Determinant of electric field distributions in superconductors via magneto-optical imaging and the Faraday law

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The determination of electric field distributions within superconductors is of fundamental importance for both basic research and applications. The analysis of electric fields allows insights into different modes of vortex dynamics in superconductors as well as the determination of local maxima of dissipation which could destabilize superconducting transport. We present an analysis of electric field distributions for thin films in magnetization experiments. We show that electric field distributions can be determined experimentally by time-resolved magneto-optical measurements of flux density distributions via the Faraday law. We give a detailed analysis of the inductive and the potential contributions to the electric fields \( E_i \) and \( E_p \), respectively. Experimental results are demonstrated for a magnetized square-shaped \( \text{YBa}_2\text{Cu}_3\text{O}_7 \) thin film.

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I. INTRODUCTION

The most remarkable phenomenon of superconducting materials is the sharp drop of the resistivity of an electric current to an unmeasurable small value at a critical temperature \( T_c \). This is inevitably related to the screening of electric fields. Since an electric field corresponds to a time change of the phase of an electron wave, the macroscopic phase coherence of the superconducting condensate is incompatible with an electric field. However, it is well known that in type-II superconductors a current-induced motion of the Abrikosov flux lines (vortices) induces electric fields.\(^1\)\(^-\)\(^3\)

The preservation of true superconducting currents requires pinning of flux lines at materials defects. In high-temperature superconductors (HTS’s) the interplay of intrinsic anisotropy and large thermal fluctuations enhances strongly the thermally activated depinning and motion of vortices. Dependent on materials properties and experimental conditions, a variety of different regimes of vortex motion, such as thermally activated flux creep (TAFc),\(^4\) current-induced plastic creep,\(^5\) or flux flow (FF) can be observed. Their relation to different modes of vortex dynamics like collective creep,\(^6\) moving vortex bundles and chains,\(^7\) flux avalanches,\(^8\) self-organized vortex motion,\(^9\) or even the occurrence of instabilities\(^10\) is topic of actual research.

There are excellent theoretical studies of electric field distributions of superconductors in magnetization\(^11\)\(^-\)\(^13\) and transport experiments.\(^14\)\(^,\)\(^15\) Gurevich and Brandt predict a universal spatial pattern of the electric field in magnetization experiments, which only depends on the sample geometry.\(^12\)\(^,\)\(^13\) For transport currents, regions of strongly inhomogeneous electric fields are predicted by Gurevich and Friesen.\(^15\) The theoretical work has in common that it is based on a materials law which is assumed to be valid in the entire superconductor.\(^16\) However, different collective modes of vortex motion (like avalanches, glassy creep, or vortex dynamics at weak links) should be inevitably related to different electric field patterns. Therefore, it is highly desirable to gain experimental insight into the electric field distributions of superconductors\(^17\) which is not based on the application of a materials law.

Work that direction on \( \text{YBa}_2\text{Cu}_3\text{O}_7 \) and \( \text{Nd}_{1.85}\text{Ce}_{0.15}\text{CuO}_4 \) single crystals has been performed by Giller \textit{et al.}\(^18\) There, a one-dimensional approximation for the calculation of the electric field \( \mathbf{E} \) from measured time evolutions of the flux density was used. For the study of the electric field and the vortex dynamics at specific locations, like current domain boundaries, grain boundaries, flux filaments at large defects, or avalanches, where anomalies of the vortex motion are expected, a more general method has to be applied. Our own first approach\(^17\) to generalize electric field imaging to two-dimensional current distributions is based on incorrect approximations. For the integration of the Faraday law, it disregards the contribution of the time evolution of the external stray field of the sample to the inductive electric field which, however, represents the dominant contribution.

In this article, we consider the true total electric field of a superconductor. The true electric field is induced by moving vortices or by temporal evolution of the supercurrent density. Theoretical work suggests the existence of an effective electric field, where thermodynamic contributions and the Bernoulli potential are considered.\(^19\)\(^,\)\(^20\) The contribution of such an effective electric field, if present, is extremely small because it is related to surface inhomogeneities of the superconducting state, such as vortex cores. If such an effective electric field would exist in a superconductor, it would be by orders of magnitude smaller than the true electric field caused by vortex motion. In particular this applies to high-\( T_c \) superconductors, which have an extremely small coherence length and the spatial extent of surface inhomogeneities is very small.

The total true electric field \( \mathbf{E} \) is comprised of the inductive \( \mathbf{E}_i \) and the potential contribution \( \mathbf{E}_p \). The determination of \( \mathbf{E}_i(r) \) via the induction law requires the measurement of the time evolution of the magnetic flux density vector \( \mathbf{B}(r) \) in the entire space inside and outside the sample. Since this is not possible for arbitrary sample geometries, the method for the determination of \( \mathbf{E}(r) \) presented in this article is restricted to magnetization experiments of thin-film superconductors with arbitrary shape. For the case of a two-dimensional current distribution \( \mathbf{j}(x,y) \), the inversion of the Biot-Savart law allows an unambiguous determination of the current density from the measured normal component of the flux density.
**II. THEORY**

In superconductors, an electric field distribution is induced by moving vortices\(^1\) or (e.g., in the vortex free Meissner phase) by a temporal evolution of supercurrents.\(^2\) As pointed out in the former section, a possibly existing thermodynamic electric field is disregarded in this article. According to the Faraday law, the electric field components \(E_{i,x}, E_{i,y}\) are induced along any contour containing \(B_i \neq 0\) (black arrows). This inductive field \(E_i\) polarizes the medium, and consequently a polarization field \(P\) (white arrows) and a charge density \(n'' = - \varepsilon_0 \mathbf{V} \cdot \mathbf{P}\) are induced in the sample (here only depicted at the sample surfaces). The induced charge density is the source of a potential electric field \(E_p = -\mathbf{P}\) in the sample.

\(B_i(x, y)\) at the surface of the superconductor.\(^2\) The determination of the time evolution of \(j(x, y, t)\) by experiment is the starting point for the calculation of the time evolution of the flux density components \(B_i(r, t) = (B_{i,x}(r, t), B_{i,y}(r, t), B_{i,z}(r, t))\) inside and outside the superconducting film and therefore for the calculation of \(E_i(r)\). The potential contribution \(E_p(r)\) can be derived from the obtained \(E_i(r)\) and \(j(x, y)\) distributions.

The article is organized as follows: In Sec. II we present the general electrodynamics of superconductors for the determination of inductive and potential contributions to \(E_i(r)\). Solutions of the equations are derived for a relaxing two-dimensional (2D) current distribution in thin films in Sec. III. Section IV presents the special case of a relaxing one-dimensional current distribution and calculates the electric field distribution in the cross section of a thin stripe. Section V presents an exemplary experimental study of the time decay of the flux density gradients of a magnetized square-shaped YBa\(_2\)Cu\(_4\)O\(_7\) thin film. The two-dimensional inductive electric field distribution in the film plane is calculated via two-dimensional Fourier transformations. In Sec. VI, it is shown that the piezoelectric polarization field \(P(r)\), the potential electric field \(E_p(r)\), and the induced charge density distribution \(n''(r)\) can be determined.

