Definability and reducibility in higher types over the reals

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October 31, 2003

Abstract

We consider sets $Ct_{\mathbb{R}}(\sigma)$ of total, continuous functionals of type σ over the reals. A subset $A \subseteq Ct_{\mathbb{R}}(\sigma)$ is reducible if A can be reduced to totality in one of the other spaces. We show that all Polish spaces are homeomorphic to a reducible subset of $\mathbb{R} \to \mathbb{R}$ and that the class of reducible sets is closed under the formation of function spaces and some comprehension.

1 Introduction

A *topological algebra* will be a topological space with some continuous functions. Technically, there will be a signature Σ of function symbols f of given arity, and then the algebra $\mathcal{A}$ will consist of a topological space A together with continuous interpretations $f^{\mathcal{A}} : A^n \to A$ of each function symbol f with arity n.
Examples of topological algebras are the natural numbers with zero and successor and discrete topology, $\mathbb{R}$ with plus, times, exponentials, trigonometric functions etc. and Eucledian topology, various Banach spaces etc.
Datatypes may be modelled as topological algebras. When this is the case, it is natural to enrich the algebra to one that also contains elements representing partial information. One way to do this is to use an algebraic or continuous domain, and represent the original space as a *quotient space*, as a set of *total elements* or both.

In this paper we will be concerned with hierarchies of functionals of finite types over the reals. If we consider the natural numbers as the base type, this hierarchy is well understood, with numerous characterisations. If we consider $\mathbb{R}$ as the base type, the situation is not that clear. There is no canonical way to represent the reals as a datatype, and it is not clear that different representations do not lead to different hierarchies of hereditarily total objects. This is discussed in Bauer, Escardó and Simpson [2] and in Normann [11].

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In Section 2, we will define the hierarchy of hereditarily total functionals over the reals, based on a standard algebraic domain representation of the real line, and on a standard domain-theoretical way of forming function spaces. This hierarchy also have numerous characterisations. It is equivalent to the one studied by Weihrauch and his group in his project on Type Two Enumerability, see Weihrauch [15]. The equivalence is proved combining the characterisations due to Schröder [13] of the Weihrauch hierarchy and to Normann [10] of the hierarchy in this paper as the hierarchy obtained in the category of Kuratowski limit spaces. In his thesis [4] De Jaeger gave a characterisation in the category of filter spaces.

In this paper we will define the concept of a *reducible set*, which technically will be a subset of a special kind of what we will call $Ct_{\mathbb{R}}(\sigma)$ for some finite type σ. The point is that when a structure $\mathcal{A}$ is represented as a reducible set in this way, we have access to internal notions of computability, like Escardó's Real PCF [5], for computing on elements in $\mathcal{A}$.
Blanck [3] showed how complete metric spaces in general can be represented as domains, and how effective metric spaces can be represented as effective domains. Polish spaces are characterised as the G_δ-subspaces of $[0,1]^{\mathbb{N}}$. In Section 4 we will use this to show that Polish spaces are homeomorphic to reducible subsets of $\mathbb{R} \to \mathbb{R}$. Moreover we will show that certain definable subsets of reducible sets will be reducible, and that the class of reducible sets, up to homeomorphic equivalence, is closed under the formation of function spaces. Thus our conclusion will be that the hierarchy of hereditarily total functionals over the reals is rich, and that many questions about higher type computability in analysis in general may be reduced to questions about this hierarchy.

2 Preliminaries

2.1 Types, domains and functionals

We will consider functionals of finite types over one base type 0.

Definition 1 0 is a type term.
If σ and τ are type terms, then $(\sigma \to \tau)$ is a type term.

We will use the standard observation that any type term will be of the form

$$(\sigma_1 \to (\sigma_2 \to \cdots (\sigma_k \to 0) \cdots))$$

where $k \geq 0$, and we will write $\sigma_1, \sigma_2, \ldots, \sigma_k \to 0$ for this.
We will work in the category of separable algebraic domains, and give a brief introduction:

A *dcpo* (*directed complete partial ordering*) is a partial ordering $(X, \sqsubseteq)$ such that each directed set is bounded and each bounded set Y has a least upper bound $\bigsqcup Y$.
An element $x_0 \in X$ is then *finitary* or *compact* if for each bounded $Y \subseteq X$, if

$x_0 \sqsubseteq \bigsqcup Y$ then there is a finite subset $Y_0 \subseteq Y$ such that $x_0 \sqsubseteq \bigsqcup Y_0$.
An algebraic domain is a *dcpo* $(X, \sqsubseteq)$ such that each $x \in X$ is the least upper bound of the finitary objects below it.
If $x_0 \in X$ is finitary, then $B^X(x_0) = \{y \in X \mid x_0 \sqsubseteq y\}$ is a *basic open set* in the *Scott Topology*. If $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ are two algebraic domains, the set of continuous functions from X to Y ordered pointwise form a new algebraic domain. If we let the morphisms be the continuous functions, the algebraic domains then form a cartesian closed category.
We will assume some familiarity with the theory for algebraic domains, see e.g Stoltenberg-Hansen & al. [14], Abramsky and Jung [1] or Gierz & al. [6].

We will consider a typed hierarchy over the reals:

Definition 2 **a)** We define the algebraic domain $R(\sigma)$ for each type term σ as follows:

- $R_0(0) = \{\mathbb{R}\} \cup \{[p, q] \mid p \in \mathbb{Q} \wedge q \in \mathbb{Q} \wedge p \leq q\}$ ordered by reversed inclusion.
- $R(0)$ is the set of ideals in $R_0(0)$ ordered by inclusion.
- $R(\sigma \to \tau) = R(\sigma) \to R(\tau)$ in the category of algebraic domains.

b) We define the set $\bar{R}(\sigma)$ of hereditarily total objects of type σ by recursion on σ as follows:

- $\bar{R}(0)$ is the set of ideals $\alpha \subseteq R_0(0)$ such that $\cap\alpha$ consists of one real number.
- $\bar{R}(\sigma \to \tau) = \{x \in R(\sigma \to \tau) \mid \forall y \in \bar{R}(\sigma)(x(y) \in \bar{R}(\tau))\}$.

The following was proved by Longo and Moggi [9] for the natural numbers. The proof works for this hierarchy as well, see e.g. Normann [10]

Proposition 1 *When x_1 and x_2 are elements of $\bar{R}(\sigma)$, let $x_1 \approx x_2$ when $x_1 \sqcap x_2 \in \bar{R}(\sigma)$.*
Then $\approx$ is an equivalence relation and, when $\sigma = \sigma_1 \to \tau$, we have

$$x_1 \approx x_2 \Leftrightarrow \forall y_1 \in \bar{R}(\sigma_1) \forall y_2 \in \bar{R}(\sigma_1)(y_1 \approx y_2 \to x_1(y_1) \approx x_2(y_2)).$$

The proof is fairly simple by induction on σ.
As a consequence, we may view the hierarchy of quotients as a typed hierarchy of extentional functionals:

Definition 3 For each type term σ we define the map ρ_σ defined on $\bar{R}(\sigma)$ and the set $Ct_{\mathbb{R}}(\sigma)$ of values of ρ_σ by

- If α is an ideal in $\bar{R}(0)$, then $\rho_0(\alpha)$ is the unique element in $\cap\alpha$.
  $Ct_{\mathbb{R}}(0) = \mathbb{R}$.

