A rich hierarchy of functionals of finite types

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Abstract

We are considering typed hierarchies of total, continuous functionals over base types that are complete, separable metric spaces. P. Urysohn [17, 18] constructed a complete, separable metric space U. One of the properties of U is that every other separable metric space can be isometrically embedded into U.
We discuss why U may be considered as the universal model of possibly infintary outputs of algorithms, and show that all our typed hierarchies may be topologically embedded, type by type, into the corresponding hierarchy over U.
Restricting our base types to effective, separable Banach spaces, we also prove a density theorem and an effective embedding theorem. These are our main technical results.

[[This is a draft version. Comments are welcome]]

1 Introduction

Acknowledgments

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2 Three kinds of data

2.1 Discussion

One of the important paradigms of the theory of computing, and of that of computability, is that we may view algorithms and programs as data. We are not going to challenge this paradigm. The paradigm is important practically in the design of digital computers, where everything, input data, programs and output data deep down are just sets of bits and bytes. It is also important theoretically, as it makes the existence of a universal algorithm possible and the unsolvability of the halting problem a mathematical statement.

However, using almost any programming language in practice, we have to distinguish between input data and output data, or at least declare what is what, and the programs are considered as syntactical entities that for most cases live separated from other kinds of data.

In this paper we will be interested in models for computing where the input data and the output data may be infinite entities. As a simple, but basic example, let us discuss the operator

$$I(f) = \int_0^1 f(x)dx$$

and how we should construct mathematical models for the kinds of data involved in computing integrals. Of course, in the world of digital computers, what we will aim at is to compute the integral as a floating point, and then the input function f has to be digitally represented in some way suitable for this aim. From the point of view of numerical analysis, this is not hard to achieve, and in fact, the computability of the integral is not a big issue. From the point of view of a conceptual analysis, it is however unfortunate to make the leap all in once from the set theoretical world of mathematical analysis to the finitistic world of digital computers. There are several reasons for this. We will discuss two of them:

1. The step from the continuous to the discrete inevitably has to violate some of the geometrical, algebraic and analytical properties of the reals. Unless one shows some care, it is not obvious that

   $$\int_0^1 (x^2 + x^5)dx = \int_0^1 x^2dx + \int_0^1 x^5dx$$

   as numerically calculated integrals, and there are certainly going to be algebraically valid identities of this sort that are not identities in the numerical interpretation. Though the practical harm of phenomena like this may be kept at a minimum, it will be nice to have a model of computability in analysis that does not suffer from such deficiencies.

2. Though technological standards for representing various kinds of data are important for the exchange of data and programs, a conceptual analysis of computability where data of the form *reals* and *real valued functions* appear, should not be restricted to a particular standard for digitalization.

It is of course impossible to view a real as the genuine output of an algorithm, since such outputs, even in a mathematical model, should be of a finitistic nature. An algebraic expression denoting a real may be considered to be such a finitistic entity, but then we will be facing the problem of the meaning of calculating the value of expressions like this. In the literature, there are essentially two ways of representing reals taking our concerns into account, representing reals as sequences of digits and representing reals via sets of approximations. It is well known since Turing [16] that the decimal representation of reals is unsuitable for modeling algorithms. There are however other ways to represent reals as sequences of natural numbers that are fruitful, see e.g. Weihrauch [20] for a further discussion. When analysis is viewed as constructive or classical 2^{nd} order number theory, the reals will normally be represented as sequences of numbers coding Cauchy sequences of rational numbers with a fixed effective modulus of convergency.

We will view output data as data of a particular kind, and we will advice some care in the choice of representing such data. Of course we have to consider more than just the set of data, we have to consider approximations to these data as well. But, and this is the core of our view, since it is the output data themselves that are of importance, the structure used to model the outputs of algorithms computing such data should contain the output data we are really interested in as a kind of substructure. We may view an algorithm computing a real as running in infinite time, producing better and better approximations as time passes, but in the end, in an ideal world, and after possibly an infinite elapse of time, the output should be the real itself.

If we consider the complete partial ordering (cpo) of all closed intervals ordered by reversed inclusion, we may identify a real x with the closed interval $[x,x]$.

If we want to stick to finitistic representations of approximations of reals, e.g. as closed intervals $[p,q]$ with rational endpoints or as closed intervals $[\frac{n}{2^k},\frac{m}{2^k}]$

with dyadic endpoints, and represent a real as an ideal of such approximations, we may canonically represent a real x as the ideal of all approximating intervals with x in the interior.
This latter kind of representation is known as a *retract domain representation*, and we will come back to this.

In our example of the integral, there are two other kinds of data that may concern us, those of the input function f and of the integration operator itself. Now, a function f will not be the final answer to an algorithm running in infinite time, since we envisage that further arguments x are given in order to obtain final outputs of the form $f(x)$. The function f may be given in many ways, and it may even be undecidable if two definitions of continuous functions define the same function or not.

Input data will also be equipped with a notion of approximating data, but as inputs may be given intentionally or extentionally and in many various ways of equal importance, we see no reason to insist that the external interpretation of an input should be identified with one particular representation. In this paper, we will assume that inputs are taken from a topological space of a nature to be specified later, and that they are represented by elements in algebraic domains of a particular kind, but our motivations for doing so will be pragmatic, this is the kind of representations that permits us to carry out the mathematical investigations we want to carry out.

The spaces used to give denotational semantics to algorithms will be substructures of function spaces. Again, our choice of representing spaces will be more pragmatic than conceptually based. Of course, any space of interpretations of algorithms may be viewed as a space of possible inputs, but the converse need not be the case.
In our example, integration is interpreted as a continuous function I from a topological space of approximations to continuous functions $f : \mathbb{R} \to \mathbb{R}$ to a space of approximations of reals such that if the argument ϕ fully determines a function f, then

$$I(\phi) = \int_0^1 f(x)dx.$$

2.2 Representing output data

Blanck [4, 5] carried out some pioneering work on the use of domain theory for representing topological spaces. Though we ad some conceptual analysis, the technical definitions and results of this section are due to Blanck. We have to assume some familiarity with basic domain theory, see e.g. Abramsky and Jung [1], Stoltenberg-Hansen & al. [14] or Amadiou and Curien [2] for introductions to the subject.

If X is a topological space, then a *domain representation* of X will consist of an algebraic domain D, a set $D^R \subseteq D$ of *representing objects* and a continuous surjection $\delta : D^R \to X$. If we let D_0 be the set of *compact* or *finitary* elements of D, we may view the elements of D_0 as *approximations* to the elements of X.

Now, if X is a set of ideal *output data*, the elements of D_0 may be chosen as the possible intermediate approximative values obtained through the computation of some element x of X. If we view this set of approximations as an extension of X, it is natural to identify each $x \in X$ with some canonical set of approximations of x, preferably a set that in some abstract sense can "be computed" from x itself. This leads us to consider the *retract representations*, i.e. representations where there is an inverse $\nu : X \to D^R$ such that $\delta(\nu(x)) = x$ for all $x \in X$.

Finally, an output should contain complete information in the sense that it should not be compatible with other output values that accept sharper approximations. This leads us to consider *upwards closed* representations, i.e. representations where if $\alpha \in D^R$ and $\alpha \sqsubseteq \beta$ then $\beta \in D^R$ and $\delta(\alpha) = \delta(\beta)$.