**FIG. 1.** Sketch of an electrically polarizable sample in an inductive electric field \(E_i\), which is induced by a local temporal evolution of \(B_i\). According to the Faraday law, electric field components \(E_{i,x}\) and \(E_{i,y}\) are induced along any contour containing \(B_i \neq 0\) (black arrows). This inductive field \(E_i\) polarizes the medium, and consequently a polarization field \(P\) (white arrows) and a charge density \(n'' = - \varepsilon_0 \mathbf{V} \cdot \mathbf{P}\) are induced in the sample (here only depicted at the sample surfaces). The induced charge density is the source of a potential electric field \(E_p = -\mathbf{P}\) in the sample.
of a finite resistivity, the component of the inductive electric field due to the relaxation of in-plane flux densities. Shown is a domain of a square-shaped film with a relaxing Bean-like current density. According to the Faraday law, in each current domain, there are inductive electric field components parallel and perpendicular to the current density. Whereas the \( j \) component \( |j| \) reflects the occurrence of a finite resistivity, the \( E_i \) component perpendicular to \( j \) induces a charge density and a polarization \( P \) which is related to inhomogeneous vortex motion. Note that the total electric field \( E_i + E_p \) (with \( E_p = -P \)) is parallel to the current density. (c) Possible occurrence of an \( E_z \) component of the inductive electric field due to the relaxation of in-plane flux densities. Shown is a \( j_z \) domain of a square-shaped film with a \( \partial B_z \). Since the total electric field must be parallel to \( j \), \( E_z \) is totally canceled by \( E_{pz} \) due to surface charge densities at the \( z = \pm d/2 \) [see (d)] and the \( z = 0 \) planes.

\[
\frac{\partial B}{\partial t} = -\nabla \times E_i. \tag{1}
\]

The inductive electric field \( E_i \) has pure rotational character and consequently \( \nabla \cdot E_i = 0 \). In electrically polarizable media or media with mobile charges, the inductive electric field \( E_i(r) \) gives rise to a dielectric polarization field \( P(r) \) (see Fig. 1 for the general case of a polarizable medium in a time-dependent field and Fig. 2 for the case of flux creep in a square-shaped superconductor in a fully penetrated state). \( P(r) \) is related to induced volume and surface charge densities \( n_i(r) \) via \( \varepsilon_0 \nabla \cdot P = -n_i \). Due to the presence of \( n_i(r) \), a potential electric field \( E_p(r) \) is generated with \( \varepsilon_0 \nabla \cdot E_p = n_i \), where \( n_i \) represents any excess or external charge density. If \( n_i = 0 \), \( E_p = -P \) inside the sample. Note, that the spatial average of the induced charge density in the entire sample \( \langle n_i(r) \rangle = 0 \). The total electric field is then (according to the Helmholtz decomposition) comprised of two contributions \( E = E_i + E_p \), where \( E_i \) has inductive and \( E_p = -\nabla \phi \) has potential character—i.e., \( \nabla \times E_p = \nabla \times \nabla \phi = 0 \) with \( \Delta \phi = -n_i / \varepsilon_0 \) and \( \nabla \cdot E_i = 0 \).

The dielectric displacement \( D = \varepsilon_0 E_i + \varepsilon_0 E_p + \varepsilon_0 P \) is given by

\[
D = \varepsilon_0 E_i + \varepsilon_0 E_p + \varepsilon_0 P, \tag{2}
\]

with

\[
\nabla \cdot D = n_i. \tag{3}
\]

Consequently, \( D \) is independent of the considered material and just depends on the electric induction field \( E_i \) and the excess charge density \( n_i \)—e.g., due to external charges.

The boundary conditions according to the Maxwell theory require the continuity of the tangential components of \( E \) at all surfaces. In contrast, the normal components of \( E \) exhibit discontinuities due to the presence of surface charge densities. In the following, we restrict our analysis to magnetization experiments of electrically isolated samples, where external charges or potentials are absent—i.e., \( \nabla \cdot D = 0 \) and \( \phi = 0 \) at the surface of the sample. The time evolution of the magnetic flux density is then comprised of \( B(r,t) = \mu_0 H_{ex}(t) + \mu_0 H_{int}(t) \), where \( \mu_0 H_{ex} \) represents the contribution of the external field and the magnetic field \( H(r) \) is generated by the flowing supercurrents:

\[
\nabla \times H = \frac{\partial D}{\partial t} + j. \tag{4}
\]

Usually, electric fields in superconductors are extremely small and thus the contribution of the time derivative of the electric displacement \( \partial D/\partial t \) to the magnetic field in Eq. (4) can be completely disregarded. For the magnetic induction one has
\[ \nabla \cdot \mathbf{B} = 0. \] (5)

In magnetization experiments, the time evolution of the flux density \( \mathbf{B}(\mathbf{r}, t) \) according to Eqs. (4) and (5) is directly related to the time evolution of the current density \( \mathbf{j}(\mathbf{r}, t) \) and the time evolution \( \mu_0 \mathbf{H}_{ex}(t) \). After ramping of \( H_{ex} \) is stopped, the time evolution of \( \mathbf{j}(\mathbf{r}, t) \) determines the time evolution of \( \mathbf{B}(\mathbf{r}, t) \), only.

The key step for the determination of the total electric field distribution \( \mathbf{E} \) in superconductors during magnetization experiments is the integration of the Faraday law [Eq. (1)] or to obtain the inductive electric field \( \mathbf{E}_i(\mathbf{r}) \). There are two alternative approaches to calculate the \( \mathbf{E}_i(\mathbf{r}) \) which exactly give the same results. One approach is the introduction of a vector potential \( \mathbf{X}(\mathbf{r}) \) with

\[ \mathbf{E}_i = \nabla \times \mathbf{X}. \] (6)

This ansatz automatically fulfills \( \nabla \cdot \mathbf{E}_i(\mathbf{r}) = 0 \) and yields (see Appendix A)

\[ \mathbf{E}_i = \frac{1}{4 \pi} \int \frac{\mathbf{B} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'. \] (7)

A second approach (for \( \partial H_{ex} = 0 \)) is obtained by applying \( \nabla \times \) to Eq. (1) and using Ampère’s law [Eq. (4)]:

\[ \mu_0 \partial_t \mathbf{j} = - \nabla \times \nabla \times \mathbf{E}, \] (8)

where \( \partial_t \) stands for \( \partial / \partial t \). A similar equation is discussed in the article of Gurevich and Friesen (see also Ref. 11), where, however, \( \partial \mathbf{j} / \partial \mathbf{E} \) and thus the nonlinear differential conductivity \( \sigma(E) = \partial j / \partial E \) appears as a material law. In our equation (8), no material law is entering.

