Non-isothermal adiabatic critical state stability of a hard superconductor

Vladimir R. Romanovskii

Abstract

A basic feature of a hard superconductor is that dissipative processes occur in them because of a thermally activated motion of vortices. The resulting dissipation of the energy stored may be accompanied by a disruption of the critical state. This instability leads to an increase in the temperature of the superconductor. Most of the carried out investigations have been based on a study of the initial stage of the magnetic flux penetration and ignore permissible heating of the superconductor. The main purpose of this work is to analyze the conditions for the onset of the thermomagnetic instability in a hard superconductor with allowance for a non-zero initial temperature heating. Conditions for adiabatic stability of the screening currents induced by an external magnetic field are formulated. It is shown that the adiabatic criteria for stability of the critical state depend strongly on the initial temperature of the superconductor. © 1998 Published by Elsevier Science B.V. All rights reserved.

Keywords: Thermomagnetic instability; Screening current; Permissible heating; Numerical analysis

1. Model

The superconducting state of hard superconductors is unstable. A local disturbance of the critical state causes the magnetic flux to change with the subsequent energy dissipation and correspondent increase in temperature. Under certain conditions, which are generally dependent on the energy stored by the screening currents, a change in the critical state may allow the superconductor to pass to the normal state. Previous investigations of the causes of a premature transition of hard superconductors to the normal state yielded a number of principles [1–5]. These principles allow one to examine the conditions of the origin of the so-called thermomagnetic instability (TMI), to set criteria for the superconducting state stability under the action of thermal disturbances. For example, the TMI theory states that the temperature rise and magnetic flux diffusion in a thermally insulated...
superconducting slab would not exhibit avalanche behavior, provided that the following condition holds:

\[ \beta = \frac{\mu_0 J_c^2 a^2}{C_c(T_{CB} - T_0)} < 3. \]  \hspace{1cm} (1)

Here, \( C_c \) is the bulk thermal capacity of the superconductor, \( a \) is the plate half-thickness, \( T_0 \) is the temperature of the coolant, \( T_{CB} \) is the critical temperature of the superconductor in a given magnetic field, and \( J_c \) is the critical current density at the temperature \( T \).

This criterion has been derived for the isothermal state of a hard superconductor because the allowable temperature variations were a priori assumed to be small compared to the coolant temperature. However, a variation in the current and/or magnetic field does in fact affect the temperature field of the superconductor. Thus, the stability conditions should also be changed.

Let us consider a thermally insulated superconducting slab, which is large in the \( y \)- and \( z \)-directions and of width \( 2a \) in the \( x \)-direction. When an external magnetic field is applied parallel to the broad face of the plate, it will induce screening currents in the superconductor. Assume that the field penetrates across the whole thickness of the slab and the critical current density depending only on the temperature. If a fluctuation has affected the initial state, then a fraction of the magnetic energy of the screening currents will be released over the whole bulk \( V \) of the superconductor. These, in turn, will lead to a temperature increase from the initial value \( T_1 \) up to \( T_2 \).

Let us estimate the amount of heat released by the change in the magnetic energy of the screening currents \( \Delta B(x, T) = \mu_0 J_c(T) x, 0 < x < a \):

\[ Q = \frac{1}{V} \int \frac{\Delta B^2}{2\mu_0} dV = \frac{1}{V} \int \frac{\Delta B^2}{2\mu_0} dV. \]

When writing this expression for the superconducting slab, the change in its enthalpy can be easily found:

\[ \int_{T_1}^{T_2} C_c dT = \frac{\mu_0}{2a} \int_0^a \left[ J_c^2(T_1) - J_c^2(T_2) \right] x^2 dx. \]  \hspace{1cm} (2)

At arbitrary temperature dependencies of \( C_c \) and \( J_c \) this expression reduces to a transcendental equation with respect to the final temperature \( T_2 \) of the superconductor. In order to rewrite the stability criteria in explicit form, we make some conventional simplifying assumptions. Spatial non-uniformity of the temperature across the slab is assumed to be small (below, in simulation of the problem more generally stated, we will not make this assumption). Let the specific heat be independent of temperature and the critical current be a function of temperature:

\[ J_c(T) = J_{c0} \frac{T_{CB} - T}{T_{CB} - T_0}. \]

Let us consider possible cases.