Blanck [5] proved that if a topological space X accepts an upwards closed retract representation, then X is a regular space, and in fact it is normal. Then, as an application of the Urysohn metrization theorem, a space like that will be metrizable. They are of course separable since each D_0 is countable.

Blanck [4] showed how we can construct a representation of each separable metric space, and this representation will indeed be an upwards closed retract representation. Since we in later sections will want to refer to Blanck's construction, we give some of the details here.

Definition 2.1 Let $\langle X, d\rangle$ be a separable metric space with a countable dense subset $\{a_n \mid n \in \mathbb{N}\}$.

a) For each $n \in \mathbb{N}$ and positive rational number r, let

$$B_{n,r} = \{x \in X \mid d(x, a_n) \leq r\},$$

i.e. the *closed* ball of radius r around a_n.

b) Let $D_0 = D_0^X$ be the set of finite sets of such closed balls, such that whenever $B_{n,r}$ and $B_{m,q}$ are in the set, then $p + q \leq d(a_n, a_m)$. (The balls at least have a potential of a non-empty interval.)

c) If K and L are in D_0, we let $K \sqsubseteq L$ if all balls in L are contained in some balls in K.

d) An ideal $\mathcal{I}$ in D_0 *represents* $x \in X$ if

   i) $x \in B_{n,r}$ whenever $B_{n.r} \in K \in \mathcal{I}$.

   ii) For each $\epsilon > 0$ there is a $K \in \mathcal{I}$ such that all balls in K have radii $< \epsilon$.

e) We let $D = D^X$ be the ideal completion of D_0, i.e. the set of ideals ordered by inclusion.

This construction may seem unnecessary complicated, but something of this complexity is required if one wants to construct an effective domain representation uniformly from an effective metric space.
Like all domains, D^X is equipped with the *Scott Topology*, where a typical element of the basis will consist of all ideals containing some fixed element of D_0. Then the map sending a representative for $x \in X$ to x will be continuous. Now, an element $x \in X$ may have more than one representative, but there will always be a least one in the inclusion ordering of the ideals. If $x \in X$ and x is in the interior of all balls in $K \in D_0$, then K must be an element of all ideals representing x. Since the set of such clusters of balls forms an ideal representing x, it is the least representing ideal. What is important to us is that the map sending $x \in X$ to the least representing ideal is continuous. Thus, X is homeomorphic to a subspace of the representing space D^X.
Also observe that if $\mathcal{I} \subseteq \mathcal{J}$ are two ideals, and if $\mathcal{I}$ represents $x \in A$, then $\mathcal{J}$ represents x. Moreover, due to the fact that metric spaces are Hausdorff, the same ideal may not represent two different elements of A. Blanck's construction is that of an upwards closed retract representation.

A simpler approach

If we are not concerned with effectivity, we may construct the representing domain based on non-empty finite intersections of those closed balls we used. Then we automatically get a dense retract representation that is upwards closed. This approach will be taken in Section 6.

3 The Urysohn Space

In section 2 we were primarily interested in mathematical models for data-types where the data could be viewed as the ultimate outputs of algorithms running in infinite time, and we observed that we may use separable metrizable or metric spaces for this purpose. Given some metric spaces as basic data-types, we will then be interested in derived data-types, where the objects in a sense are operators with ultimate values in metric spaces. In this paper, we will be mainly interested in hereditarily total objects of this kind, but of course, if one is interested in functional programming where such base types are involved, the hereditarily partial operators are essential for the construction of denotational semantics. Urysohn [17, 18] showed that there is a richest separable metric space, the so called *Urysohn space*, and the main aim of this paper is to show that any space of hereditarily total continuous functionals over any set of complete separable metric spaces can be topologically embedded into a space of functionals of the same type, but now over just the Urysohn space.

In order to be able to prove our results, we have to refer to the basic original properties of this space and to some of the more recent results about it.

Definition 3.1 Let X be a metric space. We call X *finitely saturated* if whenever $K \subset L$ are finite metric spaces, and $\phi : K \to X$ is a metric-preserving map, then ϕ can be extended to a metric-preserving map ψ from L to X.

Remark 3.2 The word *saturated* is common in model theory for this kind of phenomenon, so we adopt it here.

Urysohn proved that there exists a complete, separable metric space U that is finitely saturated, and that, up to isometric equivalence, there is exactly one such space. This space is known as the *Urysohn space*, although the term *a Urysohn Space* is used with a different meaning, a concept that is irrelevant to us.

Urysohn actually gave an explicit construction, as the completion of a countable metric space where all distances are rational numbers, and which is saturated with respect to pairs of finite spaces with rational distances. As constructed this way, U will be an effective metric space.

One important aspect of the Urysohn space is that every other separable

metric space A can be embedded isometrically into U, and if X is an effective space, the embedding will be effective.
There has been a renewed interest in the Urysohn space over the last 25 years. Two results are of importance to us:
We say that a metric space X is *compactly saturated* if whenever $C \subseteq Y$ are two separable metric spaces with C compact, and $\phi : C \to X$ is an isometric embedding, then ϕ can be extended to an isometric embedding of Y into X.
U is compactly saturated. This was proved by Huhunaishvili [9], see alo Bogatyi [6, 7]. The proof of this theorem can be made effective. Uspenskij [19] shows that U as a topological space is homeomorphic to the Hilbert space l_2, and in fact to any other separable Hilbert space of infinite dimension. Uspenskij depends on a characterization of the class of topological spaces homeomorphic to Hilbert spaces due to Toruńczyk [15].

The combined Toruńczyk - Uspenskij proof leaves no information about if this result is constructive in any sense.

In the case of choosing a domain representation for the Urysohn space, the two approaches discussed in Section 3 are equivalent.

4 A category of *qcb*'s

Adopting the convention from Battenfeld, Schröder and Simpson [3] we say that a topological space X is a *qcb*-space if it is T_0 and can be viewed as the quotient space of an equivalence relation on a space with a countable base. The corresponding category QCB is, in some sense, the richest category of topological spaces that can be handled with decency using domain theory. Schröder introduced the concept of a pseudobase, see e.g [12]:

Definition 4.1 Let X be a topological space. A *pseudobase* for X is a family $\mathbb{P}$ of non-empty subsets of X closed under finite nonempty intersections such that whenever $x = \lim x_n$ in X and $x \in O$ where $O \subseteq X$ is open, there is an element $p \in \mathbb{P}$ such that

i) $x \in p \subseteq O$

ii) $x_n \in p$ for almost all $n \in \mathbb{N}$.

A topological space is *sequential* if the topoogy is the finest one where the convergent sequences indeed are convergent. Schröder showed that all *qcb*-spaces will admit countable pseudobases, and that this is a characterization for sequential T_0-spaces. If we consider the Blanck representation of separable metric spaces, we may form a pseudobase from the set of finitary objects, which is a set of clusters of closed balls, by letting the pseudobase elements be all nonempty intersections of such clusters. These pseudo-base elements will be closed.

In Section 2 we admitted that we would be more pragmatic when modeling input spaces and spaces of interpretations of algorithms than when modeling output spaces. We are going to work with a subcategory of QCB.

Definition 4.2 Let $\mathcal{Q}$ be the class of sequential Hausdorff spaces that permit a countable pseudobase of closed sets.