**III. \( \mathbf{E}_i \) FOR 2D CURRENT DISTRIBUTIONS**

In this section, the reconstruction of the inductive electric field in superconductors with a relaxing two-dimensional current distribution \( \mathbf{j}(x, y, t) = j_x(x, y, t)e_x + j_y(x, y, t)e_y \) is considered. This is the case in thin-film samples in a perpendicular external magnetic field. The \( z \)-dependence of the current density \( \mathbf{j}(x, y) \) can be disregarded for film thickness \( d \ll \lambda \), where \( \lambda \) represents the London penetration depth—i.e., the length scale of the spatial variation of the superconducting current density. This represents a completely two-dimensional problem for the current density. However, with respect to the magnetic flux density and the electric field distribution, the problem remains three dimensional.

**A. In-plane components of \( \mathbf{E}_i \)**

After two-dimensional Fourier transformation of Eq. (7) (see Appendix B), the Fourier coefficients of the \( \mathbf{E}_{i,x}(\mathbf{r}) \) component of the electric field are given by

\[ \mathbf{E}_{i,x}(k_x, k_y, z) = \mathbf{E}_{i,x}^{(1)} + \mathbf{E}_{i,x}^{(2)}, \] (9)

with

\[ \mathbf{E}_{i,x}^{(1)} = \int_{-\infty}^{\infty} dz' g(k_x, k_y, z, z') \tilde{B}_z(k_x, k_y, z') \] (10)

and

\[ \mathbf{E}_{i,x}^{(2)} = - \int_{-\infty}^{\infty} dz' l(k_x, k_y, z, z') \tilde{B}_z(k_x, k_y, z'). \] (11)

Similarly, the Fourier coefficients of \( \mathbf{E}_{i,y}(\mathbf{r}) \) are given by

\[ \mathbf{E}_{i,y}(k_x, k_y, z) = \mathbf{E}_{i,y}^{(1)} + \mathbf{E}_{i,y}^{(2)}, \] (12)

with

\[ \mathbf{E}_{i,y}^{(1)} = \int_{-\infty}^{\infty} dz' h(k_x, k_y, z, z') \tilde{B}_z(k_x, k_y, z') \] (13)

and
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FIG. 4. Calculated temporal change of the normal magnetic flux component $B_z(x)$ of a thin strip of thickness $d=200$ nm, width $W = 8$ μm in an external magnetic field of $\mu_0 H = 42$ mT. The strip is in a nearly fully penetrated state and carries a constant magnetization $H_z = 8000$ A/m. The key step for our thin-film method is to express the Fourier coefficients of the time derivative of the two-dimensional current density $\partial_k j_{kx}$ and $\partial_k j_{ky}$. This is possible by using a two-dimensional Fourier-transformed version of the Biot-Savart law (see Appendix C).

For the determination of the Fourier coefficients of the $E_{iz}$ component of the electric field, we then calculate the two contributions in Eq. (9) separately and obtain nonanalytical expressions (see Appendix D). However, for the $z=0$ plane, where the in-plane electric field has a global maximum, the following analytical expressions are found:

$$\bar{E}_{z}(x,k_z,z) = -\int_{-\infty}^{\infty} dz' g(k_x,k_y,z') B_z(k_x,k_y,z') \left( e^{-k d z} (dk + 1) - e^{k d z} \right)$$

(14)

For the Fourier-transformed integral kernels see Appendix B.

The Fourier coefficients of the perpendicular inductive component of the electric field $E_{iz}$ are given by (see Appendix B)

$$\bar{E}_{iz}(x,k_z,z) = \frac{\mu_0}{4k^2} \left( ik_y \partial_j j_{kx} + ik_x \partial_j j_{ky} \right) \zeta(z,k,d),$$

(20)

with

$$\zeta(z,k,d) = \int_{-\infty}^{\infty} dz'' e^{k d |z - z'|} \left( e^{-k d |z'|} - e^{-k d |z''|} \right).$$

(21)

Since $ik_y \partial_j j_{kx} + ik_x \partial_j j_{ky}$ is the Fourier-space equivalent for $\nabla \cdot j = 0$, Eq. (20) shows that an $E_{iz}$ component is present only if $\eta^m \neq 0$—i.e., if the change of the rate of flux creep in time is large enough. In this article, where our method is applied to slowly relaxing magnetized films, $\eta^m$ can be completely disregarded. Furthermore, the $\zeta$ function displays a sign reversal at the $z=0$ plane and one has $E_{iz}(z=0,k,d) = 0$ for arbitrary $\eta^m$ values.

B. Normal component of $E_i$

The Fourier coefficients of the perpendicular inductive component of the electric field $E_{iz}$ are given by

$$\bar{E}_{iz}(k_x,k_y,z) = \frac{\mu_0}{4k^2} \left( ik_y \partial_j j_{kx} + ik_x \partial_j j_{ky} \right) \xi(z,k,d),$$

(20)

and wave number $k=\sqrt{k_x^2+k_y^2}$.

We now apply the thin-film approximation $kd \ll 1$, where the lateral measurement resolution $k_{max}$ is comparable or worse than the film thickness $d$. Typically, $d \approx 300$ nm and $k_{max} = 1-3$ μm in our experiments. Using the first-order approximation $e^x \approx 1 + x$ we obtain

$$\bar{E}_{iz}(k_x,k_y,z=0) = -\frac{\mu_0 d^2}{8} \partial_{kx} j_{kx} + \frac{\mu_0 k_x}{2k^2} \sinh \left( \frac{k d}{2} \right) \bar{j}_{kx}.$$

(18)

Similarly the total inductive $E_y$ can be calculated, giving

$$\bar{E}_{iz}(k_x,k_y,z=0) = -\frac{\mu_0 d^2}{8} \partial_{kx} j_{kx} + \frac{\mu_0 k_x}{2k^2} \sinh \left( \frac{k d}{2} \right) \bar{j}_{kx}.$$

(19)

C. Alternative approach for the in-plane components of $E_i$

In this section, we show that the in-plane components of the inductive electric field can also be easily obtained if we start from Eq. (8) and apply a thin-film approximation, where the $z$ dependence of the two-dimensional current density $\mathbf{j}(x,y)$ is disregarded. Writing down Eq. (8) in components, using $B_z = -\partial_y E_{iy} + \partial_y E_{iz}$, and summarizing some terms yields

$$\partial^2_x E_{ix} = -\partial_y B_z + \mu_0 \partial_y j_x + \partial_x \partial_z E_{iz},$$

$$\partial^2_x E_{iy} = -\partial_y B_z + \mu_0 \partial_y j_x + \partial_x \partial_z E_{iz},$$

$$\partial^2_x E_{iz} = -\partial_x \partial_y E_{iz} + \partial_z \partial_y E_{iz} + \partial_x \partial_z E_{iz} = 0.$$
the following, the $E_{i,z}$ component in Eq. (22) will be disregarded. This allows a decoupling of the solution of Eq. (22) for the $E_x$ and $E_y$ components, respectively, and thus

$$\begin{align*}
\partial^2_t E_{i,x} &= -\partial_y B_z + \mu_0 \partial_z j_x, \\
\partial^2_t E_{i,y} &= \partial_x B_z + \mu_0 \partial_z j_y. 
\end{align*}$$

