Taxation and systematic risk under decreasing returns to scale

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Abstract

Lund (2002a) showed in a CAPM-type model that tax depreciation schedules affect the systematic risk of after-tax cash flows and thus required expected returns after taxes. Higher tax rates imply lower betas. The present paper extends this to decreasing returns to scale. The tax system induces a difference between marginal and average beta of equity. If future tax positions are uncertain, the beta of equity is still decreasing in the tax rate, but increasing in the underlying volatility. The results are important if market data are used to infer required expected returns, and in discussions of tax design.

KEYWORDS: Corporate tax, depreciation schedule, decreasing returns, cost of equity, uncertainty

JEL classification numbers: F23, G31, H25
1 Introduction

Much of the literature on the relationship between taxation and the cost of capital neglects uncertainty.\textsuperscript{1} The central relationship can then be written as $p = c(r)$ (King and Fullerton (1984), p. 10), where $r$ is the real market interest rate and $p$ the real cost of capital, defined as a minimum pretax rate of return.\textsuperscript{2} King and Fullerton write that the function $c$ “depends upon the details of the tax code,” and that one way of thinking about the condition is to consider $r$ as given and ask how various tax systems affect $p$ in various circumstances.

This approach is easily misleading under uncertainty. Different corporate tax systems split the risk between the firm and the government in different ways. There is not one general required expected rate of return after corporate tax which could play the role of $r$. An average will not do either. Instead one must consider how each tax system affects the risk characteristics of the after-corporate-tax cash flow. In particular, depreciation schedules, interest deductibility, and loss offset are important.

Lund (2002a) showed how a standard corporate income tax affects the systematic risk of after-tax cash flows when investments are not expensed, but tax deductible according to a depreciation schedule. That study first considered the case of full (perfect) loss offset (or more generally, a tax position known with full certainty). Perfect loss offset will be a good approximation for some large firms,\textsuperscript{3} but future tax deductions connected with the marginal investment\textsuperscript{4} are not completely risk free. As an alternative Lund (2002a) considered the CRS (constant returns to scale), stand-alone, two-period case. The marginal investment was then supposed to yield operating income in the second period which would be taxed alone. The depreciation deduction would be lost to the extent that it exceeded the second-period operating income.

More realistically, the marginal investment will be taxed together with other activities in the firm. In the present paper there are supposed to be decreasing returns to scale, while all uncertainty originates from a single stochastic variable, the second-period output price. This is just a first step towards more realism, but sufficiently complicated for a separate study. Further extensions may be a multi-period model and/or more stochastic variables, such as prices of more than one output, not perfectly correlated.
In order to get results consistent with an equilibrium in capital markets, the firm will have a market value of zero. In the CAPM jargon, the firm is on the security market line. The profitable DRS (decreasing returns to scale) project can only be obtained by paying a “fee” for it, e.g., by buying a licence or patent, or by doing research and development. This payment is supposed to be immediately tax deductible. Its magnitude is determined competitively.

Apart from DRS the assumptions will be as in Lund (2002a). Only parts of the motivation are repeated here.

It is possible to arrive at results on the effect of taxation under uncertainty which are powerful, consistent with the assumptions of the present paper, but derived in a more straightforward way. Fane (1987) is an example, relying on value additivity, considering separately the risk of each of the cash flow’s elements. The method in the present paper is more suitable not only for imperfect loss offset, but generally when the systematic risk of the net after-tax cash flow of a firm is of interest. This is consistent with the practice of most firms, and may be valuable whenever data from financial markets are used to estimate required expected rates of return for new projects.

Most of the previous literature has neglected the effects of a corporate income tax on the systematic risk of equity. While the effect of interest deductibility in reducing the cost of debt is well known, it is most often neglected that even in a fully equity financed firm, the required after-tax expected rate of return is affected by taxes.

Levy and Arditti (1973) observe that taxes with depreciation schedules affect the required expected rate of return after tax. Their model is an extension of Modigliani and Miller (M&M) (1963), introducing depreciable assets in the M&M model, but maintaining their assumptions of perpetual projects and full loss offset. Lund (2002b) discusses their model and claims that a more realistic alternative turns their results around.

Galai (1988) (very briefly, p. 81) and Derrig (1994) both discuss the effect of a corporate income tax on the systematic risk of equity based on the CAPM. They do not observe the necessity of solving for the expected rate of return of an after-tax marginal project.

Both Levy and Arditti (1973) and Derrig (1994) consider only one simple tax system, and assume that the firm is certain to be in tax position. The present paper (like Lund
(2002a)) is an extension in both respects. Galai (1988) considers both risky debt and a risky tax position, but only one tax system. None of the previous studies consider DRS.

Section 2 presents a two-period model in which the firm produces with decreasing returns to scale and pays taxes with certainty. Section 3 introduces uncertainty about whether taxes are paid. Section 4 contains additional discussion of some aspects of the model. Section 5 concludes.

2 The model under full certainty

A firm invests in period 0 and produces in period 1, only. The firm has only one investment project with decreasing returns to scale. It is free to choose the scale of investment. The optimal choice is endogenous, determined by the tax system and other parameters in each case below. In this way we also characterize the minimum required expected return to equity in each case.

Assumption 1: The firm maximizes its market value according to a tax-adjusted Capital Asset Pricing Model,

\[ E(r_i) = r_f \theta + \beta_i[E(r_m) - r_f \theta], \]

where \( r_f > 0 \) and \( \theta \in (0, 1] \).

This allows for differences in the tax treatment on the hands of the firm’s owners of income from equity and income from riskless bonds, reflected in the tax parameter \( \theta \). In a discussion of taxation and the CAPM it seems reasonable to allow for \( \theta < 1 \), but it has no consequences for the results which follow. The standard CAPM with \( \theta = 1 \) is all that is needed.

When various tax systems are considered below, these are assumed not to affect the capital market equilibrium. This will be a good approximation if they apply in small sectors of the economy (e.g., natural resource extraction), or abroad in economies which are small in relation to the domestic one. This is thus a partial equilibrium analysis.

The economy where the firm’s shares are traded may have a tax system, which is exogenously given in the analysis, and reflected in \( \theta \). This is often referred to as a “personal” tax system, even though the owners may be firms or other institutions.
A consequence of the CAPM is that the claim to any uncertain cash flow $X$, to be received in period 1, has a period-0 value of

$$\varphi(X) = \frac{1}{1 + r_f \theta}[E(X) - \lambda \theta \text{cov}(X, r_m)],$$  \hspace{1cm}(2)$$

where $\lambda = [E(r_m) - r_f \theta]/\text{var}(r_m)$. Equation (2) defines a valuation function $\varphi$ to be applied below.