For our next result, we need the concept of an admissible domain representation due to Hamrin [8], based on a similar concept due to Schröder [11, 12], see also Weihrauch [20]:

Definition 4.3 Let $\langle D, D^R, \delta\rangle$ be a representation of the space X. We call the representation *admissible* if for every dense representation $\langle E, E^R, \pi\rangle$ of a space Y and every continuous function $f : Y \to X$ there is a continuous function $\phi : E \to D$ such that ϕ maps E^R into D^R and such that

$$\delta(\phi(e)) = f(\pi(e))$$

for all $e \in E^R$.

Remark 4.4 If $\langle D, D^R, \delta\rangle$ is an admissible representation of X and $x = \lim_{n\to\infty} x_n$, there will be a convergent sequence $\alpha = \lim_{n\to\infty} \alpha_n$ in D^R with $x = \delta(\alpha)$ and $x_n = \delta(\alpha_n)$ for each $n \in \mathbb{N}$.
We call this a *lifting* of the convergent sequence, and the existence of a lifting is easy to prove given an admissible representation. This is a standard observation.

Lemma 4.5 *Every space in $\mathcal{Q}$ has an upwards closed admissible representation.*

Proof
Let $X \in \mathcal{Q}$ and let $\mathbb{P}$ be a countable pseudobase of closed subsets of X. We apply the argument from Hamrin [8], and assume w.l.o.g. that $\mathbb{P}$ is closed under finite unions. Then the ideal completion $\langle D, \sqsubseteq\rangle$ of $\langle \mathbb{P}, \supseteq\rangle$ offers an

admissible representation of X, where each $x \in X$ is represented by the elements of

$$D_x^R = \{\alpha \in D \mid \forall p \in \alpha(x \in p) \wedge \forall O \text{ open}(x \in O \Rightarrow \exists p \in \alpha(x \in p \subseteq O))\}.$$

By Hamrin [8] this is an admissible representation, and we are left with showing that D_x^R is upwards closed.
If $\alpha \in D_x^R$ and $\alpha \subseteq \beta \in D$, the second requirement for $\beta \in D_x^R$ is trivially satisfied. Now, let $q \in \beta$ and assume that $x \notin q$. Then $x \in X \setminus q$ is open, so

$$\exists p \in \alpha(x \in p \subseteq X \setminus q).$$

Then $p \cap q \in \beta$ since β is an ideal. But $p \cap q = \emptyset$ and β will only contain nonempty sets. This is a contradiction, so $x \in q$. This ends the proof of the lemma.

These spaces are sequential, which means that the topology will be the finest topology where all convergent sequences do converge. This offers a natural topology on the function spaces $X \to Y$ of continuous functions, induced by the limit-space construction

$$f = \lim_{n\to\infty} f_n \Leftrightarrow \forall(x = \lim_{n\to\infty} x_n)(f(x) = \lim_{n\to\infty} f_n(x_n)).$$

Lemma 4.6 *If X and Y are in $\mathcal{Q}$, then $X \to Y \in \mathcal{Q}$.*

Proof
Let $p_1, \ldots, p_n$ be closed pseudobase elements in X and $q_1, \ldots, q_n$ be closed pseudobase elements in Y such that for all $K \subseteq \{1, \ldots, n\}$,

$$\bigcap_{k\in K} p_k \neq \emptyset \Rightarrow \bigcap_{k\in K} q_k \neq \emptyset.$$

Let

$$B_{\{\langle p_1,q_1\rangle,\ldots\langle p_n,q_n\rangle\}} = \{f \mid \forall k \leq n(f[p_k] \subseteq q_k)\}.$$

This will form a pseudobase of closed sets for $X \to Y$.

Remark 4.7 We do not use that X is in $\mathcal{Q}$, only that X is a *qcb*.

Using continuous functions as morphisms, we may view $\mathcal{Q}$ as a category. Our key examples will be the spaces we may obtain from complete, separable metric spaces closing under the function space construction. It is known, see Schröder [13], that these spaces need not be regular (or normal) spaces. It is easy to see that all topologies are Hausdorff. We will be interested in the finest regular (or normal, this amounts to the same in this case) subtopology of the sequential one:

Definition 4.8 Let $X \in \mathcal{Q}$ and let $A \subseteq X$.
We say that A is *a zero-set* if there is a continuous map $f : X \to [0,1]$ such that

$$x \in A \Leftrightarrow f(x) = 0.$$

It is not hard to show that the class of complements of zero-sets forms a regular subtopology on X. The fact that the topology on X is hereditarily Lindelöf is useful in showing that this class is closed under arbitrary unions. These concepts will be important in Sections 6 and 7.

Summarizing our views on how to give a semantics for the three kinds of data, we get

- We will accept separable metric spaces X as spaces of complete and extensional outputs.
- We will accept any space $X \in \mathcal{Q}$ as a space of total, extensional inputs.
- We will accept subspaces of the spaces $X \to U$ that are zero-sets as spaces of extensional interpretations of algorithms, where $X \in \mathcal{Q}$ and U is the Urysohn space.

In the sequel we will use the fact that if $X \in \mathcal{Q}$ with a pseudobase $\mathbb{P}$ of closed sets, and $Y \subseteq X$, then

$$\{p \cap Y \mid p \in \mathbb{P} \wedge p \cap Y \neq \emptyset\}$$

forms a psudobase of closed sets for Y.

5 Effective density theorems

Let $A = \{a_1, \ldots, a_n\}$ be a finite set. A *probability distribution* on A is a map $\mu : A \to \mathbb{R}_{[0,1]}$ such that

$$\sum_{k \leq n} \mu(a_k) = 1.$$

We let $PD(A)$ be the set of probability distributions on A, where we assume that A comes with an enumeration. $PD(A)$ can be viewed as a convex subspace of a finite dimensional Eucledian space, and thus $PD(A)$ has a canonical topology.

Definition 5.1 Let X be in $\mathcal{Q}$.
X satisfies *density with probabilistic selection* if there are

i) A sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite sets together with maps $\nu_n : A_n \to X$.

ii) A sequence of continuous maps

$$\mu_n : X \to PD(A_n)$$

such that:

- When $x = \lim_{n\to\infty} x_n$ and for each $n \in \mathbb{N}$, $a_n \in A_n$ such that $\mu_n(x_n)(a_n) > 0$, we have that $x = \lim_{n\to\infty} \nu_n(a_n)$.

When this is the case, we call $\{\langle A_n, \nu_n, \mu_n\}_{n\in\mathbb{N}}$ *a probabilistic selection.*

If $\{A_n\}_{n\in\mathbb{N}}$, $\{\nu_n\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}$ witness that X satisfies density with probabilistic selection, then $\bigcup_{i\in\mathbb{N}} A_n$ will be dense in X.

Remark 5.2 This concept will be an important tool in showing density theorems. In order to prove embedding theorems, we will extend this concept in Section 6 to what we will call *a probabilistic projection.*
In our applications, X will be a space in the hierarchy of spaces of continuous functionals of finite type, where spaces of type zero may be certain separable metric spaces. Then A_n will consist of finite functionals of the same type, where the ground types are interpreted as finite subsets of the metric spaces in question. ν_n then represents a way to embed these finitary functionals into the space of continuous functionals. Since there in general are no continuous projections in the opposite direction, the functions μ_n will be probabilistic replacements of such projections.