In the thin-film limit, the integration of Eq. (23) yields

$$E_{i,x}(x,y,z) = \mu_0 \partial_y j_z(x,y) \left( \frac{z^2}{2} - \frac{d^2}{8} \right) - \int_{-\infty}^{z} d^2 z' \partial_z B_z;$$

$$E_{i,y}(x,y,z) = \mu_0 \partial_x j_z(x,y) \left( \frac{z^2}{2} - \frac{d^2}{8} \right) + \int_{-\infty}^{z} d^2 z' \partial_z B_z. \quad (24)$$

For a discussion of the integration method see the next section. After two-dimensional Fourier transformation in the

FIG. 5. Different contributions to the total inductive electric field in the $(x,z)$ plane of a thin-film strip being extended infinitely in the $y$ direction. The strip is in a fully penetrated state and the time evolution of the flux and current distribution are shown in Fig. 4. (a) Total inductive electric field $E_{i,y}$ according to Eq. (28) for $|\delta j_z| = 10^{10}$ A/(m$^2$ sec). In (b) and (c) the two contributions to the inductive electric field distribution according to Eqs. (31) and (32) are depicted. Note that the size of the cross section of the strip $(W/d=40)$ can be seen in the grey-scale pattern of (b).
(x, y) plane and expressing the Fourier components of B_z via the Fourier components of the current density (see Appendix C) the Fourier components of the electric field are written

\[ E_{i,x}(k_x,k_y,z) = \mu_0 j_x(0,k_x,k_y,t) \left( \frac{2}{2 - \frac{d^2}{8}} \right) \]
\[ + \frac{\mu_0 k_y}{2k^2} \sinh \left( \frac{kd}{2} \right) e^{-d|z|}, \]
\[ E_{i,y}(k_x,k_y,z) = \mu_0 j_y(0,k_x,k_y,t) \left( \frac{2}{2 - \frac{d^2}{8}} \right) \]
\[ - \frac{\mu_0 k_x}{2k^2} \sinh \left( \frac{kd}{2} \right) e^{-d|z|}, \]

(25)

where \( \alpha \) is defined in Eq. (17). Note that j=0 for \( |z| > d/2 \). For the electric field distribution in the central plane z=0 of the superconductor we recover then Eqs. (18) and (19).

IV. SOLUTIONS AND TESTS FOR 1D CURRENT DISTRIBUTIONS

In order to improve our understanding of the electric field distributions, in this section, Eq. (24) is compared to a direct integration of Eq. (1) for superconducting samples with one-dimensional current distribution \( j_x(x) \) —i.e., a thin strip which is extended infinitely in the y direction. The strip has a width \( W \) (parallel to the x direction) and a thickness \( d \) [see Fig. 3(b)]. As before, we consider only magnetization experiments. The current loop is closing at infinity and no external potentials or charges are present. The only electric field component in this case is the \( E_{i,y} \) component; \( E_{i,x} \) and \( E_{i,z} \) in this artificial geometry are zero everywhere. Since \( \nabla \cdot E = \partial E_z / \partial t \), we recover the second line of Eqs. (23)

\[ \partial^2 E_{i,y} / \partial t^2 = \partial E_z / \partial t + \mu_0 \partial j_y. \]

(29)

The integration of this equation, assuming a \( z \)-independent current density in the film, yields

\[ E_{i,y}(x,z) = E_{i,y}^{(1)}(x,z) + E_{i,y}^{(2)}(x,z), \]

(30)

with a contribution being directly related to the current relaxation

\[ E_{i,y}^{(1)}(x,z) = \mu_0 \partial j_y(x) \left( \frac{2}{2 - \frac{d^2}{8}} \right) \]

(31)

Note that \( j_y=0 \) for \( |z| > d/2 \) and therefore \( E_{i,y}^{(1)} = 0 \) outside the sample. The second contribution is nonlocal and is related to an integral of the gradient of \( \partial E_z \)

\[ E_{i,y}^{(2)}(x,z) = \int_0^z d z' \partial E_z / \partial y. \]

(32)

Note that the sum of Eqs. (31) and (32) exactly corresponds to Eq. (28). This will be visualized in the following for simple relaxing model current distributions.

A. Numerical results for model current distributions

In this section, numerical results of Eqs. (28) and (30), respectively, are calculated by using theoretical Bean-model-like current distributions for the infinitely extended thin strip. We want to shed light on the relative strength of different contributions to the total electric field in Eq. (30). Results for a partly penetrated state are presented, where the shrinking of the Meissner area during magnetic relaxation and the contri-
bution of the time evolution of the Meissner currents have to be taken into account.

In the thin-film Bean model for the strip of thickness $d$ and width $W$, being extended infinitely in the y direction [see Fig. 3(b)], the current distribution is given by simple analytical expressions.\textsuperscript{25} If $H_{ex}$ is increased from the ZFC state, one obtains, for the current density,\textsuperscript{26}

$$j_z(x) = \begin{cases} \frac{J_c}{|x|}, & Q \leq |x| \leq W/2, \\ \frac{2J_c}{\pi} \arctan \left( \frac{xc}{(Q^2 - x^2)^{1/2}} \right), & |x| < Q, \end{cases}$$

(33)

with $c = 2[(W/2)^2 - Q^2]^{1/2}/W$ and $Q$ indicating the penetration depth of the flux front measured from the strip center. The penetration depth of the magnetic flux measured from the strip edge is $P = W/2 - Q$ with

$$Q = \frac{W}{2} \cosh \left( \frac{\pi H_{ex}}{J_c d} \right).$$

(34)

This one-dimensional current distribution generates a magnetic stray field in and outside the sample, which can be calculated by the Biot-Savart’s law:

$$B_z(x, z) = \frac{\mu_0}{2\pi} \int_{-W/2}^{W/2} dx' j_z(x')$$

$$\times F(x, x', z, z = d/2) - F(x, x', z, z = -d/2),$$

with

$$F(x, x', z, z') = \frac{1}{2} \ln[(x - x')^2 + (z - z')^2],$$

$$B_x(x, z) = \frac{\mu_0}{2\pi} \int_{-W/2}^{W/2} dx' j_y(x')$$

$$\times G(x, x', z, z = d/2) - G(x, x', z, z = -d/2),$$

with

$$G(x, x', z, z') = \arctan \left( \frac{z - z'}{x - x'} \right).$$

(35)

Note that in contrast to the analytical solution in Ref. 26 for $d \to 0$, the calculation of $B$ in Eq. (35) is performed for strips of finite thickness $d$, where no analytical solution is known. This is necessary for calculating electric fields inside the strip volume.