- If $x \in \bar{R}(\sigma \to \tau)$, we let

$$\rho_{\sigma\to\tau}(x) : Ct_{\mathbb{R}}(\sigma) \to Ct_{\mathbb{R}}(\tau)$$

be defined by

$$\rho_{\sigma\to\tau}(x)(\rho_\sigma(y)) = \rho_\tau(x(y)).$$

Proposition 2 *Each ρ_σ is well defined, and for $x_1 \in \bar{R}(\sigma)$ and $x_2 \in \bar{R}(\sigma)$ we have that*

$$x_1 \approx x_2 \Leftrightarrow \rho_\sigma(x_1) = \rho_\sigma(x_2).$$

All domains $R(\sigma)$ are effective domains. This means that the set of finitary objects is countable, and that there is an enumeration of the finitary objects such that $\sqsubseteq$ and the boundedness-relation are effective, and that the least upper bound operator for finite bounded sets of finitary objects is effective.

Proposition 3 *Let σ be a type term.*
Uniformly in σ and a finitary object $x_0 \in R(\sigma)$ there is an extension $x \in \bar{R}(\sigma)$ of x_0.

This was first proved in Normann [10]. De Jaeger's characterisation in [4] of this hierarchy in the category of filter spaces is an alternative source for the proof of this proposition, which we call the *Density Theorem.*

2.2 Topology

The topology on $Ct_{\mathbb{R}}(\sigma)$ is inherited from the quotient topology of $(\bar{R}(\sigma), \approx)$ via ρ_σ. This will be a sequential topology, i.e. it is the finest topology where all convergent sequences converge. The limit structure on $Ct_{\mathbb{R}}(\sigma \to \tau)$ can alternatively be defined in the category of limit spaces by

$$f = \lim_{n\to\infty} f_n \Leftrightarrow \forall a, \{a_n\}_{n\in\mathbb{N}} \in Ct_{\mathbb{R}}(\sigma)(a = \lim_{n\to\infty} a_n \to f(a) = \lim_{n\to\infty} f_n(a_n)).$$

If A and B are metric spaces, they are topological limit spaces, and then $f = \lim_{n\to\infty} f_n$ in $A \to B$ exactly when f is the pointwise limit of the equicontinuous sequence $\{f_n\}_{n\in\mathbb{N}}$.

Let $A \subseteq Ct_{\mathbb{R}}(\sigma)$. There are two natural ways to define a topology on A. Let T_1 be the topology inherited from $Ct_{\mathbb{R}}(\sigma)$.
Let $\bar{A} = \{x \in \bar{R}(\sigma) \mid \rho_\sigma(x) \in A\}$.
Let T_2 be the quotient topology inherited from the topology on $\bar{A}$. In general, T_2 will be a finer topology than T_1.

Lemma 1 *If A is closed or if A is open, then $T_1 = T_2$.*

The proof is easy and is left for the reader.

Definition 4 We let T_2 be the *induced topology* on $A \subseteq Ct_{\mathbb{R}}(k)$.

We then have

Lemma 2 *The topology on $A \subseteq Ct_{\mathbb{R}}(\sigma)$ is generated by its convergent sequences.*

Proposition 4 *Let $A \subseteq Ct_{\mathbb{R}}(\sigma)$, and let $f : A \to Ct_{\mathbb{R}}(\tau)$ be continuous. Then there is an $\hat{f} \in R(\sigma \to \tau)$ such that*

$$\forall x \in \bar{R}(\sigma)(x \in \bar{A} \Rightarrow \hat{f}(x) \in \bar{R}(\tau) \wedge \rho_\tau(\hat{f}(x)) = f(\rho_\sigma(x)))$$

Proof
This is a special case of a theorem in Normann [10], where A may be replaced by any $\bar{X} \subseteq X$, X is a separable domain and $\bar{X}$ is closed uppwards. We use the theorem for $X = R(\sigma)$ and $\bar{X} = \bar{A}$.

Now, if $A \subseteq Ct_{\mathbb{R}}(\sigma)$ and $B \subseteq Ct_{\mathbb{R}}(\tau)$, we let $\bar{R}(A \to B)$ be the set of $\hat{f} \in R(\sigma \to \tau)$ such that

$$\forall x \in \bar{R}(\sigma)(\rho_\sigma(x) \in A \Rightarrow \hat{f}(x) \in \bar{R}(\tau) \wedge \rho_\tau(\hat{f}(x)) \in B).$$

Consistent elements of $\bar{R}(\tau)$ will be equivalent, so when $\hat{f} \in \bar{R}(A \to B)$, $\hat{f}$ determines a continuous function $f : A \to B$. We let $\rho_{A\to B}(\hat{f}) = f$ in this case. Both A and B will be topological limit spaces, so there is a limit space structure on $A \to B$.

Lemma 3 *The limit space structure on $A \to B$ defines the identification topology induced by $\rho_{A\to B}$.*

Proof
We have to prove that the following are equivalent:

1. $f = \lim_{n\to\infty} f_n$ in $A \to B$ in the identification topology.
2. There is a convergent sequence $\hat{f} = \lim_{n\to\infty} \hat{f}_n$ in $\bar{R}(A \to B)$ such that $f = \rho_{A\to B}(\hat{f})$ and $f_n = \rho_{A\to B}(\hat{f}_n)$ for each n.
3. Whenever $a = \lim_{n\to\infty} a_n$ in A, then $f(a) = \lim_{n\to\infty} f_n(a_n)$ in B,

and use the fact that the identification topology will be generated from its set of convergent sequences.
2. $\Rightarrow$ 1. is trivial.
1. $\Rightarrow$ 3. follows from the fact that application is continuous in the identification topologies on $(A \to B) \times A$ and B.
3. $\Rightarrow$ 2. is a consequence of the theorem from Normann [10] behind Proposition 4.

2.3 Coding

In this section we will show that in many cases we may restrict our attention to the pure types:

Definition 5 As type terms, let 0 denote 0 as before, and let $k+1$ denote $(k \to 0)$. These types are called the *pure types*.

It is well known that there exist embeddings and projections between the pure types. Thus the next definition is mainly meant to settle the notation:

Definition 6 Let $i, k \in \mathbb{N}$. We will define the maps $\Phi_{i,k} \in \bar{R}(i \to k)$ as follows:

- $\Phi_{k,k}$ is the identity on $R(k)$.
- $\Phi_{0,1}(x) = \lambda y.x$.
- $\Phi_{1,0}(f) = f(0)$.
- $\Phi_{k,k+1}(x) = \lambda y.x(\Phi_{k,k-1}(y))$ when $k > 0$.
- $\Phi_{k+1,k}(F) = \lambda z.F(\Phi_{k-1,k}(z))$ when $k > 0$.
- If $|i-k| > 1$, define $\Phi_{i,k}$ as the shortest possible composition of those above.

The following is folklore:

Proposition 5 *If $i < k$, then $\Phi_{k,i}(\Phi_{i,k}(x)) = x$ for each $x \in R(i)$. Moreover, the maps $\Phi_{i,k}$ are total for all i and k.*

It is well known that $\mathbb{N}$ and $\mathbb{N}^k$ can be put in a 1-1 correspondence via a computable bijection.This can be used to justify that we only restrict attention to pure higher types, since mixed types of total functionals over $\mathbb{N}$ can be reduced to pure types.
A *retraction* on a topological space X is a continuous map $f : X \to X$ such that $f = f^2$. A *retract* of X will then be the image of a retraction on X. A retract of a Hausdorff-space will be closed. Thus the induced topology on any retract of $Ct_{\mathbb{R}}(\sigma)$ will be the subset topology.
It is well known that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic, and that $\mathbb{R}^2$ is not homeomorphic to any retract of $\mathbb{R}$. This shows that we do not have the same nice coding mechanisms at bottom level for the reals as for the natural numbers. We will see that much of the machinery needed for coding can be obtained at type 1, and that it then extends to higher types.
There will be alternative ways to define topologies on a cartesian product $\prod_{i=1}^{n} Ct_{\mathbb{R}}(k_i)$, like for subsets of $Ct_{\mathbb{R}}(k)$. The alternatives are the product topology and the one induced from $\prod_{i=1}^{n} \bar{R}(k_i)$. We will always use the latter. E.g. application will not be continuous otherwise.

Lemma 4 $(\mathbb{R} \to \mathbb{R}) \times \mathbb{R}$ *is homeomorphic to a retract of* $(\mathbb{R} \to \mathbb{R})$.

Proof
Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ be given.
We let $g = \langle f, a \rangle$ be defined by

$g(x) = f(x)$ when $x \leq 0$.

$g(x) = f(x-1) + a$ when $x \geq 1$.

$g(x) = x \cdot a$ if $0 \leq x \leq 1$.

It is easy to see that the image of this map is a retract of $\mathbb{R} \to \mathbb{R}$.
By the same method we see that $(\mathbb{R} \to \mathbb{R}) \times (\mathbb{N} \to \mathbb{R})$ and $(\mathbb{R} \to \mathbb{R})^2$ can be realized as retracts of $\mathbb{R} \to \mathbb{R}$, just cut the graphs into pieces, and glue the pieces together into one function.

This method actually extends to higher types:
Let $k > 1$. Let z_0 be the constant zero element of type $k-2$ ($z_0 = 0$ if $k = 2$).
Let $V_k = \{F \in Ct_{\mathbb{R}}(k-1) \mid F(z_0) = 0\}$, $\mathbf{1}_{k-1}$ the constant 1 functional of type $k-1$. Then every $F \in Ct_{\mathbb{R}}(k-1)$ can be uniquely described as $F = F_0 + a \cdot \mathbf{1}_{k-1}$, where $a \in \mathbb{R}$ and $F_0 \in V_k$.
There is a canonical bijection Θ between $Ct_{\mathbb{R}}(k)$ and $V_k \to (\mathbb{R} \to \mathbb{R})$ by

$$\Theta(\phi)(F_0)(a) = \phi(F_0 + a \cdot \mathbf{1}_{k-1}).$$

As an example, let us extend

$$\langle\ ,\ \rangle : (\mathbb{R} \to \mathbb{R}) \times \mathbb{R} \to (\mathbb{R} \to \mathbb{R})$$

constructed above to a function

$$\langle\ ,\ \rangle_k : Ct_{\mathbb{R}}(k) \times \mathbb{R} \to Ct_{\mathbb{R}}(k)$$

for $k \geq 2$.
Let $\phi \in Ct_{\mathbb{R}}(k)$, $a \in \mathbb{R}$. Let

$$\langle \phi, a\rangle_k (F_0 + x \cdot \mathbf{1}_{k-1}) = \langle \lambda b \in \mathbb{R}.\Theta(\phi)(F_0, b), a\rangle(x)$$

where $F_0 \in V_k$ and $x \in \mathbb{R}$. Simple calculation shows that this works.
This was just an example. Use of the same method shows:

Theorem 1 *Let $1 \leq n \leq k$.
Then $Ct_{\mathbb{R}}(k) \times Ct_{\mathbb{R}}(n)$ is homeomorphic to a retract of $Ct_{\mathbb{R}}(k)$.*

2.4 An approximation lemma

If $A \subseteq Ct_{\mathbb{R}}(k)$ and $f : A \to \mathbb{R}$ is continuous, we will use the density theorem for $Ct_{\mathbb{R}}(k+1)$ to show that f can be approximated on A by a sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions in $Ct_{\mathbb{N}}(k+1)$. In general, f cannot be extended to a total, continuous function on $Ct_{\mathbb{R}}(k)$. The approximation lemma may be used instead of an extension theorem in many situations. Our main application will be in Section 3.2. Further applications will be found in the forthcomming Normann [12].
We need a more accurate description of the finitary elements of $R(k+1)$ in the proof of the approximation lemma. If σ is a finitary element in $R(k)$ and $[p, q] \in R_0(0)$, then the pair $(\sigma, [p, q])$ defines the function $f_{(\sigma,[p,q])}$ defined by

- $f_{(\sigma,[p,q])}(x) = [p, q]$ if $\sigma \sqsubseteq x$

- $f_{(\sigma,[p,q])}(x) = \bot (= \mathbb{R})$ otherwise.

The approximation lemma will be

Theorem 2 *Let $A \subseteq Ct_{\mathbb{R}}(k)$. Then continuously in $f : A \to \mathbb{R}$ there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n : Ct_{\mathbb{R}}(k) \to \mathbb{R}$ such that whenever $x \in A$, each $x_n \in Ct_{\mathbb{R}}(k)$ and $x = \lim_{n\to\infty} x_n$, then $f(x) = \lim_{n\to\infty} f_n(x_n)$.*

Proof
Let $\bar{A} = \{x \in \bar{R}(k) \mid \rho_k(x) \in A\}$.
As discussed in section 2.2, if $f : A \to \mathbb{R}$ is continuous, there will be an $\hat{f} \in R(k+1)$ such that $\hat{f}$ maps $\bar{A}$ to $\bar{R}(0)$ and such that $f(\rho_k(x)) = \rho_0(\hat{f}(x))$ whenever $x \in \hat{A}$.

The naïve idea behind the construction is as follows:
Let $\{x_i\}_{x\in\mathbb{N}}$ be an enumeration of a dense subset of $\bar{A}$.
Depending on $\hat{f}(x_0), \ldots, \hat{f}(x_n)$ we will let certain finitary elements in $R(k+1)$ be the n'th approximation to f with some probability. Taking the weighted sum of the total extensions of these finitary elements will give us the approximation f_n to f.

First we will develop some general machinery independent of f:
Let Σ_A be the set of finitary elements σ in $R(k)$ such that σ has an extension in $\bar{A}$.
The *basic pairs* will be the set of pairs $(\sigma, [p,q])$ where $\sigma \in \Sigma_A$ and $p < q$ are rational numbers. Let $\{(\sigma_i, [p_i, q_i])\}_{i\in\mathbb{N}}$ be an enumeration of the basic pairs.
Let $X \subseteq \{(\sigma_i, [p_i, q_i])\}_{i\leq n}$. X need not be consistent, i.e. there may be $i, j \leq n$ such that σ_i and σ_j have a joint extension in $R(k)$, but $[p_i, q_i] \cap [p_j, q_j] = \emptyset$. We will employ the construction behind the lifting theorem in Normann [10] to modify X such that it becomes consistent. So let X be as above.

Let $Mod(n, X)$ be the set of $(\sigma_j, [p_j, q_j])$ such that for some $m \leq n$

- $j \leq n$
- $(\sigma_m, [p_m, q_m]) \in X$
- $\sigma_m \sqsubseteq \sigma_j$
- whenever $i < m$ and $[p_i, q_i] \cap [p_j, q_j] = \emptyset$ then σ_i and σ_j are inconsistent (as elements in $R(k)$).

Claim
$Mod(n, X)$ is consistent.

Proof
Let $(\sigma_{j_i}, [p_{j_i}, q_{j_i}]) \in Mod(n, X)$ for $i = 1, 2$ and let $m_1 \leq n$ and $m_2 \leq n$ witness that these basic pairs are in $Mod(n, X)$. We may assume that $m_1 \leq m_2$.
If $m_1 = m_2 = m$ we have that $[p_m, q_m] \subseteq [p_{j_1}, q_{j_1}] \cap [p_{j_2}, q_{j_2}]$, so there is nothing to worry about.
If $m_1 < m_2$ and $[p_{j_1}, q_{j_1}] \cap [p_{j_2}, q_{j_2}] = \emptyset$, then in particular $[p_{m_1}, q_{m_1}] \cap [p_{j_2}, q_{j_2}] =$

$\emptyset$, and then σ_{j_2} is inconsistent with σ_{m_1}. It follows that σ_{j_2} is inconsistent with σ_{j_1}.
This ends the proof of the claim.

Using Proposition 3 we let $\alpha_{n,X}$ be an extension of $Mod(n,X)$ to an element in $\bar{R}(k+1)$.
Let Π be the set of pairs (i,j) such that $i<j$ and such that σ_i and σ_j have a joint extension in $\bar{A}$ while $[p_i,q_i]\cap[p_j,q_j]=\emptyset$. Let Π_n be the set of pairs $(i,j)\in\Pi$ with $j\leq n$.
Let $(i,j)\in\Pi$. Given f, it is impossible that f extends both $(\sigma_i,[p_i,q_i])$ and $(\sigma_j,[p_j,q_j])$, but we cannot, in a continuous way, give an absolute preference to one of them. We will let f induce a probability measure on the set Δ_n of preference maps defined on Π_n. From each $\delta\in\Delta_n$, we will define the set X_δ being the approximation to some function suggested by δ. Finally we will let f_n be the weighted sum of the corresponding total objects α_{n,X_δ}.
It is about time to be more precise.

Let Δ_n be the set of maps δ defined on Π_n such that $\delta(i,j)\in\{i,j\}$.
For each $\delta\in\Delta_n$, we let X_δ be the set of basic pairs $(\sigma_i,[p_i,q_i])$ such that $i\leq n$ and such that $\delta(i,j)=i$ whenever $(i,j)\in\Pi_n$ and such that $\delta(j,i)=i$ whenever $(j,i)\in\Pi_n$. Thus X_δ is the set of $(\sigma_i,[p_i,q_i])$ such that i is always prefered by δ when this is an option.
Now, if $(i,j)\in\Pi_n$, let $x_{i,j}\in\bar{A}$ be a joint extension of σ_i and σ_j, and let $y_{i,j}=\rho_k(x_{i,j})$.
Let $g_{i,j}:\mathbb{R}\to[0,1]$ be continuous such that $g^{-1}_{i,j}(0)=[p_i,q_i]$ and $g^{-1}_{i,j}(1)=[p_j,q_j]$.
Given f, let the probability of $\delta(i,j)=j$ be $g(f(y_{i,j}))$ and let the probability of $\delta(i,j)=i$ be $1-g(f(y_{i,j}))$.
We may view Δ_n as a product

$$\Delta_n=\prod_{(i,j)\in\Pi_n}\{i,j\}$$

so f induce the product probability measure $\mu_{f,n}$ on Δ_n.
Finally, we let

$$f_n=\sum_{\delta\in\Delta_n}\mu_{f,n}\cdot\alpha_{n,X_\delta}.$$

Note that the only element of the construction that depends on f is the probability measure $\mu_{f,n}$. By construction, $\mu_{f,n}$ depends in a continuous way on f, and thus f_n depends in a continuous way on f. Actually, the construction can be seen as carried out for $f\in R(k)\to R(0)$ such that

$$\forall x\in\bar{R}(k)(\rho_k(x)\in A\Rightarrow\rho_0(f(x))\in\mathbb{R})$$

and then we construct $f_n\in\bar{R}(k+1)$ continuously in f.
It remains to show the claim on sequential continuity.
Any convergent sequence in $Ct_{\mathbb{R}}(k)$ will be the ρ_k-image of a convergent sequence

in $\bar{R}(k)$, so let $x = \lim_{n\to\infty} x_n$ be a convergent sequence from $\bar{R}(k)$ with limit in $\bar{A}$.
Let $\epsilon > 0$ be given.
Let $(\sigma, [p, q]) \sqsubseteq f$ be such that $q - p < \epsilon$ and $\sigma \sqsubseteq x$.
For some i, $(\sigma, [p, q]) = (\sigma_i, [p_i, q_i])$. Let this i be fixed for the rest of the proof.

Claim
Let $n \geq i$.

$$\mu_{f,n}(\{\delta \in \Delta_n \mid (\sigma, [p, q]) \in X_\delta\}) = 1.$$

Proof
If $(\sigma, [p, q]) \notin X_\delta$, there must be some $j \leq n$ such that $(i, j) \in \Pi$ (or $(j, i) \in \Pi$, but we will only consider the first case) and $\delta(i, j) = j$.
But then $\mu_{f,n}(\{\delta\})$ is a product where one factor is 0, since $f(x_{i,j}) \in [p_i, q_i]$.
This proves the claim.

The modification of X_δ to $mod(n, X_\delta)$ is like the construction used to prove that a continuous function from A to $\mathbb{R}$ can be realised as a partial continuous function from $R(k)$ to $\mathbb{R}$. The rest of this proof is an adjustment of that argument:

Let $j < i$ be such that $[p_i, q_i] \cap [p_j, q_j] = \emptyset$.
Then x must be inconsistent with σ_j, since otherwise $x \sqcup \sigma_j$ is a common extension y of σ_i and σ_j with $\rho_k(x) = \rho_k(y) \in A$.
Then there is an approximation τ to x inconsistent with σ_j. We may chose τ such that τ is inconsistent with all relevant σ_j for $j < i$.
For some m_0, $(\tau, [p, q]) = (\sigma_{m_0}, [p_{m_0}, q_{m_0}])$, and for some m_1, $n \geq m_1 \Rightarrow \tau \sqsubseteq x_n$.
Choose $n \geq max\{i, m_0, m_1\}$ and $\delta \in \Delta_n$ such that $\mu_{f,n}(\{\delta\}) > 0$.
Then we see that $(\tau, [p, q]) \in mod(n, X_\delta)$.
f_n will be the weighted sum of functions α_{n,X_δ} each of them sending x_n into $[p, q]$, so $f_n(x_n) \in [p, q]$.
This shows that $f(x) = \lim_{n\to\infty} f_n(x_n)$, and the proof is complete.

Corollary 1 *Let $A \subseteq Ct_{\mathbb{R}}(k)$ and let $f : A \to Ct_{\mathbb{R}}(m)$ be continuous.
Then there are functions $f_n : Ct_{\mathbb{R}}(k) \to Ct_{\mathbb{R}}(m)$ uniformly continuous in f, such that $f = \lim_{n\to\infty} f_n$ pointwise and equicontinuously on A.*

Proof
Apply the approximation lemma to $A \times Ct_{\mathbb{R}}(m - 1)$ in case $m > 0$.

3 Reducibility

3.1 Reductions

Let $N(0) = \mathbb{N}_\perp$ and let $N(\sigma \to \tau) = N(\sigma) \to N(\tau)$ in the category of algebraic domains. We may construct the Kleene-Kreisel continuous functionals $Ct_{\mathbb{N}}(\sigma)$ of type σ over the natural numbers as the extentional collapse of the hereditarily total functionals in this hierarchy, in analogy with the construction of $Ct_{\mathbb{R}}(\sigma)$. The corresponding sets $\bar{N}(\sigma)$ will be complete Π^1_k when $k > 0$ and the type rank of σ is $k+1$. This fact, and the constructions behind it, turned out, together with the density theorem, to be powerful tools while investigating the Kleene-Kreisel continuous functionals. In this section we will develop the analogue machinery for the $Ct_{\mathbb{R}}(\sigma)$-hierarchy.

Definition 7 Let $\mathbb{R}^+ = [0, \infty)$, i.e. the set of non-negative reals. Let R^+_0 consist of all closed non-empty intervals with non-negative rational endpoints, included the unbounded ones, and let R^+_0 be ordered by reversed inclusion.
The ideals in R^+_0 form an algebraic domain R^+.
We let $R^+(0) = R^+$ and we let $R^+(k+1) = R(k) \to R^+$
Some ideals in R^+ will contain a bounded interval, and then there will be a corresponding ideal in $R(0)$. By abuse of notation, we will consider these ideals to be equal, and we let $\bar{R}^+ = R^+ \cap \bar{R}(0)$.
Moreover, there will be an ideal generated by $\{[n, \infty) \mid n \in \mathbb{N}\}$. We will denote this ideal by ∞ and let $\bar{R}^+_\infty = \bar{R}^+ \cup \{\infty\}$.
We let $x \in R^+(k+1)$ be *total* if $x(y) \in \bar{R}^+$ whenever $y \in \bar{R}(k)$ and we let $x \in R^+(k+1)$ be *weakly total* if $x(y) \in \bar{R}^+_\infty$ whenever $y \in \bar{R}(k)$.

Definition 8 Let $A \subseteq Ct_{\mathbb{R}}(k)$ and let $\phi : R(k) \to R^+(k'+1)$ be continuous. We call ϕ a *reduction* if

- $\phi(x)$ is weakly total for each $x \in \bar{R}(k)$.
- For each $x \in \bar{R}(k)$ we have that $\rho_k(x) \in A \Leftrightarrow \phi(x)$ is total.

We then say that A is *reducible* to $Ct_{\mathbb{R}}(k'+1)$.

Lemma 5 *If A is reducible to $Ct_{\mathbb{R}}(k'+1)$, then A is reducible to $Ct_{\mathbb{R}}(k''+1)$ whenever $k' \leq k''$.*

Proof
Recall the projections $\Phi_{k'',k'}$ from Section 2.3. If ϕ is a reduction of A to $Ct_{\mathbb{R}}(k'+1)$, then

$$\psi(x) = \lambda y \in R(k'').\phi(x)(\Phi_{k'',k'}(y))$$

is a reduction of A to $Ct_{\mathbb{R}}(k''+1)$.

Lemma 6 *If $A \subseteq Ct_{\mathbb{R}}(k)$ and $B \subseteq Ct_{\mathbb{R}}(k)$ are reducible to $Ct_{\mathbb{R}}(k'+1)$, then $A \cap B$ is reducible to $Ct_{\mathbb{R}}(k'+1)$.*

Proof
Let ϕ_A and ϕ_B be the reductions.
Then $\psi(x) = \phi_A(x) + \phi_B(x)$ is a reduction of $A \cap B$.

For the rest of this paper, let $\{\xi^k_n\}_{n \in \mathbb{N}}$ be an effective enumeration of the dense subset of $\bar{R}(k)$ obtained from the proof of the density theorem.

Definition 9 Let $f \in Ct_{\mathbb{R}}(k+1)$. The *trace* of f is the function $h_f \in \mathbb{R}^{\mathbb{N}}$ defined by

$$h_f(n) = f(\xi_n^k).$$

$\mathbb{R}^{\mathbb{N}}$ is a metric space. We will use a bounded metric d on $\mathbb{R}^{\mathbb{N}}$, e.g. the product metric induced by $min\{1, |x-y|\}$ on $\mathbb{R}$.
Via the trace, this metric induce a metric for a weak topology on $Ct_{\mathbb{R}}(k+1)$.

Definition 10 $x \in R^+(k+1)$ is *adequate* if x is weakly total and $x(\xi_n^k) \in \bar{R}(0)$ for each $n \in \mathbb{N}$.

Lemma 7 *There is a continuous map* $\Xi_k : R^+(k+1) \to R^+(k+1)$ *such that*

- *If x is weakly total, then $\Xi_k(x)$ is adequate.*
- *x is total if and only if $\Xi_k(x)$ is total.*

Proof
We will give separate proofs for $k = 0$ and $k > 0$.
$k = 0$: $\xi_n^0 \in \mathbb{Q}$ for each $n \in \mathbb{N}$. Let $a \in [0,1]$ be irrational. Let $x \in R(0) \to R^+(0)$ and $y \in R(0)$.
We let $\Xi_0(x)(y)$ be defined as follows: Let y_0 be the maximal integer with $y_0 \leq y$, and let $z = y - y_0$. (If the data is not sufficiently accurate to identify y_0, the outcome will be $\bot$.)
If $z \leq a$, let

$$\Xi_0(x)(y) = min\{\frac{a}{a-z}, sup\{x(v) \mid y_0 \leq v \leq y\}\}.$$

If $a \leq z$, let

$$\Xi_0(x)(y) = min\{\frac{1-a}{z-a}, sup\{x(v) \mid y \leq v \leq y_0 + 1\}\}.$$

It is easy to verify that Ξ_0 is continuous and satisfies the required properties.
We have formulated the construction as if each input is total, assuming that it is clear what to do when a subconstruction based on partial input gives a partial output.

Now let $k > 0$. We will use that $Ct_{\mathbb{R}} \times Ct_{\mathbb{R}}(k)$ is homeomorphic to a retract of $Ct_{\mathbb{R}}(k)$ via $\phi : R(k) \times R(k) \to R(k)$ and $(\psi_0, \psi_1) : R(k) \to R(k) \times R(k)$. Let $y_0 \in \bar{R}(k)$ be such that y_0 is inconsistent with $\psi_0(\xi_n^k)$ for each n. Let

$$\Xi_k(x)(y) = min\{\frac{1}{d(h_{y_0}, h_{\psi_0(y)})}, x(\psi_1(y))\}.$$

The general comment about the construction for $k = 0$ is still valid.

Definition 11 A reduction $\phi \in R(k+1 \to k'+1)$ is *adequate* if $\phi(x)$ is adequate for each $x \in \bar{R}(k+1)$.

By the previous lemma, each reduction of a set A may be transformed into an adequate reduction.

Lemma 8 *If $A \subseteq Ct_{\mathbb{R}}(k)$ can be reduced to $Ct_{\mathbb{R}}(k'+1)$, then $Ct_{\mathbb{R}}(k) \setminus A$ can be reduced to $Ct_{\mathbb{R}}(k'+2)$.*

Proof
Let $\phi : R(k) \to R(k'+1)$ be an adequate reduction of A. let

$$\psi(x) = \lambda y \in R(k'+1) \frac{1}{d(h_y, h_{\phi(x)})}.$$

Let $x \in \bar{R}(k)$. If $\rho_k(x) \in A$, then $\phi(x)$ is total. Let $y = \phi(x)$. Then $\psi(x)(y) = \infty$.
If $\rho_k(x) \notin A$, then $h_{\phi(x)}$ is total since $\phi(x)$ is adequate. Since $\phi(x)$ is not total, it follows that $h_{\phi(x)} \neq h_y$ for all total y (since otherwise $y(z) = \infty$ whenever $\phi(x)(z) = \infty$ for total z). It follows that $\psi(x)$ is total.

Lemma 9 *Inequality on $Ct_{\mathbb{R}}(k)$ is reducible to $Ct_{\mathbb{R}}(0)$.*

Proof

$$x = y \Leftrightarrow h_x = h_y \Leftrightarrow \frac{1}{d(h_x, h_y)} = \infty.$$

The concept of reduction may, as above, be extended to several variables. So $A \subseteq Ct_{\mathbb{R}}(k_1) \times \cdots \times Ct_{\mathbb{R}}(k_n)$ is reducible to $Ct_{\mathbb{R}}(k)$ if there is a continuous map

$$\phi : R(k_1) \times \cdots \times R(k_n) \to R^+(k)$$

such that whenever $(x_1, \ldots, x_n) \in \bar{R}(k_1) \times \cdots \times \bar{R}(k_n)$ we have that

- $\phi(x_1, \ldots, x_n)$ is weakly total
- $\phi(x_1, \ldots, x_n)$ is total $\Leftrightarrow (\rho_{k_1}(x_1), \ldots, \rho_{k_n}(x_n)) \in A$.

We then observe

Lemma 10 *if $A \subseteq Ct_{\mathbb{R}}(k_1) \times \cdots \times Ct_{\mathbb{R}}(k_n) \times Ct_{\mathbb{R}}(k_{n+1})$ is reducible, and*

$$(x_1, \ldots, x_n) \in B \Leftrightarrow \forall y \in Ct_{\mathbb{R}}(k_{n+1})(x_1, \ldots, x_n, y) \in A,$$

then B is reducible.

Example 1 *The set of linear operators in $Ct_{\mathbb{R}}(2)$ is reducible.*

3.2 Function spaces

We will now see that the set of reducible sets is, up to homeomorphisms, closed under the formation of function spaces.

Theorem 3 *Let $A \subseteq Ct_{\mathbb{R}}(k)$ be reducible via ϕ to $Ct_{\mathbb{R}}(k'+1)$. Let $B \subseteq Ct_{\mathbb{R}}(k'')$ Then $A \to B$ with the sequential topology is homeomorphic to a set*

$$D \subseteq Ct_{\mathbb{R}}(max\{k+1, k'+1, k''\}).$$

Moreover, if also B is reducible, then D is reducible.

Proof
Let ξ_n be brief for $\xi_n^{k'}$. Assume that ϕ is adequate. Then $\phi(x)(\xi_n) \in \bar{R}(0)$ for all n and all $x \in \bar{R}(k)$.
Let $\hat{f} \in \bar{R}(A \to B)$ (defined in Section 2.2).
By the approximation lemma there is, continuously in $\hat{f}$ a sequence $\{\hat{f}_n\}_{n\in\mathbb{N}}$ such that $f = \lim_{n\to\infty} f_n$ pointwise and equicontinuously on A, where $f = \rho_{A\to B}(\hat{f})$ and $f_n = \rho_{k\to k''}(\hat{f}_n)$. Now, let $x \in \bar{R}(k)$ and $y \in \bar{R}(k'+1)$ be given. We define the measure $\mu_{x,y}$ on $\mathbb{N}$ as follows:

If $\sum_{i\leq n} |\phi(x)(\xi_i) - y(\xi_i)| < 1$, let

$$\mu_{x,y}(n) = |\phi(x)(\xi_n) - y(\xi_n)|.$$

If $\sum_{i\leq n} |\phi(x)(\xi_n) - y(\xi_i) \geq 1$, let

$$\mu_{x,y}(n+1) = 0.$$

If $\sum_{i<n} |\phi(x)(\xi_i) - y(\xi_i)| < 1$, but $\sum_{i\leq n} |\phi(x)(\xi_i) - y(\xi_i)| \geq 1$, let

$$\mu_{x,y}(n) = 1 - \sum_{i<n} |\phi(x)(\xi_i) - y(\xi_i)|.$$

Let $\psi(\hat{f})(x,y) = \hat{f}(x)$ if $(\forall i \in \mathbb{N})(\phi(x)(\xi_i) = y(\xi_i))$, and let $\psi(\hat{f})(x,y) = \sum_{i=0}^{\infty} \mu_{x,y}(i) \cdot \hat{f}_i(x)$ otherwise.

Claim 1
ψ is total and continuous in $(\hat{f}, x, y) \in \bar{R}(A \to B) \times \bar{R}(k) \times \bar{R}(k'+1)$.

Proof
Let $z \in \bar{R}(k''-1)$. (We may ignore z in this argument if $k'' = 0$.)
First assume that x and y are such that for some n, $\phi(x)(\xi_n) \neq y(\xi_n)$.
Then, for infinitely many i,

$$|\phi(x)(\xi_i) - y(\xi_i)| > \frac{|\phi(x)(\xi_n) - y(\xi_n)|}{2}$$

so

$$\sum_{n=0}^{\infty} |\phi(x)(\xi_n) - y(\xi_n)| = \infty.$$

Let $\epsilon > 0$ be given and assume that $\epsilon < 1$. Let n_0 be such that

$$\sum_{i=0}^{n_0} |\phi(x)(\xi_i) - y(\xi_i)| > 1.$$

Then

$$\psi(\hat{f})(x,y) = \sum_{i=0}^{n_0} \mu_{x,y}(i) \cdot \hat{f}_i(x) \in \mathbb{R}.$$

Thus $\psi(\hat{f})(x,y)$ is total in this case.
There will be approximations σ_0 and τ_0 to x and y such that the *partial real*

$$\sum_{i=0}^{n_0} |\phi(\sigma_0)(\xi_i) - \tau_0(\xi_i)| > 1.$$

From now on, in this proof, we let $\hat{f}'$ range over $\bar{R}(A \to B)$, x' over $\bar{R}(k)$, y' over $\bar{R}(k'+1)$ and z' over $\bar{R}(k''-1)$.
There is an approximation δ to $\hat{f}$, σ_1 to x and π_1 to z such that for all $i \leq n_0$, all $\hat{f}'$ extending δ, all x' extending σ_1 and all z' extending π_1 we have that

$$|\hat{f}_i(x)(z) - \hat{f}'_i(x')(z')| < \frac{\epsilon}{2}.$$

Let

$$M = 1 + max(\{\hat{f}_i(x)(z) \mid i \leq n_0\} \cup \{\hat{f}(x)(z)\}).$$

Given two measures μ_1 and μ_2 on $\mathbb{N}$, we let the *symmetric difference* $\mu_1 \bigtriangleup \mu_2$ be defined by

$$(\mu_1 \bigtriangleup \mu_2)(i) = |\mu_1(i) - \mu_2(i)|.$$

Then $0 \leq (\mu_1 \bigtriangleup \mu_2)(\mathbb{N}) \leq 2$.
By continuity of the construction there is an approximation σ_3 to x and an approximation τ_3 to y such that whenever x' extends σ_3 and y' extends τ_3, then

$$(\mu_{x,y} \bigtriangleup \mu_{x',y'})(\mathbb{N}) < \frac{\epsilon}{2M}.$$

It follows that if $\hat{f}'$ extends δ, x' extends $\sigma_1 \sqcup \sigma_2 \sqcup \sigma_3$, y' extends $\tau_1 \sqcup \tau_3$ and z' extends π_1, then

$$|\psi(\hat{f})(x,y)(z) - \psi(\hat{f}')(x',y')(z')|$$

$$= |\sum_{i=0}^{\infty} \mu_{x,y}(i) \cdot \hat{f}_i(x)(z) - \sum_{i=0}^{\infty} \mu_{x',y'}(i) \cdot \hat{f}'_i(x')(z')|$$

$$\leq \sum_{i=0}^{\infty} |\mu_{x,y}(i) \cdot \hat{f}_i(x)(z) - \mu_{x',y'}(i) \cdot \hat{f}'_i(x')(z')|$$

$$= \sum_{i=0}^{n_0} |\mu_{x,y}(i) \cdot \hat{f}_i(x)(z) - \mu_{x',y'}(i) \cdot \hat{f}'_i(x')(z')|$$

$$\leq (\mu_{x,y} \bigtriangleup \mu_{x',y'}) \cdot max\{|\hat{f}_i(x)(z) - \hat{f}'_i(x')(z')| \mid i \leq n_0\}$$

$$+ max\{|\hat{f}_i(x)(z) - \hat{f}'_i(x')(z')| \mid i \leq n_0\}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

This shows continuity in this case. Totality in this case is trivial

Now assume that $\phi(x)(\xi_n) = y(\xi_n)$ for all n. Then $\rho_k(x) \in A$. Let $z \in \bar{R}(k''-1)$ (which we still may ignore if $k'' = 0$).
Let $\epsilon > 0$ be given.
Let $n_1 \in \mathbb{N}$, $\delta_1 \sqsubseteq \hat{f}$, $\sigma_1 \sqsubseteq x$ and $\pi_1 \sqsubseteq z$ be such that if $n \geq n_1$, $\sigma_1 \sqsubseteq x'$, $\pi_1 \sqsubseteq z'$ and $\delta_1 \sqsubseteq f'$ then

$$|\hat{f}'_n(x')(z') - \hat{f}(x)(z)| < \frac{\epsilon}{2}.$$

Let $M = max\{\hat{f}_n(x), \hat{f}(x) \mid n \leq n_1\} + 1$.
Let $\sigma_2 \sqsubseteq x$ and $\tau_2 \sqsubseteq y$ be such that if $\sigma_2 \sqsubseteq x'$ and $\tau_2 \sqsubseteq y'$ then

$$\sum_{n \leq n_1} |\phi(x')(\xi_n) - y(\xi_n)| < \frac{\epsilon}{4M}.$$

It follows that if $\delta_1 \sqsubseteq \hat{f}'$, $\sigma_1 \sqcup \sigma_2 \sqsubseteq x'$, $\tau_2 \sqsubseteq y'$ and $\pi_1 \sqsubseteq z'$ then

$$|\psi(\hat{f})(x,y)(z) - \psi(\hat{f}')(x',y')(z')|$$

$$= |\hat{f}(x)(z) - \sum_{i=0}^{\infty} \mu_{x',y'}(i) \cdot \hat{f}'_i(x')(z')|$$

$$\leq \sum_{i=0}^{\infty} \mu_{x',y'}(i) \cdot |\hat{f}(x)(z) - \hat{f}'_i(x')(z')|$$

$$\leq \sum_{i=0}^{\infty} \mu_{x',y'}(i) \cdot |\hat{f}(x)(z) - \hat{f}'_i(x')(z')|$$

$$= \sum_{i=0}^{n_1} \mu_{x',y'}(i) \cdot |\hat{f}(x)(z) - \hat{f}'_i(x')(z')|$$

$$+ \sum_{i=n_1+1}^{\infty} \mu_{x',y'}(i) \cdot |\hat{f}(x)(z) - \hat{f}'_i(x')(z')|$$

$$\leq \frac{\epsilon}{4M} \cdot 2M + \frac{\epsilon}{2} = \epsilon.$$

This proves continuity in this case, and Claim 1 is proved.

From now on, let $\psi \in R((k \to k'') \to (k, k'+1 \to k''))$ be continuous and as constructed on $\bar{R}(A \to B) \times \bar{R}(k) \times \bar{R}(k'+1)$. Let

$$\tilde{C} = \{\lambda x \in R(k).\lambda y \in R(k'+1).\psi(\hat{f})(x,y) \mid f \in \bar{R}(A \to B)\}.$$

and let

$$C = \{\rho_{k,k'+1\to k''}(g) \mid g \in \tilde{C}\}.$$

Let $\bar{C}$ be the set of elements in $\bar{R}(k, k'+1 \to k'')$ equivalent to an element in

$\tilde{C}$.
For $u \in \bar{C}$, let $\hat{t}_u(x) = u(x, \phi(x))$. Then $\hat{t}_u \in \bar{R}(A \to B)$. The function $u \mapsto \hat{t}_u$ is continuous, and whenever $\hat{f} \in \bar{R}(A \to B)$ we have that $\hat{t}_{\psi(\hat{f})}$ and $\hat{f}$ are consistent, so

$$\rho_{A \to B}(\hat{t}_{\psi(\hat{f})}) = \rho_{A \to B}(\hat{f}).$$

But then C and $A \to B$ will be homeomorphic via the continuous maps represented by ψ and $u \mapsto \hat{t}_u$.

C is a set in $(Ct_{\mathbb{R}}(k) \times Ct_{\mathbb{R}}(k'+1)) \to Ct_{\mathbb{R}}(k'')$.
This set is homeomorphic to a retract of $Ct_{\mathbb{R}}(max\{k+1, k'+2, k''\})$, and we let D be the image of C under this homeomorphism.

Now assume that B can be reduced to $Ct_{\mathbb{R}}(k''')$ via ϕ'. In order to prove that D also is reducible, we show that D is definable in the apropriate way. We will introduce some notation and some functions:
Let $n = max\{k+1, k'+2, k''\}$.
Let

$$\Xi : ((R(k) \times R(k'+1)) \to R(k'')) \to R(n)$$

and

$$\Xi^d : R(n) \to ((R(k) \times R(k'+1)) \to R(k''))$$

be total such that $\Xi^d(\Xi(g)) \approx g$ for all total g in $R(k, k'+1 \to k'')$.
Then

$$D = \{\rho_n(\Xi(g)) \mid \rho_{k,k'+1 \to k''}(g) \in C\}.$$

For $h \in \bar{R}(n)$ we then have

$$\rho_n(h) \in D \Leftrightarrow \rho_{k,k'+1 \to k''}(\Xi^d(h)) \in C \wedge h \approx \Xi(\Xi^d(h)). \tag{1}$$

The second part of (1) is reducible. The first part of (1) is equivalent to

$$(\exists f \in R(k) \to R(k''))(\rho_{A \to B}(f) \in A \to B \wedge (\Xi^d(h) \approx \psi(f))). \tag{2}$$

Let $g_h = \Xi^d(h)$. If there is a f satisfying (2), then it can be recovered up to equivalence from h as $f_h(x) = g_h(x, \phi(x))$, so (2) is equivalent to

$$f_h \in \bar{R}(A \to B) \wedge g_h \approx \psi(f_h). \tag{3}$$

The second part of (3) is reducible. The first part of (3) is equivalent to

$$\forall x \in \bar{R}(k)(\rho_k(x) \notin A \vee \phi'(f_h(x)) \in \bar{R}(k''')), \tag{4}$$

which is equivalent to

$$(\forall x \in \bar{R}(k''))(\forall y \in \bar{R}(k''))\exists m(\phi(x)(\xi_m^{k'}) \neq y(\xi_m^{k'}) \vee \phi'(f_h(x)) \in \bar{R}(k''')). \tag{5}$$

Using the approximation lemma, let the sequence $\{f_{h,n}\}_{n \in \mathbb{N}}$ be obtained by applying the construction in the proof of the lemma to f_h. Let $\mu_{x,y}^h$ be defined from f_h as above. We let

$$\psi'(h)(x,y) = \sum_{i=0}^{\infty} \mu_{x,y}^h(i) \cdot \phi'(f_{h,i}(x))$$

if $\exists m(\phi(x)(\xi_m^{k'}) \neq y(\xi_m^{k'}))$, and we let

$$\psi'(h)(x,y) = \phi'(f_h(x))$$

otherwise.
We can show in analogy with previous arguments that ψ' is continuous and is a reduction of

$$\{\rho_n(h) \mid h \in \bar{R}(n) \wedge f_h \in \bar{R}(A \to B)\}.$$

This ends the proof of the theorem. Though the constructions of the reductions are explicit, we have made no effort in bringing the type level down.
There is a clear connection between our reductions and constructions of realizers, but we have made no deep exploration of this connection.

4 Polish spaces

A Polish Space is a separable topological space that admits a complete metric. We reccomend Hoffmann-Jørgensen [7] or Kechris [8] as standard references on Polish spaces.
A useful characterisation is that the Polish spaces are exactly the topological spaces homeomorphic to G_δ subsets of $[0,1]^{\mathbb{N}}$. Moreover, a G_δ subspace of a Polish space will be Polish.
$[0,1]^{\mathbb{N}}$ is homeomorphic to a G_δ subspace of $\mathbb{R} \to \mathbb{R}$, which itself is Polish, so the Polish spaces may as well be characterised via the G_δ subspaces of $\mathbb{R} \to \mathbb{R}$.

Lemma 11 *Let $A \subseteq Ct_{\mathbb{R}}(1)$.*
Then A is reducible to $Ct_{\mathbb{R}}(1)$ if and only if A is G_δ.

Proof
Let A be G_δ, i.e.

$$A = \bigcap_{n \in \mathbb{N}} A_n$$

where each A_n is open. We may assume that $A_{n+1} \subseteq A_n$.
Let $B_n = (\mathbb{R} \to \mathbb{R}) \setminus A_n$. Consider the sequence $\{\frac{1}{d(f,B_n)}\}_{n \in \mathbb{N}}$ for $f \in \mathbb{R} \to \mathbb{R}$.
If $f \in A$, this defines a sequence of reals, while if $f \not\in A$ this sequence is ∞ from one n on.
We code this sequence into a function $\phi(f)$ by

- $\phi(f)(x) = 0$ for $x \leq 0$.
- $\phi(f)(k \cdot \frac{\pi}{2}) = \frac{1}{d(f,B_n)}$.
- Between $k \cdot \frac{\pi}{2}$ and $(k+1) \cdot \frac{\pi}{2}$ the function $\phi(f)$ increases like the tangent function until it reach the value $\frac{1}{d(f,B_{n+1})}$.

Then A is reducible via ϕ

In order to prove the converse, let A be reducible to $Ct_{\mathbb{R}}(1)$ via ϕ. Then

$$f \in A \Leftrightarrow \forall x(\phi(f)(x) < \infty) \Leftrightarrow \forall n \exists m \forall x(|x| \leq n \Rightarrow |\phi(f)(x)| < m).$$

It is well known that the topology on $\mathbb{R} \to \mathbb{R}$ is the compact-open topology, so

$$Q_{n,m} = \{f \mid |x| \leq n \Rightarrow |\phi(f)(x)| < m\}$$

is open. The lemma follows.

We then have all the ingredients needed to prove

Theorem 4 *Let $P_1, \ldots, P_n$ be Polish spaces.*
Let X be a higher type space obtained in the category of limit spaces by closing $\{P_1, \ldots, P_n\}$ under the formation of function spaces. Then X is homeomorphic to a reducible subset of some $Ct_{\mathbb{R}}(k)$.

Remark 1 If we start with Polish algebras instead, we may also close the hierarchy under definable subspaces.

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