Lemma 5.3 *Let X be a separable metric space. Then X satisfies density with probabilistic selection.*

Proof
Let d be the metric on X, and let $\{a_0, a_1, \ldots\}$ be a countable dense subset of X. Let $A_n = \{a_0, \ldots, a_n\}$ with ν_n the identity function on A_n.
For any $x \in X$, let $d(A_n, x) = \min\{d(x, a_i) \mid i \leq n\}$.
If u and v are non-negative reals, let $u \dot{-} v = \max\{u - v, 0\}$.
For each $x \in X$ and $a \in A_n$, let

$$\mu_n(x)(a) = \frac{d(x, A_n) + \delta_n \dot{-} d(x, a)}{\sum_{b\in A_n}[d(x, A_n) + \delta_n \dot{-} d(x, b)]},$$

where δ_n is the minimum of 2^{-n} and all distances $d(a,b)$ for $a \neq b$ in A_n The required properties are easy to verify.

Definition 5.4 Let $\{\langle A_n, \mu_n\rangle\}_{n\in\mathbb{N}}$ be a sequence of finite subsets A_n of a metric space X together with probability distributions μ_n on each A_n.
Let $x \in X$. We say that

$$x = \lim_{n\to\infty} A_n \, mod \, \mu_n$$

if whenever we select $a_n \in A_n$ with $\mu_n(a_n) > 0$ for each $n \in \mathbb{N}$, then $x = \lim_{n\to\infty} a_n$.

Lemma 5.5 *Let U be the Urysohn space. Let $V = \{v_1, \ldots, v_n\} \subseteq U$ be a finite set with an enumeration. Then there is a continuous*

$$h_V : PD(V) \to U$$

such that whenever μ is a probability distribution on V, then

$$diam\{v_i \mid \mu(v_i) > 0\} = diam(\{v_i \mid \mu(v_i) > 0\} \cup \{h_V(\mu)\}.$$

Proof
We may let ϕ embed V isometrically into $\mathbb{R}^n$ with the max-norm and we may let ψ embed $\mathbb{R}^n$ into U isometrically such that $\psi(\phi(v_i)) = v_i$ for all $i \leq n$.
Then let

$$h_V(\mu) = \psi(\sum_{i=1}^{n} \mu(v_i) \cdot \phi(v_i)).$$

It is easy to see that this works.

Lemma 5.6 *Let U be the Urysohn space, and let $U_n \subseteq U$ be finite for each n.*
Let h_{U_n} be as in Lemma 5.5.
If $\mu_n \in PD(U_n)$ for each n and $u \in U$ such that

$$u = \lim_{n\to\infty} U_n \, mod \, \mu_n$$

then

$$u = \lim_{n\to\infty} h_{U_n}(\mu_n).$$

Proof
Use Lemma 5.5

Remark 5.7 In the proofs of Lemmas 5.5 and 5.6, we might use any normed vector space instead of U. In fact, any reasonable local convexity property allowing a continuous barycenter function will suffice. The proof of the following lemma is contained in the proof above:

Lemma 5.8 *Let* $\{A_n, \mu_n\}_{n \in \mathbb{N}}$ *be a sequence of finite sets with probability distributions in a normed vector space* A *and assume that* $b = \lim_{n\to\infty} A_n$ *modulo* μ_n.
Then

$$b = \lim_{n\to\infty} \sum_{a \in A_n} \mu_n(a) \cdot a.$$

The proof of the next theorem will use Lemma 5.6, and again we may replace U with any separable metric space A satisfying the conclusion of this lemma. The point of giving this proof is that it is effective. Later we will prove the same conclusion under weaker conditions, but then the proof will be non-effective.

Theorem 5.9 *Let* X *be a* $\mathcal{Q}$*-space that admits density with probabilistic selection, and let* U *be the Urysohn space.*
Then $X \to U$ *admits density with probabilistic selection.*

Proof
Let $\{u_n \mid n \in \mathbb{N}\}$ be a countable dense subset of U and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of finite sets with maps $\nu_n : A_n \to X$ and continuous functions

$$\mu_n : X \to PD(A_n)$$

forming a probabilistic selection.
Let $U_n = \{u_1, \ldots, u_n\}$ with id_n being the identity function on U_n. Let $B_n = A_n \to U_n$, and let $\lambda_n : U \to PD(U_n)$ be such that $\{\langle U_n, id_n, \lambda_n\rangle\}_{n\in\mathbb{N}}$ forms a probabilistic selection.
Let $h_n : PD(U_n) \to U$ satisfy the conclusion of Lemmas 5.5 and 5.6.

Let $\phi \in B_n$. First we will see how we will construct a continuous $\nu_n^* : X \to U$:
Let $x \in X$. For each $u \in U_n$ let $\mu_n^{-1}(u) = \mu_{n,x,\phi}^{-1}(u)$ be

$$\mu_n^{-1}(u) = \mu_n(x)(\phi^{-1}(u))$$

and let

$$\nu_n^*(\phi)(x) = h_n(\mu_n^{-1}).$$

We will see how the sets B_n together with the maps ν_n^* from B_n to $X \to U$ can be organized to a probabilistic selection.

Let $f : X \to U$ be continuous. We will define the probability distribution $\eta_n(f)$ on B_n as a product measure and prove the required properties. Let

$$\eta_n(f)(\phi) = \prod_{a \in A_n} \lambda_n(f(a))(\phi(a)).$$

We have to show

Claim
Let $f = \lim_{n\to\infty} f_n$ and assume that

$$\eta_n(f_n)(\phi_n) > 0$$

for each n.
Then $f = \lim_{n\to\infty} \nu_n^*(\phi_n)$.

Proof of Claim
Since we are operating in the category of sequential topological spaces, this amounts to showing that if $x = \lim_{n\to\infty} x_n$ in X, then $f(x) = \lim_{n\to\infty} \nu_n^*(\phi_n)(x_n)$ in U.
This will follow from the construction of the ν_n^*'s, the properties of the h_n's and the following

Subclaim

$$f(x) = \lim_{n\to\infty} U_n \; mod \; \mu_{n,x_n,\phi_n}^{-1}.$$

Proof of subclaim
Let $\mu_{n,x_n,\phi_n}^{-1}(v_n) > 0$ for each n.
Then there is an $a_n \in A_n$ with $\phi_n(a_n) = v_n$ and $\mu_n(x_n)(a_n) > 0$.
$x = \lim_{n\to\infty} a_n$ since we have probabilistic selection on X, so

$$f(x) = \lim_{n\to\infty} f_n(a_n).$$

Since $\eta(f_n)(g_{\phi_n}) > 0$ we must have that

$$\lambda_n(f_n(a_n))(\phi_n(a_n)) > 0$$

so

$$f(x) = \lim_{n\to\infty} \phi_n(a_n),$$

or, in other words

$$f(x) = \lim_{n\to\infty} v_n.$$

This ends the proof of the subclaim, the claim and the theorem.

The proof of Theorem 5.9 is effective, and the theorem can be viewed as the induction step in the proof of the main corollary.