Examples of the flux and current distributions are depicted in Figs. 4 and 7 for a fully penetrated and a partly penetrated state. For the calculation of the electric field, it is assumed that the time derivative of the relaxing current density $\partial j_z(x) = [j_z(x, t_2) - j_z(x, t_1)]/\Delta t \approx j_z(x, t_1)$, where $j_z(x)$ is given by Eq. (33). The corresponding time derivatives of the magnetic flux density components are then calculated by Eq. (35). Finally, the total electric field is calculated by Eq. (28) and is compared with the two different contributions to Eq. (30)—i.e., Eqs. (31) and (32), respectively.

**B. Fully penetrated state**

The electric field distribution during relaxation of the fully penetrated state in Fig. 4 is depicted in Fig. 5(a). It shows a spatial pattern which is characterized by the following properties: (i) The electric field is maximal in the cross section of the superconducting strip: however, its spatial extend in the $z$ direction is much larger than the thickness of the sample and is of the order of $W$. (ii) The electric field is parallel to the current density. Consequently, due to the sign change of the current density, $E_{i,y}(x)$ displays a sign reversal at $x = 0$, too, and has an antisymmetric structure with respect to $x = 0$. (iii) Within one current domain, $|E_{i,y}(x)|$ has a maximum value at a distance of $x = \frac{2}{3}W/2$ from the sample center. This is approximately the position of the so-called neutral line—i.e., the location where $B_z = 0$ and $B_y = \mu_0 H_{ex}$.

Very similar results have been obtained before by Gurevich and Brandt\textsuperscript{12} and Brandt\textsuperscript{13} by a nonlocal diffusion equation for the electric field with a nonlinear current-electric-field material law $E(j) \propto j^m$. They showed that a universal shape of $E_{i,y}(x)$ occurs for strongly nonlinear $E(j)$ ($m \gg 1$) which only depends on the geometry of the sample. In both references, also analytical results for the $x$ dependence of the electric field are given in the limit $m \gg 1$ in the creep regime. The main new step of this work is that no materials law is applied for the determination of electric field patterns.

The different contributions $E_{i,y}^{(1)}$ and $E_{i,y}^{(2)}$ to the total inductive electric field according to Eqs. (31) and (32) are depicted in Figs. 5(b) and 5(c). As can be nicely seen from a comparison of Fig. 5(a) with Fig. 5(c), the shape of $E_{i,y}^{(2)}$ is similar to the spatial pattern of the total $E_{i,y}$. For the fully penetrated state, $E_{i,y}^{(1)}$ represents only a minor contribution to the inductive electric field.

Since the spatial extend of the magnetic self-field of the strip increases with the sample width $W$ in both the $x$ and $z$ directions, a strong dependence of the total electric field on $W$ is expected.\textsuperscript{12} The increase of $E_{i,y}$ with increasing sample width $W$ is shown in Fig. 6, where the electric field distribution in the cross section of three strips with different $W$ is depicted. Note that the thickness is constant ($d = 200$ nm) and all other parameters are unchanged compared to Fig. 4. Independently of $W$, $E_{i,y}(x, z)$ displays a universal pattern as predicted in Refs. 12 and 13. Whereas $d$ remains constant in Fig. 6, the extent of the $E_{i,y}$ pattern in the $z$ direction increases $\propto W$.

**Partly penetrated state**

The time evolution of the flux and current density and the corresponding electric field distribution due to current relaxation in a partly penetrated state of a strip strongly differs from that one of a fully penetrated state. Figure 7 shows the change of the flux and current density according to Eqs. (33) and (35). After relaxation of the $j_z$ value by 10%, the penetration depth $P$ of the flux front is increased. Consequently, the Meissner screening current density in the flux-free region.
is rearranged in a way to screen the new flux distribution from the Meissner area. This results in an enhancement of the local current density values in the Meissner area which is induced by the decrease of the current density in the flux-filled regions.

Figure 8 shows the corresponding inductive electric field distribution together with the contributions \( E_{i,x} \) and \( E_{i,y} \) according to Eqs. (31) and (32). In the flux filled current domains, a similar \( E_{i,y}(x) \) pattern is present as in the current domains of a fully penetrated state, with a maximum of \( |E_{i,y}(x)| \) which is slightly moved from the sample edge towards the interior of the sample. The electric field in the Meissner area exhibits a sign reversal compared to the adjacent flux-filled current domain with the same current direction. The sign reversal is due to the sign reversal in \( \partial j_x(x) \) and reflects the increase of the local current density in the Meissner area. Consequently, the electric field is parallel to the current direction in the flux-penetrated regions, whereas it is antiparallel in the area of Meissner currents.

In our own first paper\(^{17}\) on the determination of electric fields in superconducting films, it was assumed that the electric field in the film is generated only by the moving flux inside the superconductor. The contribution of the temporal change of the magnetic stray field outside the film to the electric field in the film plane was completely disregarded. This approximation considers therefore only the \( E_{i,x} \) contri-
FIG. 7. Calculated temporal change of the normal magnetic flux component \( B_z(x) \) of a thin strip in a partly penetrated state. The thickness of the strip is \( d=200 \, \text{nm} \), the width \( W=4 \, \mu\text{m} \) and \( \mu_0 H =10 \, \text{mT} \). The current density in the flux penetrated area is relaxed from \( j_x(t_1)=\pm 2 \times 10^{11} \, \text{A/m}^2 \) to \( j_x(t_2)=\pm 1.8 \times 10^{11} \, \text{A/m}^2 \). \( \Delta B_z =B_z(t_2)-B_z(t_1) \) denotes the corresponding change of the flux density. For a clearer representation, \( \Delta B_z \) was shifted by a value of \(-0.005\,\text{T}\).

The measured light intensities are calibrated into magnetic flux densities as described in Ref. 25.

Figure 9(a) depicts the \( B_z(x,y) \) distribution 1.2 sec after application of the external field. The superconducting film is in a partly penetrated state with a Meissner area (dark grey) in the central region. The corresponding current density distribution is shown in Fig. 9(b). One observes a typical Bean-like current density profile with approximately constant critical current density in the flux-filled areas and a Meissner screening current density \( j_{cs}\leq j_c \) in the flux-free area. Note that the small peaks of the current density at the sample edges are well-known experimental artifacts due to the coupling of in-plane stray fields of the superconductor to the iron garnet indicator.\(^{30,31}\) This slight error cancels in the time derivative of the current density \( \partial_t j \) which is used for the calculation of \( \mathbf{E}_i \) and therefore does not affect the \( \mathbf{E}_i(t) \) pattern.

The thermally activated flux creep can be nicely seen in Fig. 10(a), where the temporal change of the flux density \( \Delta B_z(x,y)=B_z(x,y,t=1.2\,\text{sec})-B_z(x,y,t=0.7\,\text{sec}) \) is depicted as a grey-scale image. From the time evolution of \( B_z(x,y) \), the time evolution of the two-dimensional current density \( j(x,y) \) is obtained by inversion of Biot-Savarts law. According to Eqs. (18) and (19), the in-plane electric field components can be obtained from the time dependence of \( j(x,y) \) in Fourier space. The time evolution of the current density determines the 3D time evolution of the flux distribution in full space outside and inside the sample.