Case 1: complete release of the stored energy (\( T_2 > T_{CB} \)).

In this case the superconductor is in the normal state, and the expression (2) yields the final temperature the superconductor will be heated up due to the origin of the TMI:

\[ T_2 = T_1 + \beta \frac{(T_{CB} - T_1)^2}{6(T_{CB} - T_0)}. \]  \hspace{1cm} (3)

If the initial temperature of the superconductor equals the coolant temperature (the assumption on which condition (1) is based), then the final superconductor temperature reaches value \( T_2 = T_0 + \beta/6(T_{CB} - T_0) \).
Eq. (3) shows that neglecting the possible change in the initial temperature of the superconductor may cause a considerable distortion of the stable-state range bounds. The dependence $T_s(T_i)$, holding even for adiabatic conditions, allows the initial temperature to increase by a finite amount up to the critical temperature. From Eq. (3), a non-isothermal criterion for the transition to the normal state can be easily derived. Provoked by a fluctuation, the final superconductor temperature will exceed the critical value, provided that, in the general case

$$\beta > 6 \frac{T_{CB} - T_0}{T_{CB} - T_1}, \tag{4}$$

and at $T_1 = T_0$

$$\beta > 6. \tag{5}$$

The non-isothermal conditions for the TMI development reveals the significant part that the initial temperature of the superconductor plays: the initial temperature rise causes the stable-state range to extend. Condition (5), which refers to the equal temperature of the superconductor and coolant, is thus stronger than condition (4), from the viewpoint of the restrictions imposed on the parameter $\beta$. At the same time, a comparison between Eq. (1) and Eq. (5) shows that criterion Eq. (5), and consequently Eq. (4), leads to more favorable conditions for the existence of superconducting states than follows from the previously formulated criterion Eq. (1). This fact admits a simple physical treatment: in the development of instabilities, an incomplete release of stored energy, which does not allow the temperature to exceed the critical value, may occur. Therefore a characteristic initial temperature exists

$$T' = T_{CB} - \frac{6}{\beta}(T_{CB} - T_0)$$

above which the energy stored by the screening currents is insufficient to completely destroy the superconducting state. Until now, the existence of such states has not been taken into account. Let us consider them.

Case 2: a partial release of the stored energy ($T_z < T_{CB}$).

In this case for all $\beta < 6(T_{CB} - T_0)/(T_{CB} - T_1)$ or $T_1 > T'$ the final temperature of a superconductor, prompted by the TMI development, is defined by the expression

$$T_z = \begin{cases} 
2T_{CB} - T_1 - \frac{6}{\beta}(T_{CB} - T_0), & T_1 < T'' \\
T_1, & T_1 > T'' 
\end{cases}$$

where

$$T'' = T_{CB} - \frac{3}{\beta}(T_{CB} - T_0).$$

i.e., beginning from a certain characteristic value of a superconductor initial temperature $T''$, the final temperature ceases to rise. This means that the TMI does not exist under such conditions. It is easy to see that such states satisfy the inequality

$$\beta < 3 \frac{T_{CB} - T_0}{T_{CB} - T_1}, \tag{6}$$

which passes to Eq. (1) at $T_1 = T_0$.

As in the above case, condition (6) for the absence of the TMI also depends on the initial temperature, and the assumption of the equal temperatures $T_0$ and $T_1$ leads to reduced allowable values of the parameter $\beta$. 

2. Numerical experiment

To illustrate the above discussed dependence of the stability conditions on the thermal prehistory of the sample, Figs. 1–3 represent the results of numerical modeling of possible superconductor states. The magnetic field was introduced \( \frac{d B}{d t} = 0.01 \ T/s \) into a thermally insulated slab, for a number of its thickness. At any time instant \( t \), the spatial distribution of the temperature \( T(x,t) \), electric field \( E(x,t) \) and magnetic field \( B(x,t) \) within the cross-section of slab were obtained by a numerical solution of the set of equations

\[
C_s \frac{dT}{dt} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + EJ, \quad \frac{\partial B}{\partial x} = \mu_0 J, \quad \frac{\partial E}{\partial x} = \frac{\partial B}{\partial t}, \quad J = J_c + \frac{1}{\rho_s} E, \quad J_c(T) = J_{c,T} \frac{T_{cb} - T}{T_{cb} - T_0}
\]

with the initial-boundary conditions

\[
T(x,0) = T_1, \quad E(x,0) = 0, \quad \frac{\partial T}{\partial x} (0,t) = 0, \quad \frac{\partial T}{\partial x} (a,t) = 0, \quad E(x_p,t) = 0, \quad \frac{\partial E}{\partial x} (a,t) = \frac{d B}{d t} = \text{const}.
\]

Here, \( \rho_s \) is the resistivity of the superconductor, \( x_p \) is the depth of the magnetic field penetration into the slab and is defined by the equation

\[
\mu_0 \int_{x_p}^a J(x) \, dx = \frac{d B}{d t} t.
\]

In specifying the initial distribution of the magnetic field within the superconductor, two typical cases were examined. First, the TMI development was investigated on the assumption that the slab cross-section was completely filled by the field at the initial moment \( (x_p = 0) \) at \( t = 0 \), Figs. 1 and 2. Second, the superconductor temperature change was also determined when introducing the magnetic field from the zero initial value \( (x_p = a) \) at \( t = 0 \), Fig. 3) up to the value when the penetration edge reaches the slab midline, with no further variations of

![Graph](image-url)

Fig. 1. The superconductor temperature changing in time under the slow rate of the external magnetic field in the case when the initial temperature of the superconductor is equal to the coolant temperature: \( \beta > 6 \): 1—\( a = 8 \cdot 10^{-5} \) m, 2—\( a = 6 \cdot 10^{-5} \) m, 3—\( a = 4 \cdot 10^{-5} \) m. 3—\( \beta < 6 \): 4—\( a = 3 \cdot 10^{-5} \) m, 5—\( a = 2.8 \cdot 10^{-5} \) m, 6—\( a = 2.7 \cdot 10^{-5} \) m. \( \beta < 3 \): 7—\( a = 2.6 \cdot 10^{-5} \) m.
Fig. 2. (a) Increase in the superconductor temperature as a result of the TMI development at different initial temperature at \( a = 2.7 \times 10^{-3} \) m \((T^* = 4.284 \text{ K})\): 1—\( T_s = 8.9 \) K, 2—\( T_s = 8.8 \) K and at \( a = 3.7 \times 10^{-3} \) m \((T^* = 3.98 \text{ K})\): 3—\( T_s = 4.422 \) K, 4—\( T_s = 5.2 \) K, 5—\( T_s = 7.2 \) K. (b) Increase in the superconductor temperature as a result of the TMI development at different initial temperature at \( a = 1 \times 10^{-4} \) m \((T^* = 8.313 \text{ K})\): 1—\( T_s = 8.99 \) K, 2—\( T_s = 8.7 \) K, 3—\( T_s = 8.2 \) K, 4—\( T_s = 4.4 \) K.

In carrying out the calculations, the initial parameters were taken as follows:

\[ C_s = 10^3 \frac{J}{m^3 K}, \quad \lambda = 0.01 \frac{W}{mK}, \quad \rho_s = 5 \times 10^{-7} \Omega m, \quad J_{c0} = 4 \times 10^9 \frac{A}{m^2}, \quad T_{CB} = 9 \text{ K}, \quad T_0 = 4.2 \text{ K}. \]

The curves \( T_s = T(0, t) \), presented in Figs. 1–3, clarify the features of the TMI development. Not only do they illustrate the dependence on the initial temperature, they also emphasize the essential role that the magnetic field on the surface. The corresponding temperature dependences that represent the development of the TMI after the field introduction is terminated (\( dB/dt = 0 \)) are shown in Fig. 3 by dashed lines.
Fig. 3. Increase in the superconductor temperature under a gradual introduction of the magnetic field to the slab midplane when at the initial moment the slab was cooled down to the coolant temperature: 1—a = 6 \times 10^{-5} \text{ m}, 2—a = 5 \times 10^{-5} \text{ m}, 3—a = 4 \times 10^{-5} \text{ m}, a = 2 \times 10^{-5} \text{ m}.

self-consistent change in the temperature and magnetic field plays in contributing to the critical state stability. It is evident that the unstable states in Fig. 1 (with arbitrary initial temperature and magnetic field distributions) become stable, as it follows from Eqs. (4) and (6) (Fig. 3), providing that the TMI development takes into account the joint behavior of the temperature and magnetic field.

3. Adiabatic stability criterions for some practical cases

When the specific heat of the superconductor depends on temperature the condition under which there is no instability has the form

$$\frac{\mu_0 a^2 J_{C_0}^2}{C(T_1)(T_{CB} - T_0)} < 3 \frac{T_{CB} - T_0}{T_{CB} - T_1}$$

and magnetic field drop in the slab $B_m$ must satisfy

$$B_m < \sqrt{3 \mu_0 C(T_1)(T_{CB} - T_1)}.$$

In the case

$$\int_{T_1}^{T_{CM}} C_1 dT < \frac{\mu_0 a^2 J_{C_0}^2}{6} \left(\frac{T_{CB} - T_1}{T_{CB} - T_0}\right)^2$$

the critical state is fully unstable. So the upper bound on the magnetic field drop in the slab is equal

$$B_m > \left(6 \mu_0 \int_{T_1}^{T_{CM}} C_1 dT\right)^{1/2}.$$

This spontaneous heating of the superconductor is accompanied by a destruction of the superconducting state.
When the external field is applied parallel to the axis of a long cylinder, one may write the following stability criteria. The ultimate catastrophe occurs if

\[
\int_{T_1}^{T_{CB}} C_i \, dT < \frac{\mu_0 a^2 J_0^2}{4} \left( \frac{T_{CB} - T_1}{T_{CB} - T_0} \right)^2
\]

and the smallest disturbance will cause the temperature to rise up to the superconductor critical temperature. A flux jump in cylindrical superconductor will be absent if

\[
\frac{\mu_0 a^2 J_0^2}{C_i(T_1)(T_{CB} - T_0)} < 2 \frac{T_{CB} - T_0}{T_{CB} - T_1}.
\]

When we have an infinite slab superconductor and a normal metal as a laminated composite conductor (with a filling factor \(\eta\), specific heat of the normal metal \(C_n\)), the unstable state in an external magnetic field applied in the direction parallel to the surface will happen if

\[
\int_{T_1}^{T_{CB}} \left[ \eta C_i + (1 - \eta) C_m \right] \, dT < \frac{\mu_0 a^2 J_0^2 \eta^3}{6} \left( \frac{T_{CB} - T_1}{T_{CB} - T_0} \right)^2
\]

and the temperature will rise to its value \(T > T_{CB}\). The stability criterion for screening currents in this case has the form

\[
\frac{\mu_0 a^2 J_0^2 \eta^3}{[\eta C_i(T_1) + (1 - \eta) C_m(T_1)](T_{CB} - T_0)} < 3 \frac{T_{CB} - T_0}{T_{CB} - T_1}.
\]

It is necessary to note essential details in stability criterions for a superconducting composite. In these formulas of an apart from additional temperature term, a filling factor \(\eta\) has the third degree. At the same time in the existing theory formulated by means of a non-isotropic medium model with averaged physical properties, this parameter has the second degree. For instance, the similar averaged models do not allow adequately describing processes happening in composite mediums.

4. Conclusion

The analysis of stability conditions performed with the collective nature of thermal and electrodynamic processes shows that characteristic temperatures exist which influence on mechanism of critical state destruction. Therefore the existing criterion for adiabatic stability leads to a restriction of the range of allowable parameters at which a hard superconductor retains its superconducting properties under the action of the thermomagnetic instability. The expansion of this range is due to the possibility of the partial release of the energy stored by the screening currents, with the correspondent heating of the superconductor to a temperature below the critical temperature value. A strong dependence of the stability conditions on the thermal prehistory of the superconductor may also extend the region, allowing the overheating to be finite and provide a stabilizing effect. Hence, a correct definition of the thermomagnetic instability origin conditions should be based on an analysis involving interrelated processes of magnetic flux diffusion into the superconductor and its correspondent change in temperature.

Acknowledgements

The work was supported by the Russian Foundation for Basic Research (Project No 98-02-16046).
References