Corollary 5.10 *Let $X_1, \ldots, X_k$ be either effective (separable) normed vector spaces or the Urysohn space U. Let X be constructed from $X_1, \ldots, X_k$ by (iterated) use of the function space construction. Then there is an effective sequence of finite sets A_n, an effective sequence of finite maps $\nu_n : A_n \to X$ and an effective sequence of continuous maps $\mu_n : X \to PD(A_n)$ such that $\{\langle A_n, \nu_n, \mu_n\rangle\}_{n\in\mathbb{N}}$ forms a probabilistic selection.*

Remark 5.11 In many respects, it is the effective density theorem implicit in this result that is of importance.

6 An embedding theorem

In this section we will prove a theorem that is strictly topological in formulation, but where the motivation for proving it comes from the wish to understand the spaces used in the semantics of functional programming. We let $V_1, \ldots, V_n$ be formal variables for complete, separable metric spaces, and we define the formal *types* as the least set of expressions containing each variable V_i and closed under the syntactical operation $\sigma, \tau \vdash (\sigma \to \tau)$.
If $X_1, \ldots, X_k$ are separable, complete metric spaces and σ is a type expression, its interpretation $\sigma(X_1, \ldots, X_k)$ is given in the category of topological sequential spaces.

We will prove the following:

Theorem 6.1 *Let σ be a type in the variables $V_1, \ldots, V_k$ and let $X_1, \ldots, X_k$ be complete, separable metric spaces.*
Then $\sigma(X_1, \ldots, X_k)$ is homeomorphic to a zero-set in $\sigma(U, \ldots, U)$, where U is the Urysohn space.

In order to prove this theorem, we have to work with a combination of the concept of an embedding-projection pair and probabilistic selection. An embedding-projection pair between Y and X is normally a continuous

function $\varepsilon : Y \to X$ and a continuous function $\pi : X \to Y$ such that $\pi(\varepsilon(y)) = y$ for all $y \in Y$. If we have two typed structures, one with base type Y and one with base type X, one standard way to show that we may embed the first into the second type by type is to establish an embedding-projection pair between Y and X and then show that this generates an embedding-projection pair at each type.
Sometimes it is topologically impossible to have a continuous projection from X to Y. We will see that for many important cases, we can replace the use of the projection with a sequence of probabilistic approximations.

Definition 6.2 Let X and Y be in $\mathcal{Q}$.
A *probabilistic embedding-projection-pair* between Y and X consists of

i) A sequence $\{A_n\}_{n \in \mathbb{N}}$ of finite sets together with maps $\nu_n : A_n \to Y$.

ii) A continuous map $\varepsilon : Y \to X$.

iii) A sequence of continuous maps

$$\mu_n : X \to PD(A_n)$$

such that:

> When $x = \lim_{n \to \infty} x_n$ in X with $x = \varepsilon(y)$ for some $y \in Y$, and $a_n \in A_n$ for each $n \in \mathbb{N}$ such that $\mu_n(x)(a_n) > 0$, we have that $y = \lim_{n \to \infty} \nu_n(a_n)$.

We will call a sequence $\{\langle A_n, \nu_n, \mu_n \rangle\}_{n \in \mathbb{N}}$ like this a *probabilistic projection*.

This clearly implies that ε is injective.

Lemma 6.3 *Let X be a separable metric space, and let Y be isometric to a subspace of X via $\varepsilon : Y \to X$.*
Then ε is the embedding-part of a probabilistic embedding-projection-pair between Y and X.

Proof
We use the construction from the proof of Lemma 5.3, replacing the enumeration of a dense subset of X with an enumeration of a dense subset of Y, and relating $x \in X$ to the ε-range of finite parts of the dense subset of Y. There are no new technical aspects of the proof.

The key lemma in proving Theorem 6.1 is

Lemma 6.4 *Let $X \in \mathcal{Q}$, Y homeomorphic to a zero-set in X via an embedding $\varepsilon : Y \to X$. Let $A \subseteq U$ be a closed subset of the Urysohn space U. If ε is the embedding-part og a probabilistic embedding-projection pair between Y and X, then $Y \to A$ is homeomorphic to a zero-set Z in $X \to U$ admitting a probabilistic embedding-projection-pair between $Y \to A$ and $X \to U$.*

Remark 6.5 We restrict ourselves to $\mathcal{Q}$ everywhere, also in cases where the proof works for *qcb*-spaces in general, or even in a greater generality

Theorem 6.1 is proved by induction on the type, referring to Lemma 6.3 for the base cases and Lemma 6.4 for the induction steps using the function space constructor. This proof-sketch is not complete, since extra care has to be taken in the case $\sigma \to \tau$ where τ is not a base type.

Proof of Lemma 6.4
For each n let $A_n \subseteq Y$ be finite, $\nu_n : A_n \to Y$ and $\mu_n : X \to PD(A_n)$ be continuous such that the sequences form a probabilistic projection.
Let $f : X \to [0,1]$ be continuous such that

$$\varepsilon[Y] = f^{-1}(\{0\}).$$

First we will show how to embed $Y \to A$ into $X \to U$. We will use that U is homeomorphic to l_2, see Uspenskij [19], and the linear operations below are carried out via this homeomorphism.
Let $g : Y \to A$ be continuous and let $x \in X$. We let

> $\varepsilon^*(g)(x) = g(\varepsilon^{-1}(x))$ if $x \in \varepsilon[Y]$
>
> $\varepsilon^*(g)(x) =$
> $(1-\lambda)\cdot\sum_{a\in A_n}\mu_n(x)(a)\cdot g(\nu_n(a)) + \lambda\sum_{b\in A_{n+1}}\mu_{n+1}(x)(b)\cdot g(\nu_n(b))$, where $n \in \mathbb{N}$ and $\lambda \in [0,1)$ are unique such that
>
> $$f(x) = \frac{1}{n+\lambda},$$
>
> otherwise.

We have to show that $\varepsilon^*(g) \in X \to U$ is continuous and that

$$\varepsilon^* \in (Y \to A) \to (X \to U)$$

is continuous.
Since we are working with sequential spaces, this amounts to showing

Claim 1
If $g = \lim_{n\to\infty} g_n$ in $Y \to A$ and $x = \lim_{n\to\infty} x_n$ in X then

$$\varepsilon^*(g)(x) = \lim_{n\to\infty} \varepsilon^*(g_n)(x_n).$$

Proof of Claim 1

There will be two cases

Case 1 $x \notin \varepsilon[Y]$: Then $f(x) \neq 0$ and $f(x_n) \neq 0$ for almost all n. Then, locally around x, everything is continuous.