The resulting inductive electric field distribution is visualized in Fig. 10(b). The image shows the absolute value \( E_i(x,y)=[|E_{x,i}(x,y)|+|E_{y,i}(x,y)|]^{1/2} \) of the electric field at \( t=1.2\,\text{sec} \) after external field sweep. Similarly to the flux domain structure, one observes four domains in the electric field pattern. In each domain, the electric field has a local maximum near the sample edge. The electric field at the current domain boundaries exhibits a local minimum. However, the values remain finite and well above the noise level of our measurement of \( E_{\text{noise}}=10^{-10} \, \text{V/m} \). The electric field pattern is similar to the theoretical patterns obtained by Brandt\(^{11}\) for flux creep in a square-shaped thin film.

Figure 11 depicts the distributions of the electric field components \( E_{x,i}(x,y) \) and \( E_{y,i}(x,y) \) after fast-Fourier transform back-transformation of the Fourier coefficients in Eqs. (18) and (19). The electric field distribution in Fig. 11 is characterized by the following features: (i) In the flux-filled domains, the electric field \( \mathbf{E}_i \) has the same sign as the current density. Due to opposite current directions, the electric field exhibits a sign reversal in opposite flux domains. (ii) Within an individual flux domain, the electric field is inhomogeneous. \( |E_i(x,y)| \) shows a maximum near the middle of each film edge and decays monotonously from the edges towards the flux front. (iii) As already observed for the calculated data [Figs. 7 and 8] for Meissner regions in thin strips, the electric field displays an additional sign reversal at each flux front, where, however, the direction of the current is unchanged. This sign reversal is related to a different relaxation behavior in the Meissner- and flux-penetrated phases, respectively. Whereas the current density decreases during magnetic relaxation in the flux-penetrated phase, the Meissner

V. \( \mathbf{E}_i \) DISTRIBUTIONS IN 2D FILMS FROM MAGNETO-OPTICAL IMAGING EXPERIMENTS

In this section, we present two-dimensional electric field distributions which are obtained from time resolved magneto-optical imaging experiments. The measurements were done at a square-shaped \( \text{YBa}_2\text{Cu}_3\text{O}_{6.95} \) film with 500 nm thickness and a width of 1.2 mm. After zero-field cooling to \( T=8\,\text{K} \), an external magnetic field of \( \mu_0 H_{ex} =86\,\text{mT} \) was applied with a ramp rate of 0.3 mT/sec. A series of images of the normal component of the magnetic flux density were taken with a time resolution of 120 msec. For a clearer representation, \( j_x \) was shifted by a value of \(-0.005\,\text{T} \).
screening current locally increases when the flux front moves towards the sample center.

VI. DETERMINATION OF $E_p$

It is an interesting observation that the determined vector $E_p(x,y)$ of the local inductive electric field distribution is not parallel to the local current density $j(x,y)$. We observe systematic deviations between the direction of both vectors. They can be seen, e.g., in Fig. 11, where nonvanishing inductive electric field components $E_{i,x}$ and $E_{i,y}$ are also present in the current domains which are dominated by the $j_x$ and $j_y$ components, respectively (see also Fig. 2 for a comparison with theory). As we will show in this section, the deviation between the inductive in-plane electric field vector $E_i(x,y)$ and the direction of the current density $j(x,y)$ can be...
utilized for the determination of the potential contribution \( E_p(x,y) \).

In general, such deviations could have different physical origins, such as a flux-creep Hall effect, where the vortex velocity field is not perpendicular to the current density. Further possible effects are vortex dragging by the superflow, vortex sliding at planar defects, or an anisotropy of the flux-creep resistivity due to the anisotropic orthorhombic crystal structure of the superconductor. However, all these effects can be clearly ruled out because of the following reasons: (i) The observed symmetry of the transverse components of \( E_i \) with two mirror planes \( \Gamma_x \) and \( \Gamma_y \) defined by \( x=0 \) and \( y=0 \), respectively, contradicts such effects. (ii) The observed temporal evolution pattern of \( B_z(x,y) \) in Fig. 10 clearly shows that the vortex velocity is perpendicular to the direction of \( j(x,y) \). A significant flux-creep Hall effect would be visible in the symmetry breaking of the \( B_z \) pattern with respect to the \( \Gamma_x \) and \( \Gamma_y \) mirror planes, respectively (Fig. 10). (iii) The orthorhombic crystal structure of the thin films is averaged on length scales below the spatial resolution of our method due to twinning. In addition to the symmetry argument, this
rules out any possible effect due to an in-plane anisotropy of the flux-creep resistivity.

Furthermore, a state with approximately constant rate of flux creep represents a quasistationary state. Any total $\mathbf{E}$ with a direction deviating from the current direction would give rise to a rearrangement of the current density pattern in that a way to fulfill $\mathbf{E} \parallel \mathbf{j}$. Consequently, the total electric field $\mathbf{E}(x,y)$ must be parallel to $\mathbf{j}(x,y)$ and the vector of the potential electric field must be perpendicular to $\mathbf{j}$, because any $\mathbf{E}_p$ component parallel to $\mathbf{j}$ would change the spatial evolution of the current relaxation and consequently would in fact contribute to $\mathbf{E}_r$. 

FIG. 10. (a) Time change of the flux density $\Delta B_\parallel(x,y)$ $= B_\parallel(x,y,t=1.2$ sec $)-B_\parallel(x,y,t=0.7$ sec $)$ due to thermally activated flux creep. (b) Grey-scale map of the absolute value of inductive electric field distribution in the $z=0$ plane at $t=1.2$ sec of the same state of the sample shown in Fig. 9. The profiles of the electric field are taken along the black lines.
With $\theta$ denoting the deviation angle between $\mathbf{E}_i$ and $\mathbf{j}$, the total electric field is $|\mathbf{E}| = |\mathbf{E}_i| \cos \theta$. The deviation angle can be determined via $\cos \theta = \mathbf{j} \cdot \mathbf{E}_i / (|\mathbf{j}| |\mathbf{E}_i|)$, giving, for the total electric field,

$$\mathbf{E} = \mathbf{j} |\mathbf{E}_i| \frac{\mathbf{j} \cdot \mathbf{E}_i}{|\mathbf{j}| |\mathbf{E}_i|},$$

with $j=|\mathbf{j}|$. For the in-plane components of the potential electric field $\mathbf{E}_p$, follows

$$E_{p,x} = j_x (\mathbf{E}_i \cdot \mathbf{j}) / j^2 - E_{i,x},$$

$$E_{p,y} = j_y (\mathbf{E}_i \cdot \mathbf{j}) / j^2 - E_{i,y}.$$  \hspace{1cm} (36)

The resulting potential electric field distributions are depicted in Fig. 12 as grey-scale images. The induced charge density $\mathbf{n}^{in} = \varepsilon_0 \nabla \cdot \mathbf{E}_p = -\varepsilon_0 \nabla \cdot \mathbf{P}$ is depicted in Fig. 13.