A product price, $P$, will most likely not have an expected rate of price increase which satisfies the CAPM. A claim on one unit of the product will satisfy the CAPM, however, so that the beta value of $P$ should be defined in relation to the return $P/\varphi(P)$,

$$\beta_P = \frac{\text{cov}(\frac{P}{\varphi(P)}, r_m)}{\text{var}(r_m)}.$$  \hspace{1cm}(3)$$

It is possible to express this more explicitly, without the detour via $\varphi(P)$, namely as

$$\beta_P = \frac{1 + r_f \theta}{E(P) \frac{\text{var}(r_m)}{\text{cov}(P, r_m)} - [E(r_m) - r_f \theta]},$$  \hspace{1cm}(4)$$

cf. equation (4) of Ehrhardt and Daves (2000).

**Assumption 2:** In period 0 the firm invests an amount $I > 0$ in a project. In period 1 the project produces a quantity $Q = f(I) > 0$ to be sold at an uncertain price $P$. The production function $f$ has $f' > 0$, $f'' < 0$. The joint probability distribution of $(P, r_m)$ is exogenous to the firm, and $\text{cov}(P, r_m) > 0$. There is no production flexibility; $Q$ is fixed after the project has been initiated.

The assumption of $\text{cov}(P, r_m) > 0$ can easily be relaxed. It is only a convenience in order to simplify the verbal discussions below.

In order to simplify results in what follows, the specification

$$f(I) = \xi I^\alpha,$$  \hspace{1cm}(5)$$

is introduced, with $\xi > 0$ and $\alpha \in (0, 1)$, so that $\alpha = f'(I)/f(I)$, a constant elasticity.

**Assumption 3:** The fee which is paid for the right to undertake the investment project is $M_0$. This is competitively determined among firms with the same leverage (see Assumption 4) and tax position (see Assumption 5), so that the net value after taxes to the firm
of paying this fee, borrowing, undertaking the project in optimal scale, and paying taxes, is zero. The sequence of events in period 0 is as follows: (a) The authorities determine the tax system for both periods. (b) The firm pays the fee $M_0$ (and possibly borrows some fraction of this, see Assumption 4). (c) The firm determines how much to invest, $I$ (and possibly borrows some fraction of this, see Assumption 4).

**Assumption 4**: A fraction $(1-\eta) \in [0,1)$ of the financing need in period 0 is borrowed. This fraction is independent of the investment decision and of the tax system. The loan $B$ is repaid with interest with full certainty in period 1.

The financing need is equal to $M_0 + I$ minus the immediate tax relief for these costs, if any.$^{13}$

Debt is introduced only because of the prominence of debt in the traditional literature on taxes and the cost of capital. In the present paper the results can be derived with zero debt. The assumptions of independence between financing and investment, and between (after-tax) financing and taxes, are those underlying the standard derivation of the WACC, and therefore the appropriate set of assumptions here.$^{14}$

It should be kept in mind that Assumption 4 concerns the formal project-related borrowing by the firm. When applied to the subsidiary of a multinational, this may be tax motivated and differ from the net project-related borrowing undertaken by the multinational and its subsidiaries taken together. The lender to one subsidiary is often another subsidiary of the same multinational. Also, the possibility of transfer pricing is neglected here.

The assumption of default-free debt is a common simplification, and should be acceptable for the purposes of this paper. Although the firm’s operating revenue may turn out too low to repay the debt, it is realistic in many cases that the loan is effectively guaranteed by a parent company. Galai (1988) focuses on risky debt in a similar model.

It will be clear below that there may be tax advantages to debt. When the firm decides on the optimal investment $I$, the fee $M_0$ is already paid, and a fraction of this is borrowed. These are given magnitudes when the investment decision is made, so this decision is independent of the tax advantage of the $M_0$-related borrowing.
Assumption 5: A tax at rate $t \in [0, 1)$ will be paid with certainty in the production period. The tax base is operating revenue less $(grT B + cI)$. There is also a tax relief of $t(M_0 + aI)$ in period 0. The constants $g, c, a$ are in the interval $[0, 1]$.

This general formulation allows for accelerated depreciation with, e.g., $a > 0$ and $a + c = 1$, or a standard depreciation interpreted (since there is only one period with production) as $a = 0, c = 1$. There is usually full interest deduction, i.e., $g = 1$, but the Brown (1948) cash flow tax has $g = 0$, and some transfer pricing regulations might require $0 < g < 1$.

Assumption 5 implies that a negative tax base gives a negative tax. While this is unrealistic for most tax systems when the project stands alone, it is not at all unrealistic when the marginal project is added to other activity which is more profitable. An alternative assumption for the second period is considered in section 3. For the first period, however, no alternative is considered. This could rely on an assumption that firms only start projects in periods in which they are in tax position to benefit immediately from deductions allowed in the first period. This does not explain how most firms get started in the first place.

Since $M_0$ is immediately tax deductible with full certainty, it is really the after-tax payment $M = M_0(1 - t)$ which is of interest to us. From Assumption 3 it follows that it is this payment, plus any tax advantage connected with a related borrowing, which is adjusted competitively so that the total net value is zero. It is also $M$, not $M_0$, which determines the financing need, so that borrowing is a fraction of $M$. If the authorities change the tax rate, the relation between $M_0$ and $M$ changes, but it is still $M$ which is determined competitively, and thus $M$ appears in the equations to follow.

2.1 Case 1: No borrowing, no investment tax credit, $\theta = 1$

This first case is considered to demonstrate as simply as possible the method used in Lund (2002a), applied both to the problem solved there and to the case with decreasing returns to scale. This will show the distinction between two concepts, marginal beta and average beta, which are important in what follows. Consider the case with $\eta = 1$ (no borrowing),
\( a = 0 \) (no tax relief for \( I \) in period 0), and \( \theta = 1 \) (no tax discrimination effect in the capital market where the firm’s stock is traded).

In the case 1C (C for CRS) of a marginal project alone, considered in Lund (2002a), the cash flow to equity in period 1 is

\[
X_{1C} = PQ(1-t) + tcI,
\]

where \( Q \) in the CRS model replaces \( f(I) \) in the DRS model. For each set of tax and other parameters, \( Q \) is set so that the project is exactly marginal. The market value of a claim to this is

\[
\varphi(X_{1C}) = Q\varphi(P)(1-t) + \frac{tcI}{1 + r_f},
\]

For a marginal project the expression must be equal to the financing need after borrowing and taxes, \( I \), so that

\[
I = \varphi(X_{1C}) = Q\varphi(P)(1-t) + \frac{tcI}{1 + r_f},
\]

which implies

\[
\frac{Q\varphi(P)}{I} = \frac{1 - \frac{tc}{1 + r_f}}{1 - t}.
\]

The beta value of equity is a value-weighted average of the beta values of the elements of the cash flow. From (6) this is

\[
\beta_{X_{1C}} = \frac{Q\varphi(P)(1-t)}{\varphi(X_{1C})}\beta_P = \frac{1 + r_f - tc}{1 + r_f}\beta_P,
\]

cf. Lund (2002a), eq. (9) and (12). The main conclusion in that paper is that due to the tax depreciation schedule, the beta of equity is decreasing in the tax rate under a corporate income tax. Under a pure cash flow tax there is no such effect of the tax rate.

The intuition behind the tax effect is as follows: A pure cash flow tax does not affect the beta of equity, since it is equivalent, cash-flow-wise, to the government assuming the role of a shareholder. As compared with a cash flow tax, the typical corporate income tax postpones some deductions in the form of a tax depreciation schedule, and these will be less risky at the margin than the future operating income. Risk-wise this postponement is similar to a loan from the firm to the authorities, and thus it has the opposite effect of leverage: It reduces the systematic risk of equity. Since the result rests critically on the risk
characteristics of the tax value of depreciation deductions, the focus of the present paper is on making more realistic assumptions about the uncertainty of the firm’s tax position.

One main assumption in this and what follows is that the underlying systematic risk, $\beta_P$, is unaffected by changes in the tax system. This corresponds to the partial-equilibrium assumption made in relation to the capital market, cf. Assumption 1. The assumption is more realistic the smaller the investment project is, and the smaller the coverage of this tax system is, in relation to the market for the output.

Consider now the DRS case, 1D. Instead of technically adjusting $Q$ to find the characteristics of a marginal project, as above, there is now a first-order condition which determines $I$, and one can then solve for the fee which makes the overall addition to net value equal to zero.

The cash flow to equity in period 1 is

$$X_{1D} = Pf(I)(1 - t) + tcI,$$  \hspace{1cm} (11)

The market value of a claim to this is

$$\varphi(X_{1D}) = f(I)\varphi(P)(1 - t) + \frac{tcI}{1 + r_f}.$$  \hspace{1cm} (12)

The firm chooses the optimal scale in order to maximize $\pi_{1D}(I) = \varphi(X_{1D}) - I$. The first-order condition for a maximum is

$$f'(I)\varphi(P) = \frac{1 - \frac{tc}{1 + r_f}}{1 - t}.$$  \hspace{1cm} (13)

The fee will be set so that $M = \pi_{1D}(I) = \varphi(X_{1D}) - I$, which implies that the total payment in period 0 is

$$M + I = \varphi(X_{1D}).$$  \hspace{1cm} (14)

The beta value of equity is a value-weighted average of the beta values of the elements of the cash flow. From (11) this is

$$\beta_{X_{1D}} = \frac{Q\varphi(P)(1 - t)}{\varphi(X_{1D})} \beta_P.$$  \hspace{1cm} (15)

It is helpful here to introduce the parameterized production function, (5). This gives

$$\beta_{X_{1D}} = \frac{1 + r_f - tc}{1 + r_f - tc(1 - \alpha)} \beta_P.$$  \hspace{1cm} (16)
which again is decreasing in the tax rate as long as $c > 0$. As $\alpha$ approaches unity (i.e., CRS), $\beta_{X1D}$ approaches $\beta_{X1C}$ (and equilibrium $M$ approaches zero).

The difference between (10) and (16) is the final term in the denominator. This means that $\beta_{X1C} < \beta_{X1D}$ when $tc(1 - \alpha) > 0$. The two different expressions for the beta of equity will be called *marginal beta* and *average beta*, respectively. They are both relevant as descriptions of systematic risk within the same project. The average beta will describe the systematic risk of the project as a whole, and in particular, the systematic risk of the shares in a firm with only this project. The marginal beta is still the relevant one for decision making at the margin, which may be decentralized within the firm. After the cost $M_0$ has been sunk, the correct beta for calculating the required expected rate of return is the marginal beta.

The expressions for the two betas will be somewhat more complicated in the cases which follow, in particular when the tax position is uncertain. But the difference, the last term in the denominator, will reappear. So far we can observe that the origin of the difference is the tax depreciation schedule. Only if $tc > 0$, will the difference depend on $\alpha$. The costs $M_0$ and $I$ are treated differently by the tax system. Since $M_0$ is immediately deductible, the tax system is partly a cash flow system, partly based on a depreciation schedule. This will be realistic for many forms of “fees.” Immediate deduction is usual for licenses and patents, but also for the firm’s own R&D.

Apart from this difference, there is no fundamental distinction between $M_0$ and $I$ from the firm’s point of view. They are both paid in the same period, and the output next period, $Q$, could have been written as a function of their sum. (In order to represent the same underlying reality, this function would have the value zero for total costs less than $M_0$.) Another way to see the same point is that both betas are equal to $\beta_P$ if the tax rate is zero, but also if the tax rate is positive while $c$ is zero. In the present model with a sunk cost and DRS thereafter, it is the difference in tax treatment of these costs which creates the two different betas of equity.

When capital budgeting is presented in standard textbooks, this distinction between marginal and average beta is not mentioned. There may be good reasons for this: There are many details of projects and tax systems which have to be left out in a textbook. Lund
(2002a, 2002b) emphasizes the importance of tax systems for after-tax required expected rates of return. If firms continue to rely mainly on one such required rate for their net after-tax cash flow, they should be aware of tax effects, and not only on the value of debt, which has been the traditional focus. If they want to infer the requirement from capital market data for their own shares, they should be aware that these data (if the model is true) reflect average beta, not marginal beta. In addition to the need to “unlever” and “untax” betas, there is now a need to “unaverage” betas.

2.2 Case 2: Allow for borrowing, investment tax credit, $\theta < 1$

We now allow $\eta \in (0,1], a \in [0,1]$, and $\theta \in [0,1]$. The marginal beta is derived in equations (9) and (12) in Lund (2002a), and can be written (in a form which will be easily comparable with results to follow) as

$$\beta_{X2C} = \frac{(1 - ta)\Lambda - \frac{ct}{1 + r_f \theta}}{(1 - ta)\Lambda} \cdot \frac{\Lambda}{\eta} \beta_p,$$

where

$$\Lambda \equiv \eta + (1 - \eta) \frac{1 + r_f (1 - tg)}{1 + r_f \theta} > 0$$

gives the relative tax savings from leverage.

Consider now case 2D, the DRS version of case 2. Let the total borrowing, $B$, be the sum of two elements, $B_M = (1 - \eta)M$ and $B_I = (1 - \eta)I(1 - ta)$. The total cash flow to equity in period 1 will be

$$X_{2D} = Pf(I) - t(Pf(I) - cI - gr_f B) - (1 + r_f)B,$$

with market value in period 0 equal to

$$\varphi(X_{2D}) = \varphi(P)f(I)(1 - t) + \frac{cIt}{1 + r_f \theta} - \frac{(1 + r_f (1 - tg))B}{1 + r_f \theta}.$$

After the fee is sunk cost, however, the relevant loan is $B_I$, not $B$. The cash flow is

$$X_{2DI} = Pf(I) - t(Pf(I) - cI - gr_f B_I) - (1 + r_f)B_I,$$

with market value in period 0 equal to

$$\varphi(X_{2DI}) = \varphi(P)f(I)(1 - t) + \frac{cIt}{1 + r_f \theta} - \frac{(1 + r_f (1 - tg))B_I}{1 + r_f \theta}.$$
which can be rewritten as
\[
\varphi(X_{2D1}) = \varphi(P)f(I)(1 - t) + \frac{I}{1 + r_f\theta}[ct - (1 + r_f(1-t))](1 - \eta)(1 - ta)].
\] (23)

The net value of the project exclusive of the fee is
\[
\pi_{2D}(I) = \varphi(X_{2D1}) - I(1 - ta) + B_I = \varphi(P)f(I)(1 - t) - I \left[(1 - ta)\Lambda - \frac{ct}{1 + r_f\theta}\right].
\] (24)

This is maximized with respect to \(I\). The solution is only interesting if it yields a positive \(\pi_{2D}\). The first-order condition yields
\[
\varphi(P)f'(I) = \frac{1}{1 - t} \left[(1 - ta)\Lambda - \frac{ct}{1 + r_f\theta}\right].
\] (25)

The equilibrium after-tax value of the fee is determined by
\[
\pi_{2D}(I) = M\Lambda,
\] (26)

which is \(M\) plus the advantage (if any, or minus the disadvantage) from the \(M\)-related borrowing. It is shown in the appendix that this equilibrium equation together with the definitions given above, but without invoking the first-order condition for optimal \(I\), gives
\[
\varphi(X_{2D}) = \left[\varphi(P)f(I)(1 - t) + \frac{cIt}{1 + r_f\theta}\right] \frac{\eta}{\Lambda}.
\] (27)

This gives the denominator in the formula for the average beta in case 2. The beta is a value-weighted average of the betas of the elements of the cash flow \(X_{2D}\),
\[
\beta_{X2D} = \frac{\varphi(P)f(I)(1 - t)}{\varphi(P)f(I)(1 - t) + \frac{cIt}{1 + r_f\theta}} \cdot \frac{\Lambda}{\eta \beta_p}
\]
\[
= \left[1 + \frac{\frac{ct}{1 + r_f\theta}}{\varphi(P)f(I)(1 - t)}\right]^{-1} \frac{\Lambda}{\eta \beta_p}.
\] (28)

At this point, invoking the first-order condition (25) and introducing the constant-elasticity production function (5) gives the expression
\[
\beta_{X2D} = \frac{(1 - ta)\Lambda - \frac{ct}{1 + r_f\theta}(1 - \alpha)}{(1 - ta)\Lambda - \frac{ct}{1 + r_f\theta}(1 - \alpha)} \cdot \frac{\Lambda}{\eta \beta_p}.
\] (29)

The structure of the expression (16) for case 1 is easily recognized. So is also the difference between the DRS case and the CRS case: The difference between (29) and (17) resembles the difference between (16) and (10). Again it is clear that \(\beta_{X2D}\) is decreasing in the tax rate as long as \(c > 0\), and furthermore that \(\beta_{X2C} < \beta_{X2D}\) as long as \(ct(1 - \alpha)/(1 + r_f\theta) > 0\).
3 Extending the model: Uncertain tax position

The results for case 1 above are based on the assumption that the firm is certain to be in tax position in period 1. While the tax element $tPQ$ is perfectly correlated with the operating revenue, the values of the depreciation deduction and interest deduction were assumed to be certain, relying on the firm being in a certain tax position.

Most corporate income taxes have imperfect loss offset. If the tax base is negative one year, there is no immediate refund. The loss may under some systems be carried back or forward, but there are usually limitations to this, and the present value is not maintained. In a two-period model a realistic multi-period loss carry-forward cannot be represented in detail. An extreme assumption which may be useful as a starting point, and which is meaningful if the two-period model is taken literally, is that in these cases, there is no loss offset at all. The cash flow to equity in period 1 is then

$$PQ - B(1 + r_f) - t\chi(PQ - gBr_f - cI),$$  \hspace{1cm} (30)

where $\chi$ is an indicator variable, $\chi = 1$ when the firm is in tax position in period 1, $\chi = 0$ if not.

Lund (2002a) arrived at an analytical solution for marginal beta in this case under the assumption that the marginal investment constitutes the whole tax base for the firm.\(^{19}\) Option valuation techniques were used to find a formula for the value of the uncertain cash flow in period 1, following Ball and Bowers (1983) and Green and Talmor (1985).

At this point it is clear that a marginal beta may now take different meanings. A more realistic marginal beta recognizes that the marginal project is part of a larger activity, and that the probability of being in tax position depends on the outcome of that larger activity. This will be analyzed in line with the model of the previous section: The larger activity consists of a DRS investment project, the output of which is being sold at a single stochastic price in the single future period. An even more realistic model would include more stochastic variables (not perfectly correlated) and/or more periods.

Within this model even the marginal beta will depend on the elasticity $\alpha$. A lower elasticity means that the probability of being out of tax position is lower. But there is still a difference between the marginal and average beta.
Let case 3 denote this uncertainty case, using the simplifications \( a = 0, \eta = 1, \theta = 1 \) as in case 1. The following assumption replaces Assumption 5 above:

**Assumption 6:** The tax base in period 1 is operating revenue less \( cI \). When this is positive, there is a tax paid at a rate \( t \). When it is negative, the tax system gives no loss offset at all. The constants \( c \) and \( a \) are in the interval \([0, 1]\).

The valuation of the non-linear cash flow is specified as follows:

**Assumption 7:** A claim to a period-1 cash flow \( \max(0, P - K) \), where \( K \) is any positive constant, has a period-0 market value according to the model in McDonald and Siegel (1984). The value can be written as

\[
\varphi(P)N(x_1) - \frac{K}{1 + r_f}N(x_2),
\]

where

\[
x_1 = \frac{\ln(\varphi(P)) - \ln(K/(1 + r_f))}{\sigma} + \sigma/2, \quad x_2 = x_1 - \sigma,
\]

\( N \) is the standard normal distribution function, and \( \sigma \) is the instantaneous standard deviation of the price.\(^{20} \)

In what follows it is assumed that the exogenous variables \( \beta_P \) and \( \sigma \) can be seen as unrelated. A change in \( \sigma \) could be interpreted as, e.g., additive or multiplicative noise, stochastically independent of \( r_m \).

It is shown in the appendix that the marginal beta is

\[
\beta_{X3DM} = \frac{1 + r_f - tcN(x_{23D})}{1 + r_f} \beta_P,
\]

where \( x_{23D} \) is given by

\[
x_{23D} = \frac{1}{\sigma} \left[ \ln \left( \frac{1}{1 + r_f} \right) - \ln(1 - tN(x_{23D} + \sigma)) - \ln \left( \frac{c}{1 + r_f} \right) - \ln(\alpha) \right] - \frac{\sigma}{2}.
\]

Although this equation cannot be solved explicitly, it determines (one or more values for) \( x_{23D} \) implicitly as function(s) of \( t, c/(1 + r_f), \sigma, \) and \( \alpha \).

Furthermore it is shown that the average beta is

\[
\beta_{X3DA} = \frac{1 + r_f - tcN(x_{23D})}{1 + r_f - tcN(x_{23D})(1 - \alpha)} \beta_P.
\]
This means that the relationship between marginal and average beta is just as in the previous two cases, which had full certainty about the tax position. There is an extra term containing \( tc(1 - \alpha) \) subtracted in the denominator.

The two equations (33) and (35) should be compared with (10) and (16). Clearly the effect of the uncertainty in the tax position is similar to a reduced tax rate in period 1, reflecting that the probability of receiving the tax deductions is less than one hundred percent.

For comparison, the marginal beta in the stand-alone CRS case can be found by solving for \( x_{23C} \) from the following equation, also shown in the appendix,

\[
x_{23C} = \frac{1}{\sigma} \left[ \ln \left(1 - tN(x_{23C}) \frac{c}{1 + r_f} \right) - \ln(1 - tN(x_{23C} + \sigma)) - \ln \left( \frac{c}{1 + r_f} \right) \right] - \frac{\sigma}{2}. \tag{36}
\]

This subcase yields,

\[
\beta_{X3C} = \left(1 - tN(x_{23C}) \frac{c}{1 + r_f} \right) \beta_P. \tag{37}
\]

This is the case considered in Lund (2002a), except that equation (36) was not given there. Table 1 summarizes the seven subcases considered. The rightmost column gives the ratio of \( \beta_{Xi} \) (the beta of the cash flow to equity) to \( \beta_P \) in each subcase \( i \), defining this ratio as \( B_{Xi} \).

To find the derivatives of the betas with respect to \( t, \sigma, \alpha \) is not straightforward, since changes in \( t, \sigma, \alpha \) will affect \( x_{23D} \) and \( x_{23D} \) in ways which are hardly tractable by analytical tools. A further investigation requires numerical solutions of non-linear equations. The purpose of the investigation has been to trace out how the marginal and average betas depend on \( \alpha, t, \) and \( \sigma \).

The numerical investigation has only considered the simplified cases with \( a = 0, \eta = 1, \) and \( \theta = 1 \). The ratio \( c/(1 + r_f) \) was held fixed at 1/1.05. The central parameter configuration considered is \( t = 0.3, \sigma = 0.3 \). These are not unreasonable numbers. The five remaining equity betas for the simplified cases, divided by \( \beta_P \), are shown in Figure 1 as functions of the scale elasticity \( \alpha \). \( ^{21} \)

Figure 1 shows that the betas have the expected properties. For simplicity the verbal discussion will assume \( \beta_P = 1 \). The two dotted curves show the marginal \( \beta \) when the tax position is certain, \( \beta_{X1C} \), and also when a marginal project stands alone with uncertain
### Equations implicitly defining \( x_{23C} \) and \( x_{23D} \):

\[
\begin{align*}
x_{23C} &= \frac{1}{\sigma} \left[ \ln \left( \frac{1 - tN(x_{23C})}{1 + r_f} \right) - \ln \left( \frac{c}{1 + r_f} \right) \right] - \ln(1 - tN(x_{23C} + \sigma)) - \ln \left( \frac{c}{1 + r_f} \right) - \frac{\sigma}{2} \\
x_{23D} &= \frac{1}{\sigma} \left[ \ln \left( \frac{1 - tN(x_{23D})}{1 + r_f} \right) - \ln(1 - tN(x_{23D} + \sigma)) - \ln \left( \frac{c}{1 + r_f} \right) - \ln(1 - \alpha) \right] - \frac{\sigma}{2}
\end{align*}
\]

### Definition of \( \Lambda \):

\[
\Lambda \equiv \eta + (1 - \eta) \frac{1 + r_f(1 - t\theta)}{1 + r_f} > 0
\]

---

**Table 1:** Beta of equity for the seven subcases, divided by \( \beta_P \)
Figure 1: $B_{Xi} \equiv \beta_{Xi}/\beta_P$ as functions of scale elasticity, $\alpha$; $t = \sigma = 0.3, c/(1 + r_f) = 1/1.05$
tax position, $\beta_{X3C}$. These do not depend upon the scale elasticity, $\alpha$. They are both substantially lower than $\beta_P$, but the uncertainty of the tax position increases marginal $\beta$ from 72 percent to 83 percent. This reflects that the depreciation schedule reduces the systematic risk of equity, and that the uncertain tax position counteracts this to some extent, but not completely.

The two dashed curves show two DRS cases, the average $\beta$ when the tax position is certain, $\beta_{X1D}$, and the marginal when it is uncertain, $\beta_{X3DM}$. The former falls from unity to $\beta_{X1C}$ as $\alpha$ is increased. This simply reflects the difference in tax treatment of $M_0$ and $I$, and the fact that $M_0$ becomes relatively smaller as $\alpha \to 1^-$. $\beta_{X3DM}$, however, rises from $\beta_{X1C}$ to $\beta_{X3C}$ as $\alpha$ is increased. This follows from the increased probability of being out of tax position. When $\alpha$ is low enough, the tax position is virtually certain, and the marginal $\beta$ under DRS is not different from that under CRS and a certain tax position. But as $\alpha$ increases, so does the uncertainty about the tax position, and as $\alpha \to 1^-$, there is no gain anymore for the marginal project of being taxed together with a DRS project. It approaches the case where the marginal project is taxed alone.

The solid curve shows the average $\beta_{X3DA}$ in the DRS case with uncertain tax position. For low $\alpha$ values, the tax position is virtually certain, so there is no discernible difference from $\beta_{X1D}$ of the certain-tax-position case. Then as $\alpha$ exceeds 0.5, the effect of the uncertain tax position is that $\beta_{X3DA}$ takes on higher values than $\beta_{X1D}$, while still being decreasing in $\alpha$. For even higher $\alpha$ values, however, the curve becomes increasing, as it approaches $\beta_{X3DM}$, which is increasing. No more complete explanation is offered for the lack of monotonicity.

Clearly, even the DRS case with uncertain tax position can have betas substantially lower than $\beta_P$. In this case the marginal beta curve, $\beta_{X3DM}$, satisfies the intuition that it has less risk than the stand-alone marginal beta, $\beta_{X3C}$, as an effect of being taxed together with an infra-marginal cash flow. But the average beta, $\beta_{X3DA}$, does not exhibit this property uniformly, and in fact, the difference between marginal and average beta is just as large in this case as in the case with a certain tax position if only $\alpha$ is low enough.

Figures 2–5 show some sensitivities to changes in the tax rate, $t$, and the volatility, $\sigma$. The three non-constant curves from Figure 1 are reproduced as (similarly) dotted curves,
and the corresponding three curves for the new value of $t$ or $\sigma$ are drawn as dashed or solid. The values of the constant $\beta_{X1C}$ and $\beta_{X3C}$ are now only shown implicitly, as the endpoint values for some of the curves.\textsuperscript{22}

Figures 2 and 3 show that all betas are increased if the tax rate is lowered, and vice versa, which was also the main point in Lund (2002a) for the cases considered there. The effect on the lowest values ($\beta_{X1C}$, which is the limit of $\beta_{X3DM}$ for low $\alpha$, and of $\beta_{X1D}$ for high $\alpha$) seems to be proportional to $(1-t)$, which is almost correct when $c/(1+r_f)$ is close to unity, cf. equation (10), see also Corollary 2.2 in Lund (2002a). But the higher beta values do not change as much in absolute terms.

Figure 2: $B_{Xi} \equiv \beta_{Xi}/\beta_P$ as functions of scale elasticity, $\alpha$; varying the tax rate

\textsuperscript{22}
Figure 3: $B_{Xi} \equiv \beta_{Xi}/\beta_P$ as functions of scale elasticity, $\alpha$; varying the tax rate
Figures 4 and 5 show only one $\beta_{X1D}$ curve, as this is unaffected by a change in volatility. The figures show that a lower $\sigma$ works in the same direction as a higher $t$, except that $\beta_{X1D}$ is unaffected. But the effects of changes in $\sigma$ are only discernible for higher values of $\alpha$, and the magnitudes of the effects are not very large.

Figure 4: $B_{Xi} \equiv \beta_{Xi}/\beta_P$ as functions of scale elasticity, $\alpha$; varying the volatility

4 Discussion

The distinction between an average and a marginal beta is one of the novelties of this paper. It has been shown that this distinction should be made even if the firm pays taxes at the margin with full certainty. Since uncertainty in the model originates from only one
Figure 5: $B_{Xi} \equiv \beta_{Xi}/\beta_P$ as functions of scale elasticity, $\alpha$; varying the volatility
project-related stochastic variable, and since the project without tax has no option(-like) characteristics, there is no difference between marginal and average beta if there are no taxes. But with taxes this distinction appears, even in the simplest case with full certainty about the tax position, if there are decreasing returns to scale.

Whether the distinction between marginal and average beta is important in practice, is another question. Most firms may be happy with a rough estimate of the firm’s systematic risk, and may not worry too much about the details determining the required expected rate of return. Since different projects have different risk characteristics in practice, it is impossible to come up with an exact number to be used for a new project.

Nevertheless the mechanisms described here should be known by the practitioners, who may then evaluate if, when, and how to take them into account.

5 Conclusion

Lund (2002a) showed that even in a fully equity financed firm, the beta of equity is decreasing in the tax rate under a typical corporate income tax. The main intuition was that a tax depreciation schedule acts risk-wise in the opposite direction of leverage: It is similar to a loan from the firm to the authorities. In light of this it has been important to consider a more realistic model for the uncertainty of the firm’s tax position. The effect of a corporate income tax system on the systematic risk of equity after tax depends critically on loss offset provisions and the probability that the firm will be in tax position in future periods. This will depend on the total activities of a firm. This has been modelled as a decreasing-returns-to-scale technology, which has been acquired at an equilibrium cost, so that the total net value of the activity is zero.

When there are decreasing returns to scale after a sunk cost has been paid, and the sunk cost is immediately tax deductible (such as a license or R&D), while the subsequent investment cost is deductible according to a tax depreciation schedule, there will be a difference between the average and marginal beta of a project. The average beta will be reflected (if the model is true) in the stock market data for the firm’s stock, while the
marginal beta is relevant for each investment decision within the project. If required rates of return are to be derived from market data, this distinction has to be recognized.

When the firm is not certain to be in tax position at the margin in the future period, the valuation is similar to option valuation. Numerical techniques were used to solve for the systematic risk of equity in these cases. Even in this case the systematic risk of equity is less than the underlying systematic risk (relevant for a no-tax situation), it is decreasing in the tax rate, and increasing in the underlying volatility.

The methods and results demonstrated are crucial for discussions on reforms of corporate income taxation. Only if the authorities and firms (and other participants) agree on these methods can there be meaningful discussions. In particular, if firms continue to rely on using required expected rates of return after tax which are fixed irrespective of taxes, there may be beneficial reforms which look bad in the eyes of these firms, cf. Lund (2002c).

Appendix

Derivation of equation (27)

From equation (20) and the various definitions we get

$$
\varphi(X_{2D}) = \pi_{2D}(I) + I(1 - ta)\eta - \frac{B_M(1 + rf(1 - tg))}{1 + rf\theta}.
$$

(A1)

Using the equilibrium condition (26) for $M$ gives

$$
\varphi(X_{2D}) - I(1 - ta)\eta = M - B_M = \eta M = \frac{\eta}{\Lambda} \pi_{2D}(I).
$$

(A2)

(Thus we have been able to express $B_M(1 + rf(1 - tg))/(1 + rf\theta)$ as a fraction of $\pi_{2D}(I)$.)

Using the definition (24) of $\pi_{2D}(I)$ gives

$$
\varphi(X_{2D}) - I(1 - ta)\eta = \left[ \varphi(P)f(I)(1 - t) + I\frac{ct}{1 + rf\theta} \right] \frac{\eta}{\Lambda} - I(t - ta)\eta.
$$

(A3)

This simplifies to equation (27).
Derivation of equations (33)–(37)

This derivation starts with the average beta in case 3. In case 3 the cash flow to equity in period 1 is

\[ X_{3D} = Pf(I) - t \cdot \max(Pf(I) - cI, 0). \]  

(A4)

Under Assumption 7 the valuation, as of one period earlier, of a claim to this is

\[ \varphi(X_{3D}) = \varphi(P)f(I) - t \left[ f(I)\varphi(P)N(x_{13D}) - \frac{cI}{1 + rf}N(x_{23D}) \right], \]  

(A5)

where

\[ x_{13D} = \frac{\ln(f(I)\varphi(P)) - \ln\left(\frac{cI}{1 + rf}\right)}{\sigma} + \frac{\sigma}{2}, \]  

(A6)

and

\[ x_{23D} = x_{13D} - \sigma. \]  

(A7)

The expression in square brackets in (A5) can be rewritten in terms of the standard Black and Scholes’ formula for option pricing as \( C(f(I)\varphi(P), cI, 1, rf, \sigma) \), so that

\[ \varphi(X_{3D}) = \varphi(P)f(I) - tC(f(I)\varphi(P), cI, 1, rf, \sigma). \]  

(A8)

The \( C \) function has the derivatives \( \partial C / \partial (f(I)\varphi(P)) = N(x_{13D}) \) and \( \partial C / \partial (cI) = -N(x_{23D})/(1 + rf) \), to be used below.\(^{23}\)

The firm chooses \( I \) to maximize \( \pi_{3D}(I) \equiv \varphi(X_{3D}) - I \). From the first-order condition follows

\[ \varphi(P)f(I)(1 - tN(x_{13D})) = \frac{f(I)\left(1 - \frac{tc}{1 + rf}N(x_{23D})\right)}{f'(I)}. \]  

(A9)

Introducing the constant-elasticity production function gives

\[ \varphi(P)f(I)(1 - tN(x_{13D})) = \frac{I}{\alpha} \left(1 - \frac{tc}{1 + rf}N(x_{23D})\right). \]  

(A10)

Again, \( M \) has an equilibrium value equal to \( \pi_{3D} \), so that the total outlay for a firm to obtain the claim to the cash flow \( X_{3D} \) is \( M + I = \pi_{3D} + I = \varphi(X_{3D}) \). The claim is equivalent to holding a portfolio with \( f(I)(1 - tN(x_{13D})) \) claims on \( P \), and the rest risk free. The beta is a value-weighted average of the betas of these two elements, i.e.,

\[ \beta_{X3DA} = \frac{\varphi(P)f(I)(1 - tN(x_{13D}))}{\varphi(X_{3D})}\beta_P. \]  

(A11)
Here, the subscript $3DA$ is introduced to show that this is the average beta in case 3D. By introducing the expression for $\varphi(X_{3D})$ from (A5) and the constant-elasticity production function, this can be simplified as

$$\beta_{X_{3DA}} = \frac{1 + r_f - tcN(x_{23D})}{1 + r_f - tcN(x_{23D})(1 - \alpha)} \beta_P. \tag{A12}$$

It is also possible to express $x_{13D}$ and $x_{23D}$ in terms of exogenous variables, including the elasticity $\alpha$, avoiding the decision variables of the firm. Plug in from the first-order condition (A10) into (A6)–(A7) to find

$$x_{23D} = 1 + \frac{1}{\sigma} \left[ \ln \left( \frac{c}{1 + r_f} \right) - \ln(1 - tN(x_{23D} + \sigma)) - \ln \left( \frac{c}{1 + r_f} \right) - \ln(\alpha) \right] - \frac{\sigma}{2}. \tag{A13}$$

Although it is impossible to solve for $x_{23D}$ explicitly, equation (A13) determines (one or more values of) $x_{23D}$ implicitly as function(s) of $t, c/(1 + r_f), \sigma$, and $\alpha$.

In order to derive the marginal beta for the same case, consider first the marginal beta derived in Lund (2002a) for the case with an uncertain tax position, equation (24) in that paper. Under the simplifying assumptions $\eta = 1, a = 0, \theta = 1$, that paper’s equation (23) becomes

$$\gamma = \frac{1 - tN(x_{23C})}{1 - tN(x_{13C})} \frac{c}{1 + r_f}, \tag{A14}$$

and the marginal beta can be written

$$\beta_{X_{3C}} = \frac{1 - tN(x_{23C})}{1 - tN(x_{13C})} \frac{c}{1 + r_f} \beta_P. \tag{A15}$$

The subscript $3C$ is used here since the case considered in Lund (2002a) did not include the marginal project with some other activity, i.e., as if the case had constant returns to scale.

Again it is possible to express $x_{23C}$ in terms of the exogenous parameters. In this case there is no first-order condition for an interior profit maximum, but the definition of a marginal CRS project, which gives

$$\frac{\varphi(P)Q}{I} = \frac{1 - tN(x_{23C})}{1 - tN(x_{13C})} \frac{c}{1 + r_f}, \tag{A16}$$

cf. equations (5) and (23) in Lund (2002a). This lets us write

$$x_{23C} = 1 + \frac{1}{\sigma} \left[ \ln \left( \frac{c}{1 + r_f} \right) - \ln(1 - tN(x_{23C} + \sigma)) - \ln \left( \frac{c}{1 + r_f} \right) \right] - \frac{\sigma}{2}. \tag{A17}$$
which, not surprisingly, is the limit of (A13) as $\alpha$ tends to unity. Again, $x_{23C}$ is determined implicitly, this time as function(s) of $t, c/(1 + r_f)$, and $\sigma$.

What then about the marginal beta for the DRS case? This can be seen as a mixture of the two cases just considered. The marginal beta characterizes a small investment which has a net value of zero. Under imperfect loss offset the value will depend upon the probability of being in tax position. In particular this is crucial in case 3, for which it is assumed that after period one there are no more periods, so that the loss cannot be carried forward (nor backward). The criterion for the project being marginal looks similar to (A16), but in this case the valuation of the option-like cash flow to the marginal project in period 1 is based on the risk-adjusted probabilities $N(x_{13D})$ and $N(x_{23D})$, not $N(x_{13C})$ and $N(x_{23C})$, since they should now reflect the probabilities that the whole DRS project is in tax position at the margin. The project which invests $I$ to yield $Q$, and which is taxed together with the optimally scaled DRS project, is marginal when

$$\frac{\varphi(P)Q}{I} = \frac{1 - tN(x_{23D})c}{1 - tN(x_{13D})}.$$  \hspace{1cm} (A18)

The marginal beta in the DRS case becomes

$$\beta_{X3DM} = \frac{1 + r_f - tcN(x_{23D})}{1 + r_f} \beta_P,$$  \hspace{1cm} (A19)

with $x_{23D}$ given from (A12) above.
Notes

* This paper was written while the author was visiting the Department of Economics at Copenhagen Business School, Denmark. He is grateful for their hospitality.

1See, e.g., Hall and Jorgenson (1967), King and Fullerton (1984), and Sinn (1987).

2In corporate finance the terms weighted average cost of capital, cost of debt, and cost of equity are used for rates of return after corporate tax, and this will also be the terminology in rest of the present paper.

3Gordon and Wilson (1986) write that “depreciation deductions are normally riskfree in nominal terms” (footnote 10, p. 430).

4A marginal investment is an investment project which has zero market value if undertaken by the firm.

5The CAPM is presented in Assumption 1 and footnote 10 below.

6“In practical capital budgeting, a single discount rate is usually applied to all future cash flows,” Brealey and Myers (2003), p. 239. The survey of Graham and Harvey (2001) confirms this.

7See the discussion in Lund (2002a) p. 484 and p. 497.

8The present paper extends Lund (2002a). Lund (2002b) uses a different set of assumptions, which are not in conflict with those of Lund (2002a) or of the present paper. One main difference is that Lund (2002b) does not specify a CAPM relationship, only a more general model with value additivity, like in Modigliani and Miller (1963). It is also more general in that it does not assume multiplicative uncertainty, $PQ$ with $Q$ deterministic, as does Lund (2002a) and the present paper. On the other hand, Lund (2002b) relies on very specific multi-period profiles for production, depreciation and borrowing, and does not allow for uncertainty in tax positions or decreasing returns to scale.
The divergence between the results of Derrig (1994) and of the present paper is spelt out in Lund (2001).

Of course, $r_i$ is the rate of return of shares in firm $i$, $r_f$ is the riskless interest rate, $r_m$ is the rate of return on the market portfolio, $\beta_i \equiv \text{cov}(r_i, r_m) / \text{var}(r_m)$, and $E$ is the expectation operator. The original model is derived in Sharpe (1964), Lintner (1965), and Mossin (1966).

All variables are nominal. As long as the tax system is based on nominal values, the model is only consistent with a rate of inflation which is known with certainty, and fixed exchange rates. The underlying real CAPM would then be

$$
\frac{1 + E(r_i)}{1 + \hat{p}} = \frac{1 + r_f \theta}{1 + \hat{p}} + \beta_i \left[ \frac{1 + E(r_m)}{1 + \hat{p}} - \frac{1 + r_f \theta}{1 + \hat{p}} \right],
$$

where $\hat{p}$ is the rate of inflation.


The product price has what McDonald and Siegel (1984) call an (expected-)rate-of-return shortfall.

$\eta$ is in this sense the ratio of equity to assets after tax, as will become clear below.

Another defence for not introducing a more sophisticated theory of debt financing is that the survey by Graham and Harvey (2001) shows that most firms have a fixed target debt ratio.

The alternative valuation by elements is known as adjusted present value, APV, since Myers (1974).

To “unlever” betas is known from standard textbooks. See Lund (2002a, pp. 484, 497) on “untaxing” betas from market data when the firm operates under different tax systems, and on the mistakes which can be made if firms or authorities evaluate different tax systems using a single required expected rate of return.
If $g = 1$ (interest payments are fully tax deductible) debt may be attractive for tax reasons, but if at the same time $\theta = 1 - tg$, this exact advantage is captured in the capital market where the firm’s stock is traded, so that the firm is indifferent towards borrowing.

For simplicity, the model does not allow for different leverage of the two costs $M$ and $I$, although that might have been more realistic.

The current paper improves upon the solution for the case considered in Lund (2002a), by pointing out that the variables $x_1$ and $x_2$ used in equation (19) in that paper can be rewritten in terms of the exogenous parameters.

Lund (2002a) discusses the conditions under which the formula can be modified with $r_f\theta$ replacing $r_f$. If $\theta < 1$, this must rely on an assumption that anyone who trades in securities is more heavily taxed on their interest income than on their equity income. However, in order to simplify the presentation of the additional complications of this section, $\theta$ is set to unity in what follows.

For each numerical version of each of the nonlinear equation systems (34) and (36) the solution method identified one solution which might not be a unique solution. The program then did a grid search through 400 $x_2$ values for other solutions, but these were never found. It seems reasonable to conclude that the solutions found are likely to be unique.

So far no indications have been found that the dependency on $t$ or $\sigma$ should be non-monotonous. But these are solutions to non-linear equations, and the possibility has not been ruled out. For the parameters shown, however, there is every reason to believe that the solutions are unique.

The partial derivatives of Black and Scholes’ formula can be found, e.g., in Haug (1998), or in any textbook on option theory.
References