Case 2 $x \in \varepsilon[Y]$: Then $\varepsilon^*(g)(x) = g(\varepsilon^{-1}(x))$. We may, without serious loss of generality, assume that for every $n \in \mathbb{N}$ we have that $x_n \notin \varepsilon[Y]$ (since g is continuous on Y and $g = \lim_{n\to\infty} g_n$ as functions defined on Y in the limit space sense).
Then $\varepsilon^*(g_n)(x_n) =$

$$(1-\lambda_n)\sum_{a\in A_{m_n}} \mu_{m_n}(x_n)(a)\cdot g_n(\nu_{m_n}(a)) + \lambda_n \sum_{b\in A_{m_n+1}} \mu_{m_n+1}(x_n)(b)\cdot g_n(\nu_{m_n+1}(b))$$

where $m_n \in \mathbb{N}$ and $\lambda_n \in [0,1)$ are such that $f(x_n) = \frac{1}{m_n+\lambda_n}$.
Now, if we for each n select a_n such that $a_n \in A_{m_n}$ and $\mu_{m_n}(x_n)(a_n) > 0$ or such that $a_n \in A_{m_n+1}$ and $\mu_{m_n+1}(x_n)(a_n) > 0$, we may use that $x = \lim_{n\to\infty} x_n$ and the properties of probabilistic projections to see that $\varepsilon^{-1}(x) = \lim_{n\to\infty} a_{m_n}$.
Then $g(\varepsilon^{-1}(x)) = \lim_{n\to\infty} g_n(\nu_{m_n}(a_{m_n}))$ for each such sequence.
Since $\varepsilon^*(g_n)(x_n)$ is a weighted sum of values $g_n(\nu_{m_n}(a_{m_n}))$ and $g_n(\nu_{m_n+1}(a_{m_n+1}))$, we have that

$$\varepsilon^*(g)(x) = g(x) = \lim_{n\to\infty} \varepsilon^*(g_n)(x_n).$$

This ends the proof of Claim 1.

Note that $(\varepsilon^*)^{-1}(\gamma)$ defined by

$$(\varepsilon^*)^{-1}(\gamma)(y) = \gamma(\varepsilon(y))$$

will map $X \to U$ onto $Y \to U$, and that $(\varepsilon^*)^{-1}$ will be the inverse of ε^* on the image of ε^*. Thus ε^* is a homeomorphism on its range.

Claim 2
There is a continuous

$$h : (X \to U) \to [0,1]$$

such that $h^{-1}(\{0\})$ is the range of ε^*.

Proof of Claim 2
Let $\{y_n\}_{n\in\mathbb{N}}$ be a dense subset of Y and $\{x_m\}_{m\in\mathbb{N}}$ a dense subset of X.
Given $\gamma : X \to U$ we will let $h(\gamma)$ measure to what extent γ does not map $\varepsilon[Y]$ into A and to what extent γ will differ from $\varepsilon^*((\varepsilon^*)^{-1}(\gamma))$.
Note that the definition of $(\varepsilon^*)^{-1}(\gamma))$ makes sense since we never use that a function takes values in A in the definition or in the proof of Claim 1.
We simply let

$$h(\gamma) = \frac{1}{2}\sum_{n\in\mathbb{N}} 2^{-n} d_U(A, \gamma(\varepsilon(y_n))) + \frac{1}{2}\sum_{n\in\mathbb{N}} 2^{-1} d_U(\gamma(x_n), \varepsilon^*((\varepsilon^*)^{-1}(\gamma))(x_n).$$

This ends the proof of Claim 2.

It remains to produce the probabilistic projection.
Let $\mathbb{P}$ be a countable pseudobase for Y, see Section 4. Let $\{\xi_n \mid n \in \mathbb{N}\}$ be a countable dense subset of U. For $r > 0$, $r \in \mathbb{Q}$, we let

$$B_{n,r} = \{a \in U \mid d_A(a, \xi_n) \leq r\}.$$

Let $\{\langle p_i, B_i\rangle\}_{i\in\mathbb{N}}$ be an enumeration of all pairs $\langle p, B\rangle$ where $p \in \mathbb{P}$ and B is a non-empty finite intersection of closed neighborhoods of the form $B_{n,r}$.
Let $K \subseteq \mathbb{N}$ be finite. K is *relevant* if there is a $g : Y \to A$ such that

$$* \;\; \forall i \in K \forall y \in p_i(g(y) \in B_i).$$

If K is relevant, let g_K satisfy $*$.
If K is not relevant, let m be maximal such that $K \cap \{1, \ldots, m\}$ is relevant, and let

$$g_K = g_{K\cap\{1,\ldots m\}}.$$

Now, we assume that the enumeration $\{y_j\}_{j\in\mathbb{N}}$ of the dense subset of Y used in the proof of Claim 2 is chosen such that for all $p \in \mathbb{P}$, $\{y_j \mid y_j \in p\}$ is a dense subset of p. Then, whenever $p \in \mathbb{P}$, $B \subseteq A$ is a closed set and $g : Y \to A$ is continuous we have that

$$\forall y \in p(g(y) \in B) \Leftrightarrow \forall j \in \mathbb{N}(y_j \in p \Rightarrow g(y_j) \in B).$$

Now, let C_n be the powerset of $\{1, \ldots, n\}$. We will construct a sequence of continuous functions

$$\mu_n^* : (X \to U) \to PD(C_n).$$

For $\gamma : X \to U$ and $K \in C_n$ we define $\mu_n^*(\gamma)(K)$ as the product of the PD's $\mu_{n,i}^*(\gamma)$ on $\{\in, \notin\}$ for $i \leq n$, where we will measure the probability of $\langle p_i, B_i \rangle$ being an approximation to γ:
Let $k = k_n$ be so large that for all $i \leq n$ there is a $j \leq k$ such that $y_j \in p_i$.

> Let $\mu_{n,i}^*(\gamma)(\in) = 1$ if $\gamma(\varepsilon(y_j)) \in B_i$ for all $j \leq k$ with $x_j \in p_i$.
>
> Let $\mu_{n,i}^*(\gamma)(\in) = 0$ if $d_U(B_i, \gamma(\varepsilon(y_j))) \geq 2^{-n}$ for at least one $j \leq k$ with $y_j \in p_i$
>
> Let $mu_{n,i}^*(\gamma)(\in) = 1 - \lambda$ if
>
> $$2^{-n} \cdot \lambda = max\{d_U(B_i, \gamma(\varepsilon(y_j))) \mid j \leq k \wedge y_j \in p_i\}$$
>
> otherwise.

Let

$$\mu_n^*(K) = \prod_{i \in K} \mu_{n,i}^*(\in) \cdot \prod_{i \notin K} \mu_{n,i}^*(\notin).$$

Claim 3
Assume that $g; Y \to A$, $\gamma = \phi(g)$ and that $\gamma = \lim_{n\to\infty} \gamma_n$.
Assume further that for each $n \in \mathbb{N}$, $K_n \in C_n$ is such that $\eta_n(\gamma_n)(K_n) > 0$.
Then $g = \lim_{n\to\infty} g_{K_n}$.
Proof of Claim 3
Using the lim-space characterization it is sufficient to show that whenever $z = \lim_{n\to\infty} z_n \in Y$, then $g(z) = \lim_{n\to\infty} g_{K_n}(z_n)$ in U.
We will use the result by Hamrin [8] that we may construct an admissible domain representation from a pseudobase.
Let (D, D^R, δ) be the admissible domain representation of Y, where D consists of ideals of pseudobase elements in $\mathbb{P}$, and let (E, E^R, δ_1) be the corresponding domain representation of $Y \to U$.
Let $\alpha = \lim_{n\to\infty} \alpha_n$ be a convergent sequence from E^R representing

$$g = \lim_{n\to\infty} (\varepsilon^*)^{-1}(\gamma_n)$$

and let $\zeta = \lim_{n\to\infty} \zeta_n$ be a convergent sequence from D^R representing $z = \lim_{n\to\infty} z_n$, see Remark 4.4.