The results fully agree with the qualitative theoretical expectations in Fig. 2 which can be directly derived from Faraday’s induction law. The potential contribution to the total electric field is generated by a charge density distribution at the current domain boundaries and the sample edges. This charge density is built up by an inhomogeneous sheer creep...
of vortex columns. The obtained values of the induced charge density of up to $1.4 \times 10^{-14}$ C/m$^3$ nicely agree with the order of magnitude estimate which can be obtained from the equation $|n| = \varepsilon_0 |\delta B_{ex}|$ for external field sweep in the fully penetrated state in Ref. 11. If we replace the time derivative of the external magnetic field $\partial_t B_{ex}$ by the local quantity $\delta B_z = (0.1-1) \times 10^{-2}$ T/sec (see Fig. 10), we obtain a theoretical order-of-magnitude estimate of $n = (0.8-8) \times 10^{-14}$ C/m$^3$.

The accuracy of the determination of $E_p$ depends strongly on the accuracy of the determined $j$ and $\delta j$. Taking all calibration errors of the magneto-optical data into account, the error in $j$ is smaller than 10% with a comparable error in $\delta j$ (note that some of the errors cancel in the differential images used for the calculation of $\delta j$). Since $E_p$ depends not only on the absolute values of $E$ and $j$ but also on their directions, the error in $P$ may be larger (up to 30%). There are three minor features in Figs. 12 and 13 which are experimental artifacts due to defects in the mirror of the magneto-optical active sensor film:

- Some small spots in the image are experimental artifacts due to defects in the mirror of the magneto-optical sensor film visible also in Fig. 9(a).
- A slight asymmetry...
in the \( P_y \) pattern in the top \( j_y \) domain [Fig. 12(b)] which is related to a small asymmetry in the current stream lines in this area [see Fig. 9(b)], and (iii) whereas the sign change of \( n_{y}^{m} \) at the horizontal sample edges are correct, it is shifted at the vertical sample edges from \( y=0 \) towards the upper part of the sample. These small artifacts can be clearly identified with small calibration errors in the \( B_z(x,y) \) data due to the small Faraday effect in optical components, and they could be corrected artificially by a linear tilt of the data. The experimental results presented in the article are based on orginal and uncorrected data.

VII. SUMMARY AND CONCLUSIONS

Summarizing the results of this paper, we present a method which allows for the determination of the inductive and potential electric field distributions induced by thermally activated magnetic relaxation in thin-film superconductors of arbitrary shape. The inductive part of \( \mathbf{E}_i \) is obtained from the measured time evolution of the normal component of the flux density \( B_i(x,y) \).

The observed systematic deviations of the in-plane directions of \( \mathbf{E}_i \) from the directions of \( \mathbf{j} \) are used for the determination of the \( \mathbf{E}_p \) distribution and the induced surface charge density at the sample edges and the current domain boundaries.

We present experimental results for the thermally activated relaxation of a square-shaped YBaCuO thin film after ramping up the external magnetic field. The electric field pattern is very similar to the theoretical expectations of Brandt and Gurevich.\(^{11,12} \) The determination of the in-plane dielectric polarization field and the induced charge density shows that the sample edges and the current domain boundaries are electrically charged. This charge density seems to play an important role for the shape of the current streamlines at the current domain boundaries; in particular, its presence is crucial for the strong bending of the supercurrents. The method presented in this article represents a general basis for the analysis of the inductive and potential contributions to the total electric field in samples with complex microstructures such as large defects, current inhomogeneties, and grain boundaries.

As a final remark, it is very important to notice that the determination of electric field distributions of superconductors allows to obtain fundamental insights into other dynamic properties of moving vortex ensembles. Having determined the total electric field distribution \( \mathbf{E} = \mathbf{E}_i + \mathbf{E}_p \), it is very easy to obtain the dissipated power density \( p = \mathbf{j} \cdot \mathbf{E} \), the local flux velocity \( \mathbf{v} \) according to \( \mathbf{E} = \mathbf{B} \times \mathbf{v} \), and the local activation barrier \( U = -k_B T \ln(v/v_0) \), where \( k_B \) is the Boltzmann constant, \( T \) is the temperature, and \( v_0 \) depends only on the material constants of the superconductor. This will allow the space-resolved determination of activation barriers for flux creep which is up to now based on specific models for the \( U(j) \) and \( U(B) \) relations\(^{32} \) or restricted to one-dimensional sample geometries.\(^{33} \)

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APPENDIX A

After expressing the inductive electric field by a vector potential \( \mathbf{X}(r) \) with

\[
\mathbf{E}_i = \nabla \times \mathbf{X},
\]

one obtains

FIG. 13. Grey-scale representation of the induced surface charge density \( n_{y}^{m} = -\varepsilon_0 \nabla \cdot \mathbf{P} \) for the same conditions as in Fig. 10.
\[ \nabla \times \mathbf{E}_i = \nabla \cdot (\nabla \cdot \mathbf{X}) - \Delta \mathbf{X} = -\dot{\mathbf{B}}. \]  
(A2)
Without loss of generality, the gauge \( \nabla \cdot \mathbf{X} = 0 \) can be applied and Eq. (A2) can be integrated by means of the Green function of the Laplace operator, giving
\[ \mathbf{X} = \frac{1}{4\pi} \int d^3r' \frac{\dot{\mathbf{B}}(r')}{|r - r'|}. \]  
(A3)
For the inductive electric field, we find
\[ \mathbf{E}_i = \frac{1}{4\pi} \nabla \times \int \left[ \frac{\dot{\mathbf{B}}}{|r - r'|} \right] d^3r', \]  
(A4)
yielding directly Eq. (7).

**APPENDIX B**

For the \( E_{i,x}, E_{i,y}, \) and \( E_{i,z} \) components of the inductive electric field, we obtain, from Eq. (7),
\[ E_{i,x}(r) = \int d^3r' G(r - r') \dot{B}_x(r') - \int d^3r' L(r - r') \dot{B}_y(r'), \]  
(B1)
\[ E_{i,y}(r) = \int d^3r' H(r - r') \dot{B}_y(r') - \int d^3r' G(r - r') \dot{B}_z(r'), \]  
(B2)
and
\[ E_{i,z}(r) = \int d^3r' L(r - r') \dot{B}_z(r') - \int d^3r' H(r - r') \dot{B}_y(r'), \]  
(B3)
with the integral kernels
\[ G = \frac{1}{4\pi} \frac{z - z'}{|r - r'|^3}, \]  
(B4)
\[ H = \frac{1}{4\pi} \frac{x - x'}{|r - r'|^3}, \]  
(B5)
and
\[ L = \frac{1}{4\pi} \frac{y - y'}{|r - r'|^3}. \]  
(B6)
Applying two-dimensional Fourier transforms in the \((x, y)\) plane, one obtains, for the Fourier-transformed kernels, \[ g = \frac{1}{2} \text{sgn}(z - z') e^{-k|z - z'|}, \]  
(B7)
\[ h = \frac{i k}{2} e^{-k|z - z'|}, \]  
(B8)
and
\[ l = \frac{i k}{2} e^{-k|z - z'|}, \]  
(B9)
with wave number \( k = \sqrt{k_x^2 + k_y^2} \), and

