How taxes on firms reduce the risk of after-tax cash flows

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Most firms use one discount rate applied to expected net after-tax cash flows. The need to adjust for differences in risk, other than leverage, is commonly neglected. There can be substantial effects of taxation on after-tax risk when there are depreciation deductions. Among the few studies of these effects, even fewer identify all effects correctly. When marginal investment is taxed together with inframarginal, marginal CAPM beta differs from average. The problems identified here imply that currently suggested tax reforms may fail. Tax neutrality results rely on firms correctly discounting for risk, in particular the risk of tax deductions.

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1. Introduction

According to Summers (1987, p. 296), "prospective depreciation allowances are very nearly riskless," and should thus be discounted at the risk free rate. Summers observes that most firms follow a different practice. Two obvious questions are, are the claims true (under various circumstances), and what are the consequences? This paper determines the risk of depreciation and other investment-proportional allowances endogenously in a theoretical model adopted from financial economics. A firm faces exogenous output price uncertainty and maximizes after-tax value for its shareholders. The optimum determines the risk of depreciation allowances, the risk of net after-tax cash flows, and thus the after-tax cost of capital. One consequence is that when firms restrict attention to net cash flows, they should use different discount rates under different tax systems. Correct discount rates depend both on tax rates and allowance schedules. Another consequence is that tax reforms currently under debate are likely not to have the intended effects if firms behave differently from what the theory prescribes. The topic is in the intersection between financial and public economics, and the relevance for both fields will be outlined.

In financial economics it is well known that capital budgeting in most firms applies constant discount rates to net cash flows and does not properly take into account variations in the composition of these flows.[1] The alternative, considering differently risky cash flows separately, is known as the adjusted present value (APV) method (Myers, 1974).[2] The present paper is consistent with this method, using the Capital Asset Pricing Model (CAPM) and option valuation to find values of various elements of cash flows. With the APV method available, one could think that there is no reason to be interested in the risk of net cash flows anymore. There are nevertheless at least two reasons for this to be of interest, also within financial economics.

First, knowledge of the quantitative effects on net after-tax cash flows is necessary to

[1] See, e.g., surveys by Graham and Harvey (2001) and Bennouna et al. (2010). For a modern textbook treatment, see Brealey et al. (2011), pp. 526ff.

[2] See also Lessard (1979) and Brealey et al. (2011), section 19.4.

demonstrate the magnitude of mistakes that are made. If large, this may help motivate firms to introduce different methods.[3] Second, observed returns in equity markets reflect net cash flows and are used in practice to derive required expected returns for firms. This is known as deriving an asset beta from an equity beta. Since Hamada (1972) the need to "unlever" the observed returns (to adjust for differences in leverage) has been well known.[4] Lund (2002) shows that there is also a need to "untax" them, i.e., correct for differences in taxation.

In public economics the tax neutrality results of Fane (1987) and Bond and Devereux (1995) rely on the same idea as in Summers (1987), that a tax system may allow deductions that are very nearly riskless. According to the theory, firms will be indifferent to postponing these deductions if they accumulate interest at the risk free interest rate, provided that the deductions will be effective later with full certainty. This theory is consistent with the APV method, and the neutrality results depend critically on firms being able to assess risk of different cash flow elements.

Interest accumulation at a risk-free interest rate is policy relevant. It is an important element in several suggested tax reforms, such as the ACE system, see OECD (2007) and Auerbach et al. (2010). It has been noted by Zodrow (2006, p. 283) that "... negative cash flows should in principle be carried forward at the nominal risk-free interest rate. The determinations of this rate would inevitably be controversial." There may be several reasons for controversies. Firms may doubt that deductions are actually risk free, but they may also use capital budgeting procedures that deviate from theoretical prescriptions. Bjerkedal and Johnsen (2005, p. 170) report that a petroleum tax reform based on these principles could not be understood by oil companies using a uniform discount rate. While the intention was to move towards a neutral tax system, the companies claimed that the reformed system

[3]Bennouna et al. (2010, p. 231) state that "Varying the WACC for risk is not an easy task. Large firms in Canada and the United States are cognizant of this difficulty and many are working on developing separate divisional costs of capital". (The WACC is the weighted-average cost of capital, cf. the end of section 4 below.) But there is no mention that divisions operating under different tax systems should adjust the WACC accordingly, which is an implication of the present study.

[4]See Brealey et al. (2011), pp. 455f, pp. 512f.

gave disincentives for investment. According to the theory, they used too high discount rates and did not realize that the reform made their net cash flows less risky.

Summers (1987) observes that the deviating procedures in firms may distort investment decisions. The present study quantifies the deviations in discount rates. The resulting potential distortions are not of the type traditionally analyzed in public economics, summarized in the before-tax cost of capital. Instead, the "non-traditional" distortion occurs when firms apply a uniform after-tax cost of capital as discount rate for expected net cash flows, regardless of the risk characteristics of these flows.

While the present study concentrates on tax effects on the after-tax cost of capital, effects on the before-tax cost of capital also come out of the model as a by-product. These results are closely related to earlier results, and are reported in section 7 for comparison with those.

The present study introduces an investment opportunity with decreasing returns to scale (DRS) into the model of Lund (2002), while maintaining the other assumptions of that model. In Lund (2002) only the marginal investment was characterized, as a project with constant returns to scale (CRS). Instead, the after-tax cost of capital here comes out of a first-order condition for optimal investment, with DRS ensuring that the second-order condition is satisfied. Moreover, introducing DRS opens for four new insights.

First, there is a new concept, a distinction between average and marginal beta, i.e., a difference between the covariance measure of risk for the average and the marginal return. The difference appears as the result, solely, of the (typical) tax system, in a model in which there is no such difference in the absence of taxation. While economists in general are aware of the difference between average and marginal rates of return, this phenomenon is quite different, since it only concerns the covariance risk (disappearing if the covariance is zero), and is only due to the tax shields.

Second, this distinction gives a unified framework for understanding the existing literature on the topic, tax effects on the risk of after-tax cash flows, even in the absence of leverage. Eight earlier contributions are discussed in section 2 below. Without the marginal-average distinction, there has been some confusion.

Third, the marginal-average difference adds another complication to the problem of deriving an asset beta from an equity beta. It is shown that the observed returns should not only be "unlevered" (Hamada, 1972) and "untaxed" (Lund, 2002), but also "unaveraged." If they neglect this, firms will use the wrong betas. The quantitative importance is illustrated below.

Fourth, the DRS production function allows for an endogenization of the risk of tax deductions under imperfect loss offset. The risk of the "very nearly riskless" tax deductions (Summers, 1987) is characterized in an analytical model. They are not completely risk free. The risk of a tax deduction, and thus the risk of the marginal project, depends on inframarginal profits. More specifically, as an upper bound for the risk, under the extreme assumption that there is no loss offset at all, an analytical solution is found using option valuation. How the risk of tax deductions depends on the parameters of the model is then determined. This concerns both the total risk and the systematic (covariance) risk. Previous studies (theoretical, empirical, simulation) have no similar solutions for the risk.

While it is well known that taxes interact with leverage in determining the after-tax cost of capital of a levered firm,[5] the results here do not depend on leverage at all. Thus an all-equity firm is considered, and debt financing is only briefly discussed at the end of section 4.

The paper is organized as follows: Section 2 reviews the previous literature. Section 3 presents the model of valuation and DRS production. Section 4 introduces the tax system and gives results for the case with risk free tax shields. Section 5 considers risky tax shields. Section 6 extends the model to allow for entry costs, possibly outweighing the rent in DRS production. Section 7 has results on the required expected return before taxes. Section 8 concludes. Some proofs and additional details are in the appendices, sections 9.1–9.3.

[5] See Modigliani and Miller (1963) and any textbook in corporate finance.

2. The existing literature

A well-known textbook in finance, Brealey et al. (2011), ignores the possibility to say something systematically about how the cost of equity depends on depreciation and similar investment-related tax shields. It states (p. 528) that "Depreciation tax shields contribute to project cash flow, but they are not valued separately; they are just folded into project cash flows along with dozens, or hundreds, of other specific inflows and outflows. The project's opportunity cost of capital reflects the average risk of the resulting aggregate." This practice[6] obscures the tax-induced difference between marginal and average betas. Moreover, it is unfortunate if the firm operates under different tax systems, which implies different after-tax costs of capital.

Eight previous theoretical studies that discuss the effect of taxes on the risk of after-tax rates of return, allowing for tax effects even in the absence of debt, are Levy and Arditti (1973), Galai (1988), Jacoby and Laughton (1992), Derrig (1994), Bradley (1998), Galai (1998), Lund (2002), and Rao and Stevens (2006). More details on these are given in section 4 below, since the unified framework must first be presented. But a few remarks are in order here to set the stage.

Jacoby and Laughton (1992) and Bradley (1998) use the finance-theoretic approach originally developed for option valuation to study valuation of natural resource projects under taxation. They use Monte-Carlo simulations for specific resource extraction projects, extending the APV method. To find project values after tax there is no need for required expected returns for after-tax net cash flows. But these are found after the net expected cash flows and their market values have been calculated, technically like internal rates of return. The present paper gives an analytical model based on similar assumptions. Analytical models and simulation models supplement each other. Restricting attention to a stylized two-period model is not realistic, but reveals some basic relations, such as the distinction

[6]Brealey et al. (2011) just describe a practice, and do not endorse it. In section 19.4 they present the APV method, and on pp. 526ff they find values of depreciation tax shields using risk free interest rates. There is no discussion of how this affects the after-tax beta of equity.

between marginal and average systematic risk.