Let $\epsilon > 0$. Since α represents g and ζ represents z, there is an $m \in \mathbb{N}$ such that $\langle p_m, B_m\rangle \in \alpha$, $p_m \in \zeta$ and such that the diameter of B_m is less than ϵ. We will show that for sufficiently large n we have that $g_{K_n}(z_n) \in B_m$. This will show the claim.

Let n_0 be such that for $n \leq n_0$ we have that $\langle p_m, B_m\rangle \in \alpha_n$ and that $p_m \in \zeta_n$. Recall how we used k_n in the construction of $\mu^*_n(g)$. Let n_1 be so large that for any $i \leq m$, if $g[p_i] \not\subseteq B_i$, then there is a $j \leq k_{n_1}$ such that $x_j \in p_i$ and $g(x_j) \notin B_i$.
Select one such j_i for each relevant $i \leq m$, and then choose n_2 so large that for each $n \geq n_2$ and each relevant $i \leq m$ we have that

$$d_U(\gamma_n(\varepsilon(x_{j_i})), B_i) > 2^{-n}.$$

This is possible since $\gamma(\varepsilon(x_{j_i})) = \lim_{n\to\infty} \gamma_n(\varepsilon(x_{j_i}))$
Let $n \geq max\{n_0, n_1, n_2\}$ and let $K \subseteq \{1, \ldots, n\}$ be such that $\mu^*_n(g_n)(K) > 0$. For $i < m$ we have ensured that if $\gamma_n[\varepsilon[p_i]] \not\subseteq B_i$, then $\mu^*_{n,i}(\gamma_n)(\in) = 0$ and since $\langle p_m, B_m\rangle \in \alpha_n$ we also have that $\mu^*_{n,m}(\gamma_n)(\in) = 1$. It follows that g witnesses that $K \cap \{1, \ldots, m\}$ is relevant and contains m. This holds in particular for $K = K_n$, so $g_{K_n}(z_n) \in B_m$.

Now the proof is complete, but let us summarize what we have achieved.

> We have defined the embedding $\varepsilon^* : (Y \to A) \to (X \to U)$ and proved that it is continuous and has a continuous inverse on its range.
>
> We have proved that the range of ε^* is a zero-set.
>
> We have defined the finite set C_n and the map
>
> $$K \mapsto g_K$$
>
> from C_n into $Y \to A$. Let $\nu^*_n(K) = g_K$.
>
> For each $\gamma \in X \to U$, we have defined the probability distribution $\mu^*_n(\gamma)$ on C_n and proved that altogether, ε^* and $\{\langle C_n, \nu^*_n, \mu^*_n\rangle\}_{n\in\mathbb{N}}$ form a probabilistic embedding-projection pair between $Y \to A$ and $X \to U$.

This ends the proof of lemma 6.4.

We have not included cartesian products as one type constructor, but in order to handle types of the form $\sigma = \tau \to \delta$ in the reflection of Lemma 6.4 it will make life simpler if we view any type σ as a type $\sigma = \tau_1, \ldots, \tau_m \to V_i$ where V_i is interpreted as some separable metric space. This means that we need an extra induction step in the proof of Theorem 6.1, the case of products.
If $X_1, \ldots, X_m$ are spaces in $\mathcal{Q}$ or in *qcb* in general, the product $\prod_{i=1}^m X_i$ is not just the standard topological product, but the finest topology accepting the induced convergent sequences in the product topology as convergent. We then have

Lemma 6.6 *Let $Y_1, \ldots, Y_m$ and $X_1, \ldots, X_m$ be two sequences of spaces in $\mathcal{Q}$, and assume that there are probabilistic embedding-projection pairs between Y_i and X_i for each $i \leq m$.*
Then there is a probabilistic embedding-projection pair between $\prod_{i=1}^m Y_i$ and $\prod_{i=1}^m X_i$.

Proof
This is more an observation than a lemma:

> If ε_i is the embedding for each i, we let $\varepsilon = \prod_{i=1}^m \varepsilon_i$.
>
> If f_i witnesses that the range of ε_i is a zero-set, let
>
> $$f(x_1, \ldots, x_m) = \frac{1}{m} \sum_{i=1}^m f_i(x_i)$$
>
> witness that the range of ε is a zerp-set.
>
> If A_n^k and $\nu_n^k : A_n^k \to Y_i$ are the finite "approximations" to Y_i used for the probabilistic projections, we let A_n and ν_n be obtained by just taking products.
>
> The probability distributions of the product is just the products of the probability distributions.

It is easy to verify that all properties are preserved in this construction. This ends the proof of the lemma.

Now we have all the ingredients needed to prove Theorem 6.1:
If $X_1, \ldots, X_k$ are separable metric spaces, and σ is a type expression in the variables $V_1, \ldots, V_k$ we prove by induction on σ that there is a probabilistic

embedding-projection pair between $\sigma(X_1, \ldots, X_k)$ and $\sigma(U, \ldots, U)$, where the image of the embedding is a zero-set.
The induction start $\sigma = V_i$ is covered by Lemma 6.3.
For the induction step, we let $\sigma = \tau_1, \ldots, \tau_m \to V_j$.
We then use Lemma 6.6 and the induction hypothesis to show that there is a probabilistic embedding-projection-pair between

$$\prod_{i=1}^{m} \tau_i(X_1, \ldots, X_k)$$

and

$$\prod_{i=1}^{m} \tau_i(U, \ldots, U).$$

We then use Lemma 6.4 to complete the induction step.

Remark 6.7 This proof is noneffective. We have used that U is homeomorphic to l_2, and we do not know of any effective proof of that. There are however methods that are likely to get us around this problem, like those we used in Section 5.
However, the concept of a relevant set of natural numbers, and the choice of the functions g_K in the proof of Lemma 6.4, are not effective in a general situation, even when the metric spaces $X_1, \ldots, X_k$ are effective. Thus we may as well use the topological characterization of U as homeomorphic to l_2 in this proof.

7 Effective embedding theorems

7.1 What may go right and what may go wrong

In this subsection we will look at the embedding theorem from Section 6, in order to see what an effective version will require. We may assume that we start with effective separable, complete metric spaces $X_1, \ldots, X_k$ at base types. The problem is what we will mean with an effective version of Theorem 6.1, and then, what is needed to acquire one.

At domain level, we will have effective domain representations of the Urysohn space U and the spaces $X_1, \ldots, X_k$. If we choose the embeddings of the X_i into U with some care, very little is lost. We can, however, not expect to have embedding-projection pairs at the domain level, the best we can hope for is to have effective maps from the domain representing $\sigma(X_1, \ldots, X_k)$ into the domain representing $\sigma(U, \ldots, U)$ preserving totality and representing a topological embedding.

Lemma 7.1 *Let X be an effective complete metric space. Then there is an effective isometry between X and a closed subset A of U such that the distance function $u \mapsto d_U(u, A)$ is computable.*

Proof
We will employ the "Swizz Cheese Principle" essentially proved in the original paper Urysohn [18] stating that if we remove a finite set of open balls with computable center and radius from U, then the resulting space is effectively isometric to U.
Let $\{x_1, \ldots, x_n, \ldots\}$ be the effective dense subset of X. Let $\{u_i\}_{i\in\mathbb{N}}$ be the effective dense subset of U obtained by the original construction. We embed $\{x_n \mid n \in \mathbb{N}\}$ isometrically into U by finding $\varepsilon(x_i) \in U$ in such a way that whenever $n < i$ we have that $d_U(x_n, u_i) \geq min\{d_U(u_i, \eta(x_j)) \mid j \leq i\}$. This can be achieved effectively using the Swizz Cheese Principle.
Then the distance from u_i to the range of ε is computable, and consequently, the distance from $u \in U$ to the range of ε is computable.