The Fourier coefficients of \( E_{i,x} \) can be expressed as
\[ \tilde{E}_{i,x}(k_x, k_y, z) = \tilde{E}_{i,x}^{(1)}(k_x, k_y, z) + \tilde{E}_{i,x}^{(2)}(k_x, k_y, z) \]
\[ = \int d\bar{z}' g(k_x, k_y, z, z') \tilde{B}_y(k_x, k_y, z') - \int d\bar{z}' \tilde{l}(k_x, k_y, z, z') \tilde{B}_z(k_x, k_y, z'). \]  
(B10)

Similarly, the Fourier coefficients of \( E_{i,y} \) are given by
\[ \tilde{E}_{i,y}(k_x, k_y, z) = \tilde{E}_{i,y}^{(1)}(k_x, k_y, z) + \tilde{E}_{i,y}^{(2)}(k_x, k_y, z) \]
\[ = \int d\bar{z}' g(k_x, k_y, z, z') \tilde{B}_z(k_x, k_y, z') - \int d\bar{z}' \tilde{h}(k_x, k_y, z, z') \tilde{B}_y(k_x, k_y, z'). \]  
(B11)
and those of \( E_{i,z} \) are written
\[ \tilde{E}_{i,z}(k_x, k_y, z) = \tilde{E}_{i,z}^{(1)}(k_x, k_y, z) + \tilde{E}_{i,z}^{(2)}(k_x, k_y, z) \]
\[ = \int d\bar{z}' h(k_x, k_y, z, z') \tilde{B}_y(k_x, k_y, z') - \int d\bar{z}' \tilde{g}(k_x, k_y, z, z') \tilde{B}_z(k_x, k_y, z'). \]  
(B12)

**APPENDIX C**

Once the time evolution of the two-dimensional current density in the film is known, the time evolution of \( \mathbf{B}(r, t) \) in the entire space inside and outside the superconducting film can be determined by the Biot-Savart law:
\[ \mathbf{B}(r, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(r', t) \times (r - r')}{|r - r'|} d^3r'. \]  
(C1)

After two-dimensional Fourier transforms, we can again use the Fourier-transformed integral kernels \( g, h, \) and \( l \). We obtain, for the Fourier coefficients of the \( B_z \) component,
\[ \tilde{B}_z(k_x, k_y, z) = \frac{i \mu_0}{2} \int_{-d/2}^{d/2} d\bar{z}' e^{-k|z - z'|} \alpha, \]  
(C2)
with
\[ \alpha = \frac{k_x}{k} \tilde{j}_x(k_x, k_y, z) - k_y \tilde{j}_y(k_x, k_y, z). \]  
(C3)
Assuming that the current density does not vary in the \( z \) direction within the film thickness \( |z| \leq d/2 \), this yields
\[ \tilde{B}_z(k_x, k_y, z) = \frac{i \mu_0}{2} \alpha \begin{cases} 2 e^{-k|z| \sinh (kd/2)}, & |z| \geq d/2, \\ 1/2 (2 - e^{-k(d/2)}) - e^{k(-d/2)}, & |z| < d/2. \end{cases} \]  
(C4)
Similarly, the \( B_x \) and \( B_y \) components are given after two-dimensional Fourier transforms by
\[ \tilde{B}_x(k_x, k_y, z) = \mu_0 \int_{-d/2}^{d/2} d\bar{z}' \tilde{j}_x \]  
(C5)
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\[ \overline{B}_\gamma(k_x k_y z) = -\mu_0 \int_{-d/2}^{d/2} dz' \overline{j}_\gamma g, \] (C6)

yielding

\[ \overline{B}_\gamma(k_x k_y z) = -\frac{\mu_0 j_\gamma}{2k} \left[ e^{-k|z-d/2|} - e^{-k|z+d/2|} \right] \] (C7)

and

\[ \overline{B}_\gamma(k_x k_y z) = -\frac{\mu_0 j_\gamma}{2k} \left[ e^{-k|z-d/2|} - e^{-k|z+d/2|} \right]. \] (C8)

**APPENDIX D**

We calculate the two contributions to the Fourier coefficients of \( E_{i,\alpha} \) in Eq. (9). For the first contribution one obtains

\[ \overline{E}_{i,\alpha}^{(1)}(k_x k_y z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz' g(k_x k_y z', z') \overline{B}_\gamma(k_x k_y z') \times \left\{ e^{-k|z'-d/2|} - e^{-k|z'-d/2|} \right\}. \] (D1)

At the \( z=0 \) plane, the integral gives

\[ \frac{1}{k} \left( e^{-kd/2} (dk + 1) - e^{kd/2} \right), \] (D2)

and in the approximation \( kd \approx 1 \) the first-order approximation \( e^{kd/2} \approx 1 + kd/2 \) gives for the integral

\[ -\frac{1}{2} kd^2. \] (D3)

The second contribution in Eq. (9) is given by

\[ \overline{E}_{i,\alpha}^{(2)}(k_x k_y z, t) = \int_{-\infty}^{\infty} dz' \left( k_x k_y z', z' \right) \overline{B}_\gamma(k_x k_y z') \times \left\{ e^{-k|z'-d/2|} - e^{-k|z'-d/2|} \right\}. \] (D4)

with

\[ f_1 = \int_{-\infty}^{\infty} dz' 2 e^{-k|z'-d/2|} \sinh \left( \frac{kd}{2} \right), \]

\[ f_2 = \int_{d/2}^{\infty} dz' 2 e^{-k|z'-d/2|} \sinh \left( \frac{kd}{2} \right), \]

\[ f_3 = \int_{-d/2}^{d/2} dz' e^{-k|z'-d/2|} (2 - e^{-k|z'-d/2|} - e^{k|z'-d/2|}). \] (D5)

At the \( z=0 \) plane, the functions \( f_i \) can be evaluated analytically:

\[ f_1 = f_2 = \frac{1}{k} \sinh \left( \frac{kd}{2} \right) e^{-kd}, \]

\[ f_3 = -\frac{1}{k} \left\{ 5 e^{kd} + dke^{kd} - 1 - 4 e^{3kd/2} \right\} e^{-3kd/2}. \] (D6)

For \( kd \approx 1 \) and using the first-order approximation \( e^{kd/2} \approx 1 + kd/2 \) one obtains

\[ \overline{E}_{i,\alpha}^{(2)}(k_x k_y z = 0) = \frac{\mu_0 k_x}{2k^2} \sinh \left( \frac{kd}{2} \right) \alpha. \] (D8)
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