Galai (1998) and Lund (2002) are the papers most closely related to the present one. Galai (1998) has a theoretical two-period model with results on the systematic risk of the cash flows to the three claimants, equity, debt, and tax authorities, and on possible conflicts of interest between these. The equity beta is found to be declining in the tax rate, but the relation between the marginal and average beta is not made explicit.

More recently, like the present paper, Rao and Stevens (2006) set up a two-period model of a firm subject to taxation, with investment in the first and a risky outcome in the second period. Their model is more general by considering risky debt, and in some other respects. The pricing model is an approximate Arbitrage Pricing Theory (APT), which is robust with respect to different distributional assumptions. They improve upon the literature by a simultaneous solution to the cost of debt, the optimal level of debt, and the risks of the tax shields. But they have a different focus for their analysis from that of the present paper.[7] They give no results for effects on their endogenous variables of changes in tax rates or other tax parameters. Their model starts with some exogenous project profitability before tax, and does not identify the marginal project. It does not determine the risk of the tax shields endogenously based on the taxation of the marginal project together with an inframarginal project. The model becomes quite complicated and cannot be solved analytically.

Section 5 below supplements previous studies of effects of asymmetric taxation, with imperfect loss offset. In theoretical studies, Auerbach (1986) shows how loss carryforward provisions affect the incentives to invest under risk neutrality, while Green and Talmor (1986) consider a similar problem with assumptions close to those of the present paper. Both find that asymmetric tax treatment of profits and losses have ambiguous effects on investment incentives. They were followed by empirical studies, estimating firms' expected or effective marginal tax rates with various methods. Auerbach and Poterba (1987) show

[7] E.g., they state that "Our interest in this paper is not on the tax shield's risk *per se*" (Rao and Stevens (2006), pp. 19f). But, "The sensitivity of interest rate and tax policy changes on firms' economic balance sheets, and hence investor's wealth can, in principle, be evaluated in our model" (p. 25).

how loss carryforwards affect tax rates for large U.S. firms under the simplifying assumption that all firms face the same transition probabilities between profits and loss positions. Shevlin (1990) instead finds firm-specific estimates using a simulation method. Graham (1996) investigates the predictive power of alternative methods, suggesting proxies that may be used due to data limitations. If data allow use of the simulation method, this comes out best. Although related to the present paper, these studies focus on marginal tax rates, and none of them consider effects on the systematic risk. Auerbach and Poterba (1987) have a footnote 14 in which they acknowledge that they leave systematic risk out of their analysis.

3. The model

A firm invests in period 0 and produces in period 1, only. The only source of uncertainty for the firm is output price uncertainty in period 1. The firm considers a DRS investment project. The optimal scale of investment is endogenous, determined by the tax system and other parameters in each case below. The first assumption of the model[8] is:

Assumption 1: *The firm is fully equity financed and maximizes its market value according to the Capital Asset Pricing Model,*

$$E(r_j) = r + \beta_j[E(r_m) - r], \qquad (1)$$

where $r > 0$ *is a riskless interest rate, and* $\beta_j = \operatorname{cov}(r_j, r_m)/\operatorname{var}(r_m)$.

Here, r_j and r_m are the rates of return on the firm's shares and the market portfolio, respectively. When changes in tax rates or deductions are considered below, these are assumed not to affect the capital market equilibrium. This will be a good approximation

[8] All variables are nominal. The model is consistent with deterministic inflation, whereas stochastic inflation would require a more complicated model, especially if taxes are not inflation adjusted. The results in the paper do not rely specifically on the CAPM. They can be generalized to any valuation model with a linear risk measure. A related model in public economics is section 2.4.4 in Devereux (2004), using a covariance with a stochastic discount factor as risk measure.

for a tax that only applies to a sector that is small in relation to the capital market. With international capital market integration, it will be a reasonable assumption even for nationwide tax systems, except for tax changes in larger nations. What should be considered large sectors or nations in this connection, is an empirical question that will not be answered here.[9] We can think of the tax applying in a host nation, while shares are traded at home.

The home economy may have a tax system, fixed throughout the analysis, and possibly[10] reflected in r. A consequence of the CAPM is that the claim to any uncertain cash flow X, to be received in period 1, has a period-0 value of

$$V(X) = [E(X) - \lambda \operatorname{cov}(X, r_m)]/(1 + r), \tag{2}$$

where $\lambda = [E(r_m) - r]/\operatorname{var}(r_m)$. This defines a valuation function V to be applied below.

Results that follow rely on a constant pre-tax "asset beta." Starting with Hamada (1972), numerous authors have analyzed the relation between, on one hand, equity betas, and, on the other, the beta of a firm's assets seen as identical to the beta of an unlevered firm.[11] But in the present paper one must distinguish between the pre-tax beta of assets, for an unlevered firm not subject to taxation, and the after-tax asset beta, which is the beta of an unlevered, taxed firm. With the model's simple cash flows before tax, the pre-tax asset beta is simply the beta of a claim to one unit of output, with price P. This beta is defined[12] in relation to the return $P/V(P)$,

$$\beta_P = \frac{\operatorname{cov}(\frac{P}{V(P)}, r_m)}{\operatorname{var}(r_m)}. \tag{3}$$

[9]Even for tax changes in the U.S. one could assume that the international capital market is unaffected. Bulow and Summers (1984) suggest that this may be a reasonable assumption (their footnote 3).

[10]This r could differ from the riskless interest rate by a tax adjustment factor, see Miller (1977), Sick (1990), Benninga and Sarig (2003). Since the model has no debt financing, the variable r has only one interpretation, the intercept in the CAPM equation, which is the firm's after-tax discount rate for riskless cash flows. This is robust, consistent with tax adjustment or no tax adjustment.

[11]For a modern textbook treatment, see, e.g., Brealey et al. (2011), pp. 455f, pp. 512f.

[12]In this paper, all CAPM betas are defined with a covariance between two returns in the numerator. These are unit free. When a cash flow has several terms, the (return) beta of the net cash flow is the value-weighted average of the (return) betas of the terms.

Assumption 2: *In period 0 the firm invests an amount $I > 0$ in a project. In period 1 the project produces a quantity Q to be sold at an uncertain price P. The joint probability distribution of (P, r_m) is exogenous to the firm, and $\text{cov}(P, r_m) > 0$. Produced quantity is $Q = f(I) = \kappa I^\nu$, with $\kappa > 0$, $\nu \in (0, 1)$. There is no production flexibility; Q is fixed after the project is initiated. There is no salvage value and no operating cost in period 1.*

The exogenous distribution follows from an assumption of competition. The assumption $\text{cov}(P, r_m) > 0$ is only a convenience to simplify verbal discussions. All equations hold also for the case of a negative covariance, and most of them also for zero covariance, although not expressions for the ratio of two betas. After proposition 2 below there are some comments on the case of a negative covariance.

The simple multiplicative structure of the before-tax cash flow in period 1, PQ, with only P being stochastic, is not innocuous. If cash flows instead consist of revenues from multiple products, perhaps minus costs of period 1 inputs, the relative composition of these might change under distortive taxation. Then the covariance might change, but such effects are neglected here. Moreover, in reality a produced quantity Q is often stochastic. For Case F in section 4, the crucial assumption is not that Q is deterministic, but that it is independent of the vector (P, r_m). If independent, a stochastic Q would imply that Q in the model should be replaced by $E(Q)$, and the results in section 4 would still hold.[13]

4. Case F: Risk free tax shields

This section will arrive at two expressions for the after-tax asset beta under the assumption that the firm is certain to be in tax position in the next period, Case F (F for risk Free). The two betas will be referred to as the marginal and average beta. Section 5 will arrive at two other expressions for the after-tax asset beta, under the assumption that next period's

[13]Under independence, one has $\text{cov}(PQ, r_m) = E(Q)\,\text{cov}(P, r_m)$. The assumption that Q is independent of (P, r_m) is closely related to the "no flexibility" part of assumption 2. Clearly, managerial flexibility in period 1 could mean that Q would depend on P (in particular if there were also period 1 inputs, or if P could be negative).

tax position is uncertain, depending on the output price. These two betas will also be referred to as the marginal and average beta, for that case. In addition there will be a reference in Section 5 to the after-tax asset beta for a marginal project taxed alone, under uncertainty about the tax position, as derived in Lund (2002). There is also a generalized version of the model in section 6.

Assumption 3: *A tax at rate* $t \in [0, 1)$ *will be paid with certainty in the production period. The tax base is operating revenue less* cI. *There is also a tax relief of* taI *in period 0. The constants* a *and* $c/(1+r)$ *are in the interval* $[0, 1]$*; moreover,* $t[a + c/(1+r)] < 1$.

In this two-period model, the parameter c corresponds to a depreciation allowance, while a corresponds to investment tax credits. The requirement $t[a + c/(1+r)] < 1$ precludes "gold plating incentives," i.e., the tax system carrying more, in present value terms, than 100% of an investment cost. Assumption 3 implies that a negative tax base gives a negative tax paid out by the authorities. While unrealistic for most tax systems when the project stands alone, this is often a good approximation when a project is added to other activity that is more profitable and only weakly correlated with it. An alternative assumption for the second period is considered in section 5.

By a *marginal project* is meant a small project with zero value after taxes, if any. By a *marginal investment* is meant the investment which starts the marginal project in period zero. A marginal project may be taxed on its own, or it may appear as the "last" part of a DRS investment project, the margin determined by the first-order condition. Consider first the DRS project as a whole. The cash flow to equity in period 1 is

$$X_{FA} = Pf(I)(1 - t) + tcI. \tag{4}$$

The market value of a claim to this is

$$V(X_{FA}) = V(P)f(I)(1 - t) + tcI/(1 + r). \tag{5}$$

The firm chooses the optimal scale to maximize

$$\pi_F(I) = V(X_{FA}) - I(1 - ta). \tag{6}$$

The first-order condition for a maximum[14] is

$$V(P)f'(I) = \frac{1 - ta - tc/(1+r)}{1 - t}, \tag{7}$$

which can be rewritten, based on the analytical production function, as

$$V(P)f(I)(1 - t) = (I/\nu)\left(1 - ta - tc/(1+r)\right). \tag{8}$$

The after-tax asset beta is a value-weighted average of the (return) betas of the two elements of the cash flow, (4). Of these, the tax deduction has a beta of zero, and we are left with

$$\beta_{FA} = \frac{V(P)f(I)(1-t)}{V(P)f(I)(1-t) + Itc/(1+r)}\beta_P. \tag{9}$$

To understand the deviation of β_{FA} from the pre-tax β_P, consider the relative weights of the two terms in the denominator of (9). This depends on $I/f(I)$ and exogenous variables, of which ν is of particular interest. According to (8), the optimal value of $I/f(I)$ is proportional to ν. Consider as a thought experiment what happens when ν is reduced from unity (which is CRS). The relative weight of the last term in (5), and in the denominator in (9), is reduced, and β_{FA} approaches β_P. With large infra-marginal profits, average beta is close to the pre-tax beta.

The first-order condition and the parameterized production function together give

$$\beta_{FA} = \frac{1 - ta - tc/(1+r)}{1 - ta - t(1-\nu)c/(1+r)}\beta_P. \tag{10}$$

[14]Cf. Galai (1998), equation (17).

(The subscript FA refers to case F, average beta.) The result is summarized as follows:

Proposition 1: *Under assumptions 1–3, the after-tax asset beta is given by (10). When $tc > 0$, it is strictly decreasing in the tax rate t, in the investment tax credit rate a, in the present value of the deduction rate c, and in the scale elasticity ν.*

The proof is in appendix 9.1. The next proposition will show that the *average beta* found in (10) is not very helpful for finding the cost of capital for making decisions on optimal scale. Instead, its limit when $\nu \to 1^-$ is appropriate. This *marginal beta* is[15]

$$\beta_{FM} = \lim_{\nu \to 1^-} \beta_{FA} = \frac{1 - ta - tc/(1+r)}{1 - ta} \beta_P, \tag{11}$$

Under CRS, one cannot find the optimal scale from a first-order condition. Instead, the condition that a pair of (I, Q) for a CRS project is marginal, is that

$$V(P)Q(1-t) = I\left(1 - ta - tc/(1+r)\right). \tag{12}$$

Observe that $\beta_{FM} < \beta_{FA}$ when $tc(1-\nu) > 0$.[16]

The average and marginal betas are both relevant as descriptions of systematic risk for the same project. The average beta will describe the systematic risk of the DRS project as a whole, and in particular, the systematic risk of the shares in an equity-financed firm with only this project. Equation (10) should be used to find the pre-tax asset beta, β_P, from observations of the equity beta.

The marginal beta is the relevant one for decision making, which may be decentralized within the firm. The reason is that at the margin, the ratio Q/I is given by (12), not by (8). The result on the appropriate beta for decision-making is:

Proposition 2: *Under assumptions 1–3, the firm's optimal investment is found by maximizing its expected present value with a constant risk-adjusted discount rate based on*

[15] This is a special case of proposition 2 in Lund (2002), who also includes a multiperiod extension.

[16] Results on the ratio β_{FM}/β_{FA} could easily be derived, but are left out to save space.

β_{FM} from (11). The same can be found by maximizing the expected present value with a non-constant risk-adjusted discount rate based on the beta from (9), with beta being a function of investment, I.

The proof is in appendix 9.1. The average beta, β_{FA}, will be endogenously determined as part of the optimization. This restricts its usefulness from a managerial point of view.

The marginal beta is decreasing in the tax rate under any tax system with postponed deductions for investment outlays. Under a pure cash flow tax ($a = 1, c = 0$) there is no such effect of the tax rate. On the other hand, when $c/(1 + r)$ is close to unity and a is small, marginal beta is close to $(1 - t)\beta_P$.

Consider the typical tax systems with $c > 0$. If firms continue to discount net after-tax cash flows with the same discount rates under all tax systems, they will tend to undervalue projects with high tax rates relative to those with low tax rates. They will tend to undervalue projects taxed with ACE or similar deductions relative to those taxed with traditional depreciation allowances, and to undervalue all of these relative to projects taxed under pure cash flow taxes, $c = 0$. The need to use lower discount rates is particularly strong for firms that are subject to such high tax rates that are applied in many resource rent sectors, when those rates are combined with depreciation and sometimes uplift deductions proportional to investment. These qualitative conclusions hold throughout the paper. However, one should consider the other cases as they may be more realistic.

If $\beta_P < 0$, partial derivatives of β_{FA} and β_{FM} with respect to $t, a, c/(1 + r)$, and ν will be positive instead of negative. Changes in those four exogenous variables lead to increased betas, algebraically, i.e., betas closer to zero. As a summary, the results of proposition 1 hold for the change in the absolute values of the betas. Both $|\beta_{FA}|$ and $|\beta_{FM}|$ will be reduced when t, a, or $c/(1 + r)$ is increased, and $|\beta_{FA}|$ will be reduced when ν is increased.

Proposition 2 has important implications for all studies that consider the effect of taxation on required after-tax returns to equity. This must be based on a characterization of how the tax system determines what is a marginal investment. Levy and Arditti (1973), Galai (1998), and Lund (2002) all identify this tax effect.

Galai shows how the (average) beta of equity depends on taxation, with an expression (his equation (21S')) containing the ratio of the pre-tax project value to the shareholders' project value. But he does not specify this ratio further. In the present paper this ratio is $V(P)Q \cdot (1-t)/V(X)$, here determined as function of exogenous parameters for various cases in equations (12) and (8).

Galai (1998, p. 156) states that "the cut-off rate for the investment to be taken by shareholders is not determined" by the beta found in his equation (21S'), here called the average beta. But he does not define or give an expression for the marginal beta which determines that cut-off rate.

Some authors (Derrig, 1994, Rao and Stevens, 2006) study tax effects for any exogenously given level of profitability. Rao and Stevens (2006, p. 11) even emphasize that their "analysis accommodates both positive and negative NPV firms." They claim (e.g., middle of p. 2) that the analysis leads to the cost of capital, but they do not solve for the marginal investment. To find the cost of capital would require another round of iterations.

Some (Jacoby and Laughton, 1992, Bradley, 1998) have studied the same for specific numerical examples, with various realistic profitability levels.[17] This works well when the aim is to find tax effects on the systematic risk of a given project. But if one wants the effect on the required expected return, one must consider a project that is exactly marginal after tax. Again an extra iteration round is needed.

The term "cost of capital" should not be used for a rate of return of a DRS project that yields supranormal profits (rents). However, the expected return from equations (4) and (5), $E(X_{FA})/V(X_{FA})$, is an equilibrium expected return, and does not in itself exhibit any supranormal profit. The denominator reflects that profit. This expected return will reflect the systematic risk quantified by β_{FA}. It is (one plus) the correct risk-adjusted

[17] Jacoby and Laughton (1992, pp. 44f) and Bradley (1998, pp. 69f) do not claim to identify the required expected return. The discount rates they find are based on average betas and are appropriate for finding values of given projects. They do not explicitly recommend them for decision making.

discount rate to be used for finding the market value of X_{FA}, but only for some specific ratio $tcI/E(P)f(I)(1-t)$. In the model this is optimally chosen and depends on ν.

The tax adjustment in the cost of equity that has been derived in this section can be combined with a cost of debt to find a weighted average cost of capital, WACC. Lund (2002) derives a WACC formula in which the marginal beta of equity determines the cost of equity. The weighted average of this and the cost of debt is found in the usual way. This is based on the simplest standard assumptions of an exogenous, fixed debt/equity ratio[18] and risk free debt.[19] Using the same WACC as discount rate, Lund (2012) shows that under DRS, the maximization of the expected present value of the total cash flow to equity and debt leads to maximization of after-tax value to shareholders.

5. Case R: Risky tax shields

In this paper, tax shields are risky if there is a chance that the firm will not have sufficient taxable income to deduct all the shields. The political risk of a change in the tax system is not a topic here. To get analytical results the present section assumes that there is no debt and no loss offset at all. This means that the two-period model is taken literally and the tax code does not allow carry-backs. One purpose of the present paper is to see how much the results of Case F are modified when tax shields are risky. Thus it is relevant to consider this most extreme riskiness.[20] It turns out that even then, the beta of equity is substantially lower than the asset beta before taxes, given reasonable parameter values. The cash flow to equity in period 1 is

$$PQ - t\max(0, PQ - cI). \tag{13}$$

A marginal beta may now take different meanings. A realistic marginal beta recognizes

[18] A more satisfactory approach would allow the debt/equity ratio to depend on taxation.

[19] Interaction between risky debt and risky deductions are complicated to model, and might not allow analytical solutions. See various approaches in Galai (1988), Pitts (1997), and Rao and Stevens (2006).

[20] Shevlin (1990) describes intermediate cases.

that the marginal project is typically part of a larger activity, and that the probability of being in tax position depends on the outcome of that larger activity. This will be analyzed in line with the model of the previous sections: The larger activity consists of a DRS investment project, the output of which is being sold at a single stochastic price in the single future period.

Let Case R (for Risky deductions) denote the case with an uncertain tax position. The following assumption replaces assumption 3 above:

Assumption 4: *The tax base in period 1 is operating revenue less* cI. *When this is positive, there is a tax paid at a rate* t. *When it is negative, the tax system gives no loss offset at all. There is also a tax relief of* taI *in period 0. The constants* a *and* $c/(1+r)$ *are in the interval* $[0,1]$; *moreover,* $t[a + c/(1+r)] < 1$.

The tax cash flow is similar to a cash flow from a European call option. The valuation will use similar assumptions to value tax claims as those pioneered by Ball and Bowers (1983), Green and Talmor (1985), and Majd and Myers (1985).[21] McDonald and Siegel (1984) show how to value this option, modifying the Black-Scholes-Merton formula, when the underlying asset has a rate-of-return shortfall. The valuation of the nonlinear cash flow is specified as follows:

Assumption 5: *A claim to a period-1 cash flow* $\max(0, P - K)$, *where* K *is any positive constant, has a period-0 market value according to the model in McDonald and Siegel (1984).*

The value can be written as

$$V(P)N(z_1) - KN(z_2)/(1+r), \tag{14}$$

[21] More or less the same assumptions are used in Jacoby and Laughton (1992) and Bradley (1998), cited above. It should be noted that option-like valuation is not limited to the geometric Brownian motion that is used here and in most of the literature. Bradley (1998) considers the average beta also under an alternative process.

where

$$z_1 = \frac{\ln(V(P)) - \ln(K/(1+r))}{\sigma} + \sigma/2, \quad z_2 = z_1 - \sigma, \tag{15}$$

N is the standard normal distribution function, and σ is the instantaneous standard deviation of the price.[22] To apply an absence-of-arbitrage argument for option valuation when there is a rate-of-return shortfall, forward or futures contracts for the output must be traded, or there must exist traded assets that allow the replication of such contracts. The validity of an option valuation formula in an economy with taxation is discussed, e.g., in McDonald (2006, appendix 10.A). He concludes that "When dealers are the effective price-setters in a market, taxes should not affect prices."

The combination of the CAPM and the option pricing model relies on, e.g., the assumptions in Galai and Masulis (1976),[23] and is now standard in the real options literature.[24] The CAPM will now be a single-beta version of the intertemporal CAPM of Merton (1973). Capital markets operate in continuous time, whereas investment, production and taxes happen at discrete points in time.

In what follows it is assumed that the exogenous variables β_P and σ can be seen as unrelated as long as $\sigma > 0$, cf. footnote 14 in McDonald and Siegel (1986). A change in σ could be interpreted as, e.g., additive or multiplicative noise in P, stochastically independent of the previous (P, r_m).[25]

[22] As shown in any textbook in finance, $N(z_2)$ is a risk-adjusted probability for the option to be exercised, and $N(z_1)$ multiplies this with a conditional risk-adjusted expectation. See, e.g., McDonald (2006). The riskless discount factor is still written as $1/(1+r)$ to maintain consistency with the previous sections.

[23] An alternative would be to rely on an approximate Arbitrage Pricing Theory. Rao and Stevens (2006) rely on this for a related analysis, assuming ad hoc that the approximate valuation equation holds exactly. For the purpose of the present paper, to use the option pricing formula applied here, one must in addition assume that the output price follows a geometric Brownian motion. Leland (1999) points out weaknesses in combining option pricing models with the CAPM. The differences between standard betas and the Bs suggested by Leland are small in relation to the effects pointed out in the present paper.

[24] See, e.g., McDonald and Siegel (1986, p. 716) and Dixit and Pindyck (1994, p. 115). Taxation of real options is analyzed in, e.g., MacKie-Mason (1990), McKenzie (1994), and Niemann (1999).

[25] Of course, the method used here does not mean that σ could be zero while β_P is different from zero. Davis (2002) argues that covariances are likely to change when volatilities of commodity prices change, but does not give any arguments for his assumption that correlations are unchanged. An empirically based discussion for oil is found in Lund and Nymoen (2013), casting doubt on both the extreme assumptions, that either correlations or covariances are invariant to changes in volatility.

The after-tax asset beta is now the beta of the replicating portfolio at $t = 0$ for a claim to the after-tax cash flow, a derivative asset that includes a short position in tQ call options on P. Propositions 3–5 are shown in appendix 9.2:

Proposition 3: *Under assumptions 1, 2, 4, 5, the after-tax asset beta is given by (16).*

$$\beta_{RA} = \frac{1 - ta - tN(z_{2D})c/(1+r)}{1 - ta - tN(z_{2D})(1-\nu)c/(1+r)}\beta_P, \tag{16}$$

where z_{2D} is given by

$$z_{2D} = \frac{1}{\sigma}\ln\left(\frac{1 - ta - tN(z_{2D})c/(1+r)}{\nu[1 - tN(z_{2D}+\sigma)]c/(1+r)}\right) - \frac{\sigma}{2}. \tag{17}$$

This equation determines z_{2D} implicitly as function of $t, a, c/(1+r), \sigma$, and ν. Such an analytical solution on implicit form for the endogenous z_{2D} seems to be new in option valuation.[26] Similar problems have been solved before (Auerbach, 1986, Green and Talmor, 1986), but with more general functional forms and no solution like (17). One result that follows here is that the rate-of-return shortfall (or convenience yield) does not affect the ratio β_{RA}/β_P.

Proposition 4: *Under assumptions 1, 2, 4, 5, the beta for a marginal project taxed together with the optimally chosen DRS project is given by (18).*

$$\beta_{RM} = \frac{1 - ta - tN(z_{2D})c/(1+r)}{1 - ta}\beta_P. \tag{18}$$

The two equations (18) and (16) should be compared with (11) and (10). Clearly the effect of the uncertainty in the tax position is similar to a reduced tax rate in period 1, reflecting that the probability of earning the tax shields is less than 100%. The relation between

[26] Lu (2012) extends the result to a multiperiod setting. The beta is increasing in the number of periods, but "even for a ten-period project the after-tax betas are still significantly lower than the pre-tax asset beta" (p. 34).

marginal and average beta is similar to that of the previous case, which had full certainty about the tax position. In both there is an extra term containing $tc(1-\nu)$ subtracted in the denominator of the average beta. But also the marginal beta in (18) is affected by the parameter ν through the factor $N(z_{2D})$, which is determined in equation (17).

For comparison, the marginal beta in the stand-alone CRS case can be found. This is denoted RC because it only applies if the project actually has constant returns to scale. The probability of being in tax position is lower in this case.

Proposition 5: *Under assumptions 1, 2, 4, 5, the beta for a marginal project taxed alone is given by (19).*

$$\beta_{RC} = \frac{1 - ta - tN(z_{2C})c/(1+r)}{1 - ta}, \tag{19}$$

where z_{2C} is given by

$$z_{2C} = \frac{1}{\sigma} \ln\left(\frac{1 - ta - tN(z_{2C})c/(1+r)}{[1 - tN(z_{2C}+\sigma)]c/(1+r)}\right) - \frac{\sigma}{2}, \tag{20}$$

which is the limit of (17) as $\nu \to 1$. This is the case considered in Lund (2002), except that equation (20) was not given there.[27]

How the marginal and average betas depend on t, σ, and ν has been traced through numerical solution to the nonlinear equations. All cases considered have $a = 0$ and the ratio $c/(1+r)$ fixed at 1/1.05. The central parameter configuration considered is $t = 0.35, \sigma = 0.3$. These are not unreasonable numbers (when the time unit is one year). For simplicity the verbal discussion below will assume $\beta_P = 1$. The five after-tax asset betas, divided by β_P, are shown in figure 1 as functions of the scale elasticity ν. A sixth relevant curve for comparison would be β_P itself, horizontal at 1.0 in the diagram. This would be the unlevered beta without taxation or with pure cash flow taxation.

FIGURE 1 HERE.

[27] Even for the stand-alone CRS case, the present paper improves upon the solution for the case considered in Lund (2002), by pointing out that the variables z_1 and z_2, called x_1 and x_2 in equation (19) in that paper, can be rewritten implicitly as functions of the exogenous parameters.

Figure 1 shows that the betas have the expected properties. Consider first Case F with riskless tax shields. The sparsely dotted horizonal line gives the marginal β_{FM}, while the heavily/infrequently dashed curve gives the average β_{FA}. The first one is a constant, independent of ν. The numerical value, approximately 0.67, is close to $(1-t)\beta_P = 0.65$. The average beta declines from β_P to β_{FM} as ν goes from zero to unity. The curve is slightly convex. The upper limit, equal to β_P, comes from the fact that the relative weight on the final term in (5) goes to zero. In the limit as $\nu \to 0^+$, the future cash flow is proportional to P and has the same systematic risk as P. As $\nu \to 1^-$, β_{FA} approaches β_{FM} because the whole project approaches a marginal project at this limit.

In Case F the effect on β_{FA} of varying ν comes through the changing relative weights of two cash flow elements, one proportional to P, the other risk free. This effect is still present in Case R with risky tax shields. But here there is another, opposing effect: A higher ν reduces the probability of being in tax position in period 1. This affects both marginal and average beta in Case R. The densely dashed curve gives the β_{RM} of the marginal project taxed together with the inframarginal project. This is increasing and convex as function of ν. As $\nu \to 1^-$, the technology approaches CRS, and the risk of the tax shields increases. The upper limit is thus equal to the β_{RC} of a marginal project taxed alone. The lower limit, when $\nu \to 0^+$, is equal to the marginal β_{FM} when the tax shields are risk free. In this limit there is so much income, relative to the investment, that the probability of not paying taxes goes to zero. The solid curve gives the average β_{RA} for the case of risky tax shields. For small ν values there is no detectable difference between this and β_{FA}, since the risk is minuscule. As ν increases towards unity, β_{RA} approaches β_{RM} from above, since the DRS investment approaches a CRS investment. The feature that β_{RA} is nonmonotonic in ν is not so easy to explain (and may not hold for all parameter configurations).

The results on β_{RA} can be compared with those of Jacoby and Laughton (1992), although their numerical examples are more complicated, involving also various degrees of operating leverage. In their figure 5 the systematic risk of the net after-tax cash flow decreases monotonically with increasing rent, which is consistent with the right-hand increasing part of the β_{RA} curve shown in figure 1 here. Rents increase to the right in their figure 5, to the left in figure 1 here. The convexity is qualitatively the same in both curves. Their conclusion (p. 44) that "the larger fields will be undervalued relative to the smaller fields if all are discounted with the same discounting structure, as they would be using standard DCF methods" is true within the range they cover, but not in general, due to the possible non-monotonicity demonstrated here.

FIGURE 2 HERE.

Clearly, even the DRS case with risky tax shields can have betas substantially lower than β_P. In this case the marginal beta, β_{RM}, satisfies the intuition that it has less risk than the stand-alone marginal beta, β_{RC}, as an effect of being taxed together with an inframarginal cash flow. But the average beta, β_{RA}, does not exhibit this property uniformly, and in fact, the difference between marginal and average beta is just as large in this case as in the case with risk free tax shields. The convexity of the curves strengthens the feature that tax shields, and thus after-tax equity, have relatively low systematic risk when there are moderately decreasing returns to scale.

Figures 2 and 3 show some sensitivities to changes in the tax rate, t, and the volatility, σ. The three non-constant curves from figure 1 are reproduced, marked as β_i^0. The corresponding three curves for the new value of t or σ are marked without the superscript 0. The values of the constant β_{FM} and β_{RC} are now only shown implicitly, as the endpoint values for the curves.

FIGURE 3 HERE.

Figure 2 shows all betas decreasing in the tax rate,[28] which was also the main point in Lund (2002) for the cases considered there. The effect on the lowest values, β_{FM}, seems to be proportional to $(1-t)$, which is almost correct when $c/(1+r)$ is close to unity, see also corollary 2.2 in Lund (2002). For a given ν, the ratio β_{FM}/β_{FA} is decreasing in t.

Figure 3 shows only one β_{FA} curve, as this is unaffected by a change in volatility. The figure shows that except for this, a lower σ works in the same direction as a higher t. But

[28] The high tax rate of 70 percent can be representative for rent tax systems, although rates vary a lot.

the effects of changes in σ are only discernible for higher values of ν, and the magnitudes of the effects are not very large. The effects of σ on the ratio β_{FA}/β_P are robust results in the sense that they do not rely on the assumption made about the lack of relation between σ and β_P. However, the effects of σ on β_{FA} (separately) could also include effects via possible changes in β_P, not analyzed here.

6. Tax deduction for entry costs?

This section investigates if some conditions for industry equilibrium would undermine the results from the DRS model. The question arises since existence of rents may attract entry of new firms.[29] The question has not been raised in the studies cited above. It will be shown that marginal and average betas differ except under a combination of two conditions: Entry costs exactly outweigh quasi-rents for each firm in operation, and the tax treatment of entry costs is equal to the tax treatment of investment costs. A sunk-cost argument cannot eliminate this issue, since even sunk costs may involve future tax deductions.

Under uncertainty it is natural to assume that access to a unique resource or technology to some extent is the result of a random, risky process. One extreme assumption is that firms undertake R&D (or exploration for natural resources) with negligible costs and very low success probability. A small number of firms will have been lucky, and find themselves in the situation described by the model of the previous sections. These firms have only negligible tax shields for R&D costs. The opposite extreme is that all firms that have access to the investment opportunity, have paid the same entry cost, and that the after-tax entry cost is equal to the after-tax net value of the investment opportunity. It will be shown below that an important related question for the results of this study is to which extent there are tax shields for the entry costs, in particular in period 1.

The intention is now to find conditions under which the marginal and average after-tax

[29] If there are entry barriers so that rent exists in equilibrium, the model in the previous sections is adequate, as long as firms do not influence output or factor prices. Such market power is outside the scope of this paper.

asset betas coincide, to show possible weaknesses of the main results of this paper. The rest of this section thus considers firms that have access to the opportunity after having paid an entry cost that cancels the net profit from the opportunity, after tax.

Assumption 6: *An entry cost M is paid for the right to undertake the investment project. This is competitively determined among firms with the same tax position, so that the net value to the firm of paying this entry cost, undertaking the project in optimal scale, and paying taxes, is zero. The sequence of events in period 0 is as follows: (a) The authorities determine the tax system for both periods. (b) The firm pays the entry cost M. (c) The firm determines how much to invest, I. In addition to tax deductions defined in assumption 4, there are tax deductions bM in period 0 and hM in period 1, where b and $h/(1+r)$ are constants in the interval* $[0, 1]$.

To distinguish the case from those above, this will be called Case G (for Generalized model). The extension of Case R will be developed, while the similar extension of Case F can be found by setting the $N(\cdot)$ expressions equal to unity. The cash flow to equity in period 1 is

$$X_G = Pf(I) - t \cdot \max(Pf(I) - cI - hM, 0). \tag{21}$$

The valuation, as of one period earlier, of a claim to this is

$$V(X_G) = V(P)f(I) - t\left[V(P)f(I)N(z_{1G}) - (cI + hM)N(z_{2G})/(1+r)\right], \tag{22}$$

where

$$z_{1G} = \frac{\ln(V(P)f(I)) - \ln\left((cI + hM)/(1+r)\right)}{\sigma} + \frac{\sigma}{2}, \tag{23}$$

and

$$z_{2G} = z_{1G} - \sigma. \tag{24}$$

Proof of the following proposition is in appendix 9.3.

Proposition 6: *Under assumptions 1, 2, 4–6, the after-tax asset beta is given by (25). When there is no deduction for M in period 1 ($h = 0$), then $\beta_{GA} = \beta_{RA}$ (of equation (16)). Alternatively, when the two costs M and I are treated equally by the tax system ($a = b, c = h$), then $\beta_{GA} = \beta_{RC}$ (of equation (19)).*

The average beta in the general case is given by

$$\beta_{GA} = \frac{1 - ta - \frac{tcN(z_{2G})}{1+r}}{1 - ta - \frac{tcN(z_{2G})}{1+r}(1-\nu) + \frac{thN(z_{2G})}{1+r} \cdot \frac{(1-\nu)[1-t(a+cN(z_{2G})/(1+r))]}{[1-t(b+hN(z_{2G})/(1+r))]}} \beta_P, \tag{25}$$

where $z_{2G} =$

$$\frac{1}{\sigma}\left[\ln\left(\frac{1 - ta - \frac{tN(z_{2G})c}{1+r}}{1 - tN(z_{2G}+\sigma)}(1+r)\right) - \ln\left(c\nu + h\frac{(1-\nu)[1 - t(a + \frac{cN(z_{2G})}{1+r})]}{1 - t(b + \frac{hN(z_{2G})}{1+r})}\right)\right] - \frac{\sigma}{2}.$$

Only the two special cases mentioned will be discussed. The first case implies that b does not matter for results when $h = 0$, since equilibrium $M(1-tb)$ is determined endogenously. A higher (lower) b will lead to a higher (lower) M, keeping equilibrium $M(1-tb)$ unaffected, and when $h = 0$, only $M(1-tb)$ matters, not M separately. For instance, the two subcases ($b = 0, h = 0$) and ($b = 1, h = 0$) give the same β_{RA} despite very different tax treatment of M. With $h = 0$, the difference between marginal and average beta remains.

With equal tax treatment of the two costs ($a = b, c = h$), the firm's whole activity can be seen as a marginal project. In relation to the issues analyzed in this paper, there is nothing that distinguishes this from constant returns to scale, except that the scale of production is determined. To conclude this section: The marginal and average after-tax asset betas will differ except for firms that have paid entry costs equal to the rents they realize. But even then, Case R above covers at least two interesting possibilities: The entry cost is deductible immediately, or not at all. An additional requirement for the difference between marginal and average beta to disappear is that entry costs and investment costs are treated identically by the tax system.

7. Cost of capital before taxes

This section considers the cost of capital before corporate taxes, which is the traditional measure for the effects of the tax system on the acceptance or rejection of real (nonfinancial) investment projects. The complications of entry costs in section 6 are neglected here. The expected (one plus rate of) return before taxes is $E(P)Q/I$, rewritten as

$$\frac{E(P)Q}{I} = \frac{E(P)}{V(P)} \cdot \frac{V(P)Q}{I}. \tag{26}$$

The first fraction on the right hand side is assumed to be exogenous, and is given by (1) and (3). The second is determined by the requirement that the project be marginal after tax. For Case F above (cf. (12)), the required expected return is

$$\frac{E(P)}{V(P)} \cdot \frac{1 - ta - tc/(1+r)}{1-t} \equiv \frac{E(P)}{V(P)} \cdot \gamma_1, \tag{27}$$

defining the second fraction as γ_1. This multiplicative distortion in the expected return appears in Hall and Jorgenson (1969), p. 395. The extension of their result to uncertainty shows that with full loss offset, the multiplicative distortion is independent of risk.

The results can be compared with those of Devereux (2004), which is another extension of Hall and Jorgenson (1969) to a situation with uncertainty. His section 2.4.4 discusses a similar problem of tax distortions of the optimal scale of risky one-period investment. His risk premia are covariances with a stochastic discount factor, not with the market portfolio (cf. footnote 8 above). His model has an annual depreciation rate δ which is stochastic, so there is capital risk, i.e., a risky future value of remaining non-financial capital. If this is ignored by letting $\delta \equiv 1$ (deterministic), results on relative value distortions are the same as here. But the present paper's multiplicative form PQ of next period's pre-tax cash flow turns out to be a crucial simplification, as will be shown next.

In the notation of Devereux (2004), the stochastic next-period result of the marginal money unit invested is now $\tilde{p}+1$, valued today at $V(\tilde{p}+1)=1$, in case there are no taxes. When instead the investment is marginal after tax, the before-tax result is denoted as $\tilde{p}^T+1$. These correspond to PQ/I in this paper. From his equation (2.28) follows that the relative distortion in before-tax value of next period's result of the marginal investment is

$$\frac{V(\tilde{p}^T+1)}{V(\tilde{p}+1)}=\frac{E(\tilde{p})+1}{1+r}+\mathrm{cov}(\tilde{p}^T,\tilde{m})=\frac{1-A}{1-\tau} \tag{28}$$

in his notation.[30] This is the same as γ_1 in (27).

In the model of Devereux (2004), there is no way to pin down the "relationship between $\tilde{p}$ and $\tilde{p}^T$, where each is defined for a marginal investment" (p. 55). He observes that in "the absence of risk, these are scalars that can easily be compared. However, in the presence of risk, any comparison must take into account the whole distribution of each return" (p. 55). In the model in section 4 above, the simple multiplicative form allows for a solution for the Q/I ratio of a marginal investment in absence of taxes, $(Q/I)^*=1/V(P)$, and in presence of taxes, $(Q/I)^T=\gamma_1/V(P)$. This gives:[31]

$$\tilde{p}+1=P(Q/I)^*=P/V(P) \text{ and } \tilde{p}^T+1=P(Q/I)^T=\gamma_1 P/V(P). \tag{29}$$

The relation between the two is thus

$$\tilde{p}^T=\gamma_1\tilde{p}+\gamma_1-1, \tag{30}$$

and the risk premia are related by

$$\mathrm{cov}(\tilde{p}^T,\tilde{m})=\gamma_1\,\mathrm{cov}(\tilde{p},\tilde{m}). \tag{31}$$

[30] His τ is the (statutory) tax rate, here denoted t. His A is the tax value of tax shields from a unit investment, in present value terms, here $ta+tc/(1+r)$.

[31] Lower-case $\tilde{p}$ is taken from the notation of Devereux (2004), while upper-case P still refers to this paper's stochastic price P

It should be noted that these are properties of the before-tax cash flows of the marginal investments in the two cases. This is, of course, different from the return betas of the after-tax cash flows, that appear in (11). Those are different in part because the tax shield tcI is part of future cash flows, but also because the deduction taI in the investment period reduces the after-tax net investment required.

The simplification, PQ, comes at the expense of lower realism. The model of Devereux (2004) could allow for managerial flexibility, so that Q is dependent on P. It could also allow for a multitude of products and variable input factors, with relative amounts of these at the margin depending on taxation.

Consider now Case R with an uncertain tax position. The relevant $V(P)Q/I$ ratio for a marginal project taxed together with the inframarginal project is given in equation (51) in appendix 9.2. The required expected return is

$$\frac{E(P)}{V(P)} \cdot \frac{1 - ta - tN(z_{2D})c/(1+r)}{1 - tN(z_{1D})}. \tag{32}$$

Also in the case with no loss offset, the distortion is independent of systematic risk, but here it depends on total risk. The relevant tax rates can now be interpreted as expected rates, although with a risk-adjusted probability measure, cf. footnote 22.

The following proposition summarizes:

Proposition 7: *Under assumptions 1–3 the required expected return before corporate taxes is given by (27). It is decreasing in a and $c/(1+r)$. It is increasing in the tax rate if $a + c/(1+r) < 1$. The distortion from the tax system does not depend on total or systematic risk. Under assumptions 1, 2, 4, 5, the required expected return before corporate taxes is given by (32). This distortion from the tax system depends on total risk, but not on systematic risk as long as total risk is unchanged.*

The effects of changes in a, c, and t are not surprising, since the results are so closely related to those of Hall and Jorgenson (1969). The simplicity of the results is more of a novelty, together with the clarification of the effect of the two categories of risk. It should be stressed that the tax effects on after-tax costs of capital may (and, typically, will) exist even if there is no tax effect on the before-tax cost of capital. In (27), if $a = 0$ and $c/(1+r) = 1$, the second fraction is unity irrespective of t. The before-tax cost of capital depends on the present value of deductions, not on how the deductions are distributed in time. But β_{FM} from (11) will in that case be proportional to $(1 - t)$, and will in general reflect how deductions are distributed in time.

If firms make decisions based on the same discount rate under different tax systems, there will be distortions of the non-traditional type mentioned in the introduction, even when the traditionally measured distortion via the before-tax cost of capital is zero.[32]

8. Conclusion

The point of departure of this paper is the literature on effects of corporate taxes on the beta of an unlevered firm, and its after-tax cost of capital, in particular the theoretical models of Galai (1998), Lund (2002), and Rao and Stevens (2006), but also the previous simulation results in Jacoby and Laughton (1992) and Bradley (1998). One main new result is the tax-induced difference between the marginal and average beta of equity. The difference occurs when the tax system allows depreciation or similar deductions in years after investments have been made, and some inframarginal profits exist. This implies that deductions are proportional to the investment outlay, not to the valuation of the subsequent cash flows. This difference gives rise to the difference in the betas.

The model also allows for an analytical approach to the riskiness of depreciation tax shields. Relative to most tax systems, the approach exaggerates the riskiness by assuming no loss offset at all, apart from what is available from the same period's inframarginal profit. The valuation model originates from the theory of financial options. It reveals that

[32] The tax effect on the marginal beta (after tax) is not unrelated to the relative distortion in the before-tax cost of capital. Lund (2002), equations (12), (24), and (31), shows that it can be written as proportional to that distortion, denoted γ_i, but with other tax effects coming into the equations.

even when the tax shields are risky, they still induce reduced systematic risk of equity. Moreover, with the volatility of the firm's output price as exogenous, the model determines endogenously the probability of having a taxable profit. Another result is that the rate-of-return shortfall does not affect the ratio of the equity beta to the asset beta. The model has also been extended to consider entry costs.

The results on the marginal-average difference imply that some previous studies have failed to identify the cost of capital. Moreover, the relation between observed equity betas and the betas of assets is more complicated than previously believed. When some firms have inframarginal projects and the tax system allows depreciation deductions, not only must equity betas be unlevered, but also "untaxed" and "unaveraged."

The results are of practical importance for the large majority of firms that base their decisions on one discount rate, applied to net expected after-tax cash flows, unadjusted for differences in taxation. Under tax systems with depreciation or other postponed investment-proportional deductions, these firms will tend to undervalue projects subject to higher tax rates relative to those with lower tax rates. They will undervalue projects with generous postponed deductions relative to those with more weight on immediate deductions, like investment tax credits. The easiest way out of these problems seems to be the APV method, separate discounting of cash flows with different risk characteristics. It may also be possible to decentralize decisions based on one discount rate appropriate for each project, but this would require adjustments for taxation and rent (or quasi rent), based on rules of thumb. Summers (1987, p. 298) is skeptical of the practicability of such a discount rate reflecting "average degree of riskiness."

The results are of practical importance for tax reforms that use economic theory to predict outcomes. When authorities are convinced that future tax shields are close to risk free, it is tempting to assume that firms are able to respond accordingly, based on economic theory. Whether firms do this based on the APV method or one adjusted discount rate from this paper's model is of no importance for policy implementation. But if firms continue to use a uniform, inappropriate discount rate, the problems pointed out by Summers (1987) may be even larger than he believed, since correct discount rates for expected net cash flows are more divergent than previously known. Thus it may take time and learning before the desirable neutrality properties of some tax systems materialize.

9. Appendix

9.1. Proof of propositions 1 and 2

In this subsection, define $\hat{c} \equiv c/(1+r)$, $\Delta \equiv 1 - ta - t\hat{c}(1-\nu)$, and $\xi \equiv E(r_m) - r$.

To prove proposition 1, signs of partial derivatives are needed. The partial derivatives,

$$\frac{\partial \beta_{FA}}{\partial t} = \beta_P \frac{-\hat{c}\nu}{\Delta^2}, \quad \frac{\partial \beta_{FA}}{\partial a} = \beta_P \frac{-t^2\hat{c}\nu}{\Delta^2}, \tag{33}$$

$$\frac{\partial \beta_{FA}}{\partial \hat{c}} = \beta_P \frac{-\nu t(1-ta)}{\Delta^2}, \quad \frac{\partial \beta_{FA}}{\partial \nu} = \beta_P \frac{-t\hat{c}(1-ta-t\hat{c})}{\Delta^2}, \tag{34}$$

are all strictly negative, q.e.d.

To prove proposition 2, observe that the expected return on a claim to one unit of output satisfies the CAPM: $E(P)/V(P) = 1 + r + \beta_P\xi$. The maximand based on marginal beta is

$$\frac{E(P)f(I)(1-t) + tcI}{1 + r + \beta_{FM}\xi} - I(1-ta). \tag{35}$$

The proposition claims that maximization of this with respect to I gives the same result as (7). The first-order condition is

$$\frac{E(P)f'(I)(1-t) + tc}{1 + r + \beta_{FM}\xi} = 1 - ta. \tag{36}$$

Introduce β_{FM} from (11); for $E(P)$ introduce $V(P)(1 + r + \beta_P\xi)$; and find

$$V(P)f'(I)(1-t) = \frac{(1+r)(1-ta-t\hat{c}) + \beta_P\xi(1-ta-t\hat{c})}{1+r+\beta_P\xi} = 1 - ta - t\hat{c}, \tag{37}$$

which is (7). This proves the first part. Consider now the other part, that the average beta can be used, provided it is considered a function $\beta_{FA} = \beta_{FA}(I)$, defined by (9). The maximand is

$$\frac{E(P)f(I)(1-t)+tcI}{1+r+\beta_{FA}(I)\xi} - I(1-ta). \tag{38}$$

Introduce $E(P) = V(P)(1 + r + \beta_P\xi)$ and use equation (9) to rewrite the maximand as

$$\frac{V(P)(1+r+\beta_P\xi)f(I)(1-t)+tcI}{1+r+\frac{V(P)f(I)(1-t)}{V(P)f(I)(1-t)+It\hat{c}}\beta_P\xi} - I(1-ta), \tag{39}$$

the same maximand as in (6), q.e.d.

9.2. Proof of proposition 3–5

This derivation starts with the average beta in Case R. The cash flow to equity in period 1 is

$$X_R = Pf(I) - t \cdot \max(Pf(I) - cI, 0). \tag{40}$$

Under assumption 5 the valuation, as of one period earlier, of a claim to this is

$$V(X_R) = V(P)f(I) - t\left[V(P)f(I)N(z_{1D}) - cIN(z_{2D})/(1+r)\right], \tag{41}$$

where

$$z_{1D} = \frac{\ln(V(P)f(I)) - \ln\left(cI/(1+r)\right)}{\sigma} + \frac{\sigma}{2}, \quad z_{2D} = z_{1D} - \sigma. \tag{42}$$

The expression in square brackets in (41) can be rewritten in terms of the standard Black and Scholes' formula for option pricing as $C(V(P)f(I), cI, 1, r, \sigma)$, so that

$$V(X_R) = V(P)f(I) - tC(V(P)f(I), cI, 1, r, \sigma). \tag{43}$$

The C function has known derivatives $\partial C/\partial(V(P)f(I)) = N(z_{1D})$ and $\partial C/\partial(cI) = -N(z_{2D})/(1+r)$, to be used below.

The firm chooses I to maximize $\pi_R(I) \equiv V(X_R) - I(1-ta)$. The first-order condition is

$$V(P)f'(I) = \frac{(1 - ta - tN(z_{2D})c/(1+r))}{(1 - tN(z_{1D}))}. \tag{44}$$

Introducing the constant-elasticity production function gives

$$V(P)f(I)(1 - tN(z_{1D})) = (I/\nu)\left(1 - ta - tN(z_{2D})c/(1+r)\right). \tag{45}$$

The claim is equivalent to a portfolio with $f(I)(1-tN(z_{1D}))$ claims on P, and the rest risk free. The beta is

$$\beta_{RA} = \frac{V(P)f(I)(1 - tN(z_{1D}))}{V(X_R)}\beta_P. \tag{46}$$

RA denotes the average beta in Case R. Introduce $V(X_R)$ from (41) and the f function to find

$$\beta_{RA} = \frac{1 - ta - tN(z_{2D})c/(1+r)}{1 - ta - tN(z_{2D})(1-\nu)c/(1+r)}\beta_P. \tag{47}$$

Also, z_{1D} and z_{2D} can be written in terms of exogenous variables. Plug (45) into (42) to find equation (17). To derive the marginal beta for the same case, consider first the marginal beta with risky tax position derived in equation (24) in Lund (2002). That paper's equation (23) becomes

$$\gamma_2 = \frac{1 - ta - tN(z_{2C})c/(1+r)}{1 - tN(z_{1C})}, \tag{48}$$

and the marginal beta can be written

$$\beta_{RC} = \left(1 - ta - tN(z_{2C})c/(1+r)\right)\beta_P. \tag{49}$$

The subscript RC (C for CRS) is used here since the case considered in Lund (2002) did not include the marginal project with some other activity, i.e., as if the case had CRS. The definition of a marginal CRS project gives

$$\frac{V(P)Q}{I}=\frac{1-ta-tN(z_{2C})c/(1+r)}{1-tN(z_{1C})}, \tag{50}$$

cf. equations (5) and (23) in Lund (2002). This leads to equation (20) in the main text.

The marginal beta for the DRS case can be seen as a mixture of the two cases just considered. The valuation is based on the risk-adjusted probabilities $N(z_{1D})$ and $N(z_{2D})$, reflecting the probability that the whole DRS project is in tax position. The project that invests I to yield Q, and that is taxed together with the optimally scaled DRS project, is marginal when

$$\frac{V(P)Q}{I}=\frac{1-ta-tN(z_{2D})c/(1+r)}{1-tN(z_{1D})}. \tag{51}$$

With z_{2D} given in (17), the marginal beta in the DRS case becomes

$$\beta_{RM}=\frac{1-ta-tN(z_{2D})c/(1+r)}{1-ta}\beta_P. \tag{52}$$

9.3. Proof of proposition 6

The firm chooses I to maximize $\pi_G(I)\equiv V(X_G)-I(1-ta)$. From the first-order condition follows

$$V(P)f(I)(1-tN(z_{1G}))=\frac{f(I)\left(1-ta-tcN(z_{2G})/(1+r)\right)}{f'(I)}. \tag{53}$$

Introducing the constant-elasticity production function gives

$$V(P)f(I)(1-tN(z_{1G}))=(I/\nu)\left(1-ta-tcN(z_{2G})/(1+r)\right). \tag{54}$$

Equilibrium M is given by

$$M(1-tb)=V(X_G)-I(1-ta)=(I/\nu)\left(1-ta-tcN(z_{2G})/(1+r)\right)$$

$$+ tcIN(z_{2G})/(1+r) - I(1-ta) + thMN(z_{2G})/(1+r),$$

which can be solved for

$$M = I\frac{(1-\nu)[1-t(a+cN(z_{2G})/(1+r))]}{\nu[1-t(b+hN(z_{2G})/(1+r))]}. \tag{55}$$

Solve for $V(X_G) = (I/\nu)$

$$\times \left[1 - ta - \frac{tcN(z_{2G})}{1+r}(1-\nu) + \frac{thN(z_{2G})}{1+r} \cdot \frac{(1-\nu)[1-t(a+cN(z_{2G})/(1+r))]}{[1-t(b+hN(z_{2G})/(1+r))]}\right].$$

This gives the average beta for this case,

$$\beta_{GA} = \frac{1 - ta - \frac{tcN(z_{2G})}{1+r}}{1 - ta - \frac{tcN(z_{2G})}{1+r}(1-\nu) + \frac{thN(z_{2G})}{1+r} \cdot \frac{(1-\nu)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{1-t(b+\frac{hN(z_{2G})}{1+r})}}\beta_P, \tag{56}$$

with

$$z_{2G} = \tag{57}$$

$$\frac{1}{\sigma}\left[\ln\left(\frac{1-ta-\frac{tN(z_{2G})c}{1+r}}{1-tN(z_{2G}+\sigma)}(1+r)\right) - \ln\left(c\nu + h\frac{(1-\nu)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{1-t(b+\frac{hN(z_{2G})}{1+r})}\right)\right] - \frac{\sigma}{2}.$$

If $h = 0$, then $z_{2G} = z_{2D}$ (of (17)), and $\beta_{GA} = \beta_{RA}$ (of (16)). This follows since the fraction

$$\frac{(1-\nu)[1-t(a+\frac{cN(z_{2G})}{1+r})]}{1-t(b+\frac{hN(z_{2G})}{1+r})}, \tag{58}$$

which appears in both (56) and (57), vanishes, since it is multiplied by h.

If $a = b$ and $c = h$, then ν vanishes from (56) and (57), since the last term in large square brackets in (57) is reduced to

$$-\ln(c\nu + h(1-\nu)) = -\ln(c). \tag{59}$$

Thus we find $z_{2G} = z_{2C}$ (from (20)), and $\beta_{GA} = \beta_{RC}$ (from (19)), q.e.d.

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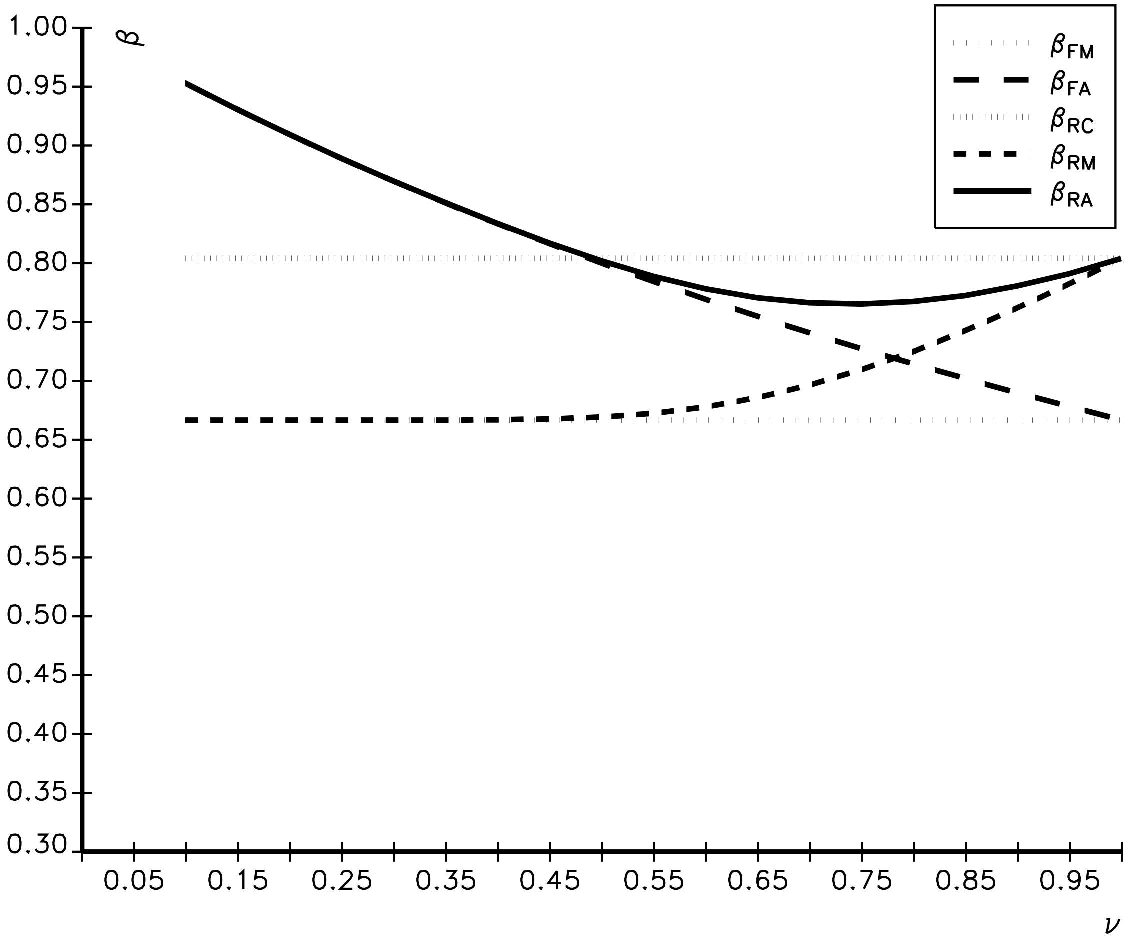
Figure 1

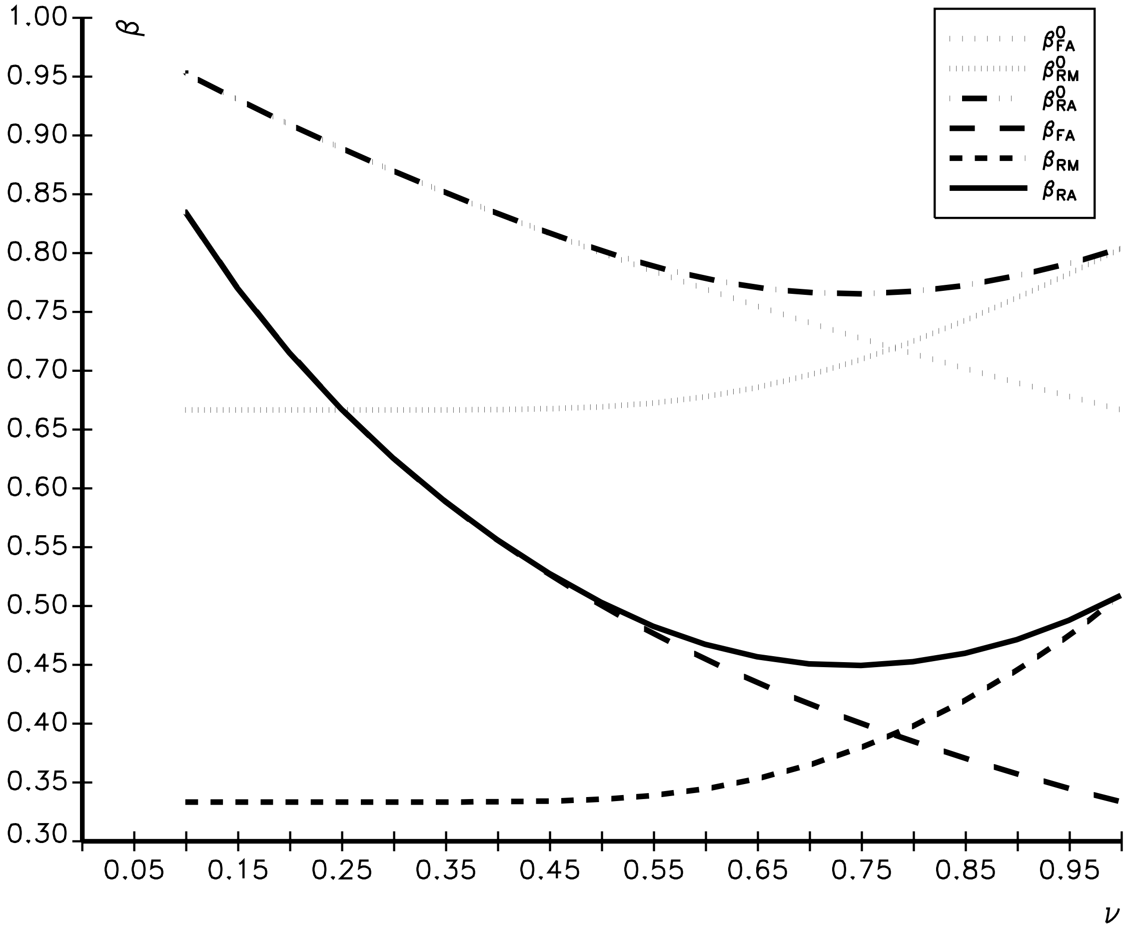
β_i/β_P as functions of scale elasticity, ν



Notes: Tax rate $t = 0.35$, volatility $\sigma = 0.3$, PV of tax shield $c/(1+r) = 1.05$.

Figure 2

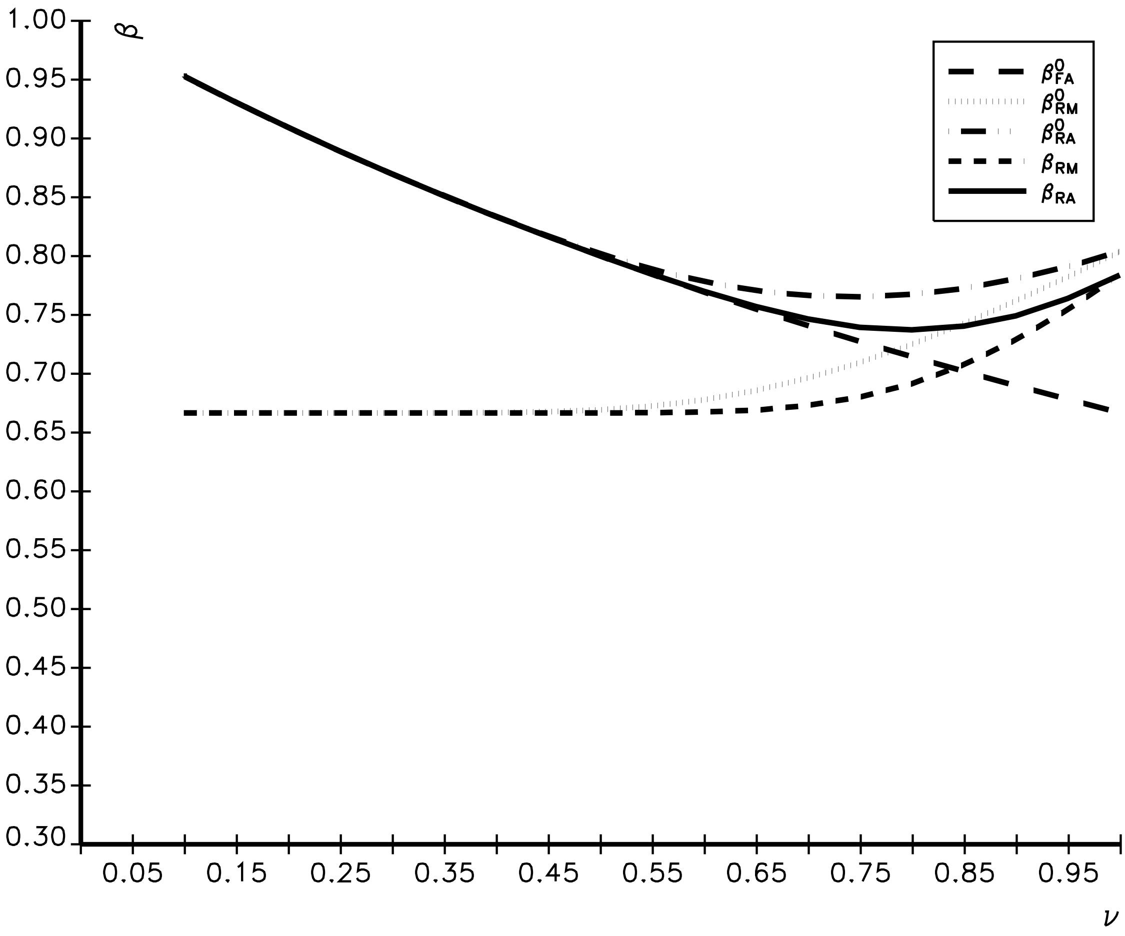
β_i/β_P as functions of scale elasticity, ν; varying the tax rate



Notes: Volatility $\sigma = 0.3$, PV of tax shield $c/(1+r) = 1.05$. For β_i^0: tax rate $t = 0.35$; for β_i without superscript: $t = 0.7$.

Figure 3

β_i/β_P as functions of scale elasticity, ν; varying the volatility



Notes: Tax rate $t = 0.35$, PV of tax shield $c/(1+r) = 1.05$. For β_i^0: Volatility $\sigma = 0.3$; for β_i without superscript: $\sigma = 0.2$