This lemma shows that every effective metric space may be viewed as a zero-set in U in an effective sense.

In the proof of our embedding theorem in section 6, we made essential use of the probabilistic projections, and of course, if we have a probabilistic projection from X to Y, we implicitly have an enumeration of a dense subset of Y. Thus any effective version of the embedding theorem, at least to the extent that our proof method applies, will involve effective enumerations of dense subsets of the smaller set. It is well known that this cannot be achieved in general, and for the sake of completeness we will offer an example.

Example 7.2 Let $A \subseteq \mathbb{N}$ be recursively enumerable but not computable, and let $f : \mathbb{N} \to \mathbb{N}$ be a computable 1-1 enumeration of A.
We will construct an effective subspace of the Banach space l_∞ of all bounded sequences of reals.
Let $a < b$ be reals, and let $[a, b]_n$ be those $f \in l_\infty$ where $f(n) \in [a, b]$ and $f(m) = 0$ for $m \neq n$.
Let X consist of the constant 0 together with all $[0, 3]_n$ for $n \in A$ and all $[1, 3]_n$ for $n \notin A$.

It is easy to see that we can effectively enumerate a dense subset of X with a computable metric, using a stage m where $f(m) = n$ to decide to extend the ongoing sub-enumeration of $[1,3]_n$ to a sub-enumeration of $[0,3]_n$. Thus X is an effective metric space.

If we have an effectively enumerated dense set $\{g_n \mid n \in \mathbb{N}\}$ of total functions in $X \to \mathbb{N}$, we see from the obvious connectedness-properties of X that

$$n \not\in A \Leftrightarrow \exists m(g(0,\ldots) \neq g(n \mapsto 2))$$

where $n \mapsto 2$ is the element in $[1,3]_n$ that takes the value 2 on n.

This would imply that A is computable, so there is no such sequence $\{g_n\}_{n\in\mathbb{N}}$

7.2 An effective embedding theorem

We will now briefly combine the effective parts of the constructions in Section 7 and the proof of the density theorem, Corollary 5.10, in Section 6 and prove the following

Theorem 7.3 *Let $X_1, \ldots, X_k$ be effective (separable) normed vector spaces. Let $X = \sigma(X_1, \ldots, X_k)$ be constructed from $X_1, \ldots, X_k$ by use of the function space construction. Then there is an effective probabilistic embedding-projection-pair between X and $\sigma(U, \ldots, U)$*

Remark 7.4 As for the proof of Corollary 5.10 we may choose $X_1, \ldots, X_k$ among all metric spaces for which our proof of density works, e.g. convex subspaces of normed vector spaces and others accepting a kind of continuous barycenter function. We do not have a proof in the case where some V_i may be interpreted as U. If U can be organized to a convex space in some effective way, our arguments would work also then.

Proof of Theorem 7.3

We use induction on the type. As a part of the induction hypothesis, we need that the function witnessing that the range of the embedding is a zero-set, indeed is effective.

We let $\vec{V} = (V_1, \ldots, V_k)$ be a list of variables, and we let $\vec{X} = (X_1, \ldots, X_k)$ be a corresponding sequence of effective, complete metric spaces. We let $\vec{U}$ denote the ordered sequence consisting of k U's.

If σ is a type (expression) in the given variables, we interpret $\sigma(\vec{X})$ as before. For $i = 1, \ldots, k$, we let $\{x_{i,n}\}_{n\in\mathbb{N}}$ be the effective enumeration of a dense subset of X_i witnessing that X_i is an effective metric space, and let $X_{i,n} = \{x_{i,1} \ldots , x_{i,n}\}$, i.e. the n first elements in the enumeration. We then let $\sigma(\vec{X}_n)$ be the interpretation of σ when V_i is interpreted as $X_{i,n}$ and we take all functions in the recursion step $\sigma = \tau \to \delta$. Note that $\sigma(\vec{X}_n)$ is a finite set. In order to prove the theorem, we will by recursion/induction on σ establish effective versions of

1. Maps $\nu_{\sigma,n} : \sigma(\vec{X}_n) \to \sigma(\vec{X})$.
2. An embedding $\varepsilon_\sigma : \sigma(\vec{X}) \to \sigma(\vec{U})$.
3. A map $f_\sigma : \sigma(\vec{U}) \to [0, 1]$ such that $f_\sigma^{-1}(\{0\})$ is the range of ε_σ.
4. Continuous maps
$$\mu_{\sigma,n} : \sigma(\vec{U}) \to PD(\sigma(\vec{X}_n))$$
such that
$$\{\langle\sigma(\vec{X}_n), \nu_{\sigma,n}, \mu_{\sigma,n}\rangle\}_{n\in\mathbb{N}}$$
is a probabilistic projection.

If $\sigma = V_i$, we let $\nu_{\sigma,n}$ be the identity function on $\{x_{i,1}, \ldots , x_{i,n}\}$, we let $\varepsilon_{V_i} : X_i \to U$ be an effective, isometric embedding as constructed in Lemma 7.1, we let
$$f(u) = min\{d_U(u, \varepsilon_{V_i}[X_i]), 1\}$$
and for $\mu_{V_i,n}$ we use the probability distribution constructed in the proof of Lemma 5.3, replacing A_n with $X_{i,n}$ and distances in A with distances to the objects $\varepsilon_{V_i}(x_{i,j})$ in U.
This gives us the induction start.
The induction step are as in the proof of Theorem 6.1 in two steps, $\tau = \tau_1 \times \cdots \times \tau_m$ and $\sigma = \tau \to V_i$. The first step offers no challenges. The proof of Lemma 6.6 can be directly translated, every operation in that proof is effective.

Now let $\sigma = \tau_1, \ldots , \tau_m \to V_i$. We let $\tau = \tau_1 \times \cdots \times \tau_m$ and assume that 1. - 4. holds for τ, with the established notation. The argument will be a mixture of the proof of Theorem 5.9 and the proof of Lemma 6.4. The extract from the proof of Theorem 5.9 is adjusted to the case where we replace U with the normed vector space X_i, and we adjust the extract from the proof of Lemma 6.4 by replacing the l_2-structure on U by the induced normed vector-space-structure on the relevant ε-image of X_i.

For $\phi : \tau(\vec{X}_n) \to X_{i,n}$ and $x \in \tau(\vec{X})$ we let

$$\nu_{\sigma,n}(\phi)(x) = \sum_{b \in \vec{X}_n} \mu_{\tau,n}(\varepsilon_\tau(x))(b) \cdot \phi(b),$$

where we use the vector space algebra in X_i.
Now, if $\gamma \in \sigma(\vec{X})$ and $y \in \tau(\vec{U})$ we define $\varepsilon_\sigma(\gamma)(y)$ using cases analogue to those used in the proof of lemma 6.4:

If $y = \varepsilon_\tