminars Oslo-Trondheim.

We also plan to organize an international conference for approximately 50 participants.

4.2. **Budget.** In a large and fast moving area like operator algebras it is necessary to attract top young researchers at postdoc level, and involve them in compelling projects. We are applying for 2 such fellowships.

We are also applying for 2 doctoral fellowships.

To maintain connections with our international collaborators, we are applying for fellowships for guest researchers, for up to 3 months per year.

As an important part of making the scientific program a success, we require funding to reduce the teaching load of Tuset (Oslo University College) by 50%. This will bring his teaching duties into line with the two other participating institutions (NTNU and UiO).

4.3. **The operator algebra group.** The group has the following permanent members:

UiO	NTNU	HiO
Erik Alfsen (emeritus)	Toke M. Carlsen	Lars Tuset
Erik Bédos	Trond Digernes	
Ola Bratteli	Magnus Landstad	
Nadia S. Larsen	Christian Skau	
Sergey Neshveyev		
Erling Størmer (emeritus)		

There are currently two Ph.D. students at UiO: Amandip S. Sangha and Olav G. Imenes. Three new Ph.D.-students will begin in 2008: one at UiO and two at NTNU.

Key personell (CV's are attached): Larsen, Neshveyev, Carlsen, Landstad, Tuset.

4.4. **International collaborators.** W. Arveson, R. Conti (Univ. of Newcastle, Australia), A. Van Daele (Univ. of Leuven, Belgia), S. Eilers (Univ. of Copenhagen, Denmark) T. Giordano (Univ. of Ottawa, Canada), M. Izumi (Kyoto Univ., Japan), S. Kaliszewski and J. Quigg (Arizona State Univ., USA), A. Kishimoto (Hokkaido Univ., Japan), M. Laca and I. F. Putnam (Univ. of Victoria, Canada), H. Matui (Chiba Univ., Tokyo, Japan), M. Müeger (Radboud Univ. Nijmegen, The Netherlands), E. Ortega (Univ. of Southern Denmark), I. Raeburn, A. Sims and S. Vittadello (Univ. of Wollongong, Australia), D. Robinson, S. Silvestrov (Lund Univ., Sweden), F. Shultz (Wellesley College, USA), V. S. Varadarajan and D. Weisbart (UCLA, USA).

References

[1] S. E. Akrami and S. Majid, *Braided cyclic cocycles and nonassociative geometry*, J. Math. Phys. **45** (2004), 3883–3911.

[2] E. Alfsen and F. Shultz, *Geometry of state spaces of operator algebras.* Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2003.

[3] W. Arveson and E. Størmer, *Asymptotic lifts of positive linear maps*, Pacific J. Math. **233** (2007), 1–14.

[4] T. Bates, T. M. Carlsen and S. Eilers, *Dimension groups associated to β-expansions*, Math. Scand. **100** (2007), 198–208.

[5] E. Bédos, *Discrete groups and simple C^*-algebras*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 521–537.

[6] E. Bédos and R. Conti, *On twisted Fourier analysis and convergence of Fourier series on discrete goups*, to appear in J. of Fourier Anal. and Appl.

[7] J.-B. Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (N.S.) **1** (1995), 411–457.

[8] R. Bowen, *Anosov foliations are hyperfinite*, Ann. of Math. (2) **106** (1977), 549–565.

[9] O. Bratteli, *Derivations, dissipations and group actions on C^*- algebras*, Lecture Notes in Mathematics, **1229**, Springer, 1986.

[10] O. Bratteli, A. Kishimoto and D. W. Robinson, *Approximately inner derivations*, to appear in Math. Scand.

[11] O. Bratteli, A. Kishimoto and D. W. Robinson, *Rohlin flows on the Cuntz algebra $\mathcal{O}_\infty$*, J. Funct. Anal. **248** (2007), 472–511.

[12] A. Connes, *On the spectral characterization of manifolds*, preprint 2008.

[13] A. Connes and M. Marcolli, *From Physics to Number Theory via Noncommutative Geometry, Part I: Quantum Statistical Mechanics of $\mathbb{Q}$-lattices*, in "Frontiers in Number Theory, Physics, and Geometry, I", 269–350, Springer Verlag, 2006.

[14] A. Connes, M. Marcolli and N. Ramachandran, *KMS states and complex multiplication*, Selecta Math. (N.S.) **11** (2005), 325–347.

[15] A. Connes, H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995), 174–243.

[16] A. Connes and E. Størmer, *Entropy of automorphisms in II_1 von Neumann algebras*, Acta Math. **134** (1975), 289–306.

[17] T. Digernes, V. S. Varadarajan and S. R. S. Varadhan, *Finite approximations to quantum systems*, Rev. Math. Phys. **6**(4) (1994), 621–648.

[18] T. Digernes, V. S. Varadarajan and D. E. Weisbart, *Schrödinger Operators on Local Fields: Self-Adjointness and Path Integral Representations for Propagators*, under review.

[19] T. M. Carlsen and S. Eilers, *Augmenting dimension group invariants for substitution dynamics*, Ergodic Theory Dynam. Systems **24** (2004), 1015–1039.

[20] T. M. Carlsen and S. Eilers, *Matsumoto K-groups associated to certain shift spaces*, Doc. Math. **9** (2004), 639–671 (electronic).

[21] T. M. Carlsen and S. Eilers, *Ordered K-groups associated to substitutional dynamics*, J. Funct. Anal. **238** (2006), 99–117.

[22] N. Fowler, *Discrete product systems of Hilbert bimodules*, Pacific J. Math. **204** (2002), 335–375.

[23] T. Giordano, I. F. Putnam, H. Matui and C. F. Skau, *The absorption theorem for affable equivalence relations*, to appear in Ergodic Theory Dynam. Systems.

[24] T. Giordano, I. F. Putnam, H. Matui and C. F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^2$-systems*, J. Amer. Math. Soc. **21** (2008), 863–892.

[25] T. Giordano, I. F. Putnam, H. Matui and C. F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems*, preprint 2008.

[26] M. Izumi, S. Neshveyev and R. Okayasu, *The ratio set of the harmonic measure of a random walk on a hyperbolic group*, Israel J. Math. **163** (2008), 285–316.

[27] E. Ha, F. Paugam, *Bost-Connes-Marcolli systems for Shimura varieties, I, Definitions and formal analytic properties*, IMRP Int. Math. Res. Pap. **5** (2005), 237–286.

[28] M. Horodecki, P. Horodecki, and R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A **223** (1996), 1–8.

[29] S. Kaliszewski, M. B. Landstad and J. Quigg, *Hecke C^*-algebras, Schlichting completions, and Morita equivalence*, to appear in Proc. Edinburgh Math. Soc.

[30] S. Kaliszewski, M. B. Landstad and J. Quigg, *Hecke C^*-algebras and semidirect products*, to appear in Proc. Edinburgh Math. Soc.

[31] A. N. Kochubei and R. M. Sait-Ametov, *Interaction Measures on the Space of Distributions over the Field of p-Adic Numbers*, Inf. Dimens. Anal. Quantum Probab. Relat. Top. **6**(3) (2003), 389–411.

[32] M. Laca, N. S. Larsen and S. Neshveyev, *Phase transition in the Connes-Marcolli GL_2-system*, J. Noncommut. Geom. **1** (2007), 397–430.

[33] M. Laca, N. S. Larsen and S. Neshveyev, *On Bost-Connes type systems for number fields*, preprint arXiv:0710.3452v2 [math.OA].

[34] M. B. Landstad and N. S. Larsen, *Generalised Hecke algebras and C^*-completions*, to appear in Internat. J. Math.

[35] M. B. Landstad and A. Van Daele,*Compact and discrete subgroups of algebraic quantum groups I*, preprint, math.OA/0702458.

[36] M. B. Landstad and A. Van Daele, *Groups with compact open subgroups and multiplier Hopf *-algebras*, to appear in Expositiones Math.

[37] N. S. Larsen and I. Raeburn, *Projective multi-resolution analyses arising from direct limits of Hilbert modules*, Math. Scand. **100** (2007), 317–360.

[38] Yu. I. Manin, *Real multiplication and noncommutative geometry*, in: The legacy of Niels Henrik Abel, 685–727, Springer, Berlin, 2004.

[39] Yu. I. Manin and M. Marcolli, *Continued fractions, modular symbols, and noncommutative geometry*, Selecta Math. (N.S.) **8** (2002), 475–521.

[40] S. Neshveyev and E. Størmer, *Dynamical entropy in operator algebras.* A Series of Modern Surveys in Mathematics, **50**, Springer-Verlag, Berlin, 2006.

[41] S. Neshveyev and L. Tuset, *Hopf algebra equivariant cyclic cohomology, K-theory and index formulas*, K-Theory 31 (2004), 357–378.

[42] S. Neshveyev and L. Tuset, *A local index formula for the quantum sphere*, Comm. Math. Phys. 254 (2005), 323–341.

[43] S. Neshveyev and L. Tuset, *The Dirac operator on compact quantum groups*, preprint arXiv: math/0703161 [math.OA].

[44] S. Neshveyev and L. Tuset, *Notes on the Kazhdan-Lusztig theorem on equivalence of the Drinfeld category and the category of $U_q(\mathfrak{g})$-modules*, preprint arXiv:0711.4302v1 [math.QA].

[45] M. V. Pimsner, *A class of C^*-algebras generalising both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Fields Institute Comm. **12** (1997), 189–212.

[46] I. Raeburn, *Graph algebras*, CMBS Regional Conference Series in Mathematics, vol. **103**, Amer. Math.Soc., 2005.

[47] J. Rosenberg, *Noncommutative variations on Laplace's equation*, preprint 2008.

[48] A. Sims and T. Yeend, *C^*-algebras associated to product systems of Hilbert bimodules*, preprint arXiv:0712.3073v1 [math.OA].

[49] E. Størmer, *Extension of positive maps into $B(H)$*, J. Funct. Anal. **66** (1986), 235–254.

[50] E. Størmer, *A reduction theorem for capacity of positive maps*, Positivity **11** (2007), 69–75.

[51] E. Størmer, *Multiplicative properties of positive maps*, Math.Scand. **100** (2007), 184–192.

[52] E. Størmer, *Separable states and positive maps*, J. Funct. Anal. **254** (2008), 2303-2312.

[53] E. Størmer, *Separable states and positive maps II*, preprint arXiv. OA/0803.4417.

[54] D. Tamarkin, B. Tsygan, *Cyclic formality and index theorems*, in: EuroConférence Moshé Flato 2000, Part II (Dijon), Lett. Math. Phys. **56** (2001), 85–97.

[55] V. S. Varadarajan and D. Weisbart, *Convergence of quantum systems on grids*, J. Math. Anal. Appl., **336**(1) (2007), 608–624.

[56] I. V. Volovich, *Number theory as the ultimate physical theory*, Preprint CERN-TH-4781-87, 1987.

[57] I. V. Volovich, *p-adic string*, Classical Quantum Gravity, **4**(4) (1987), L83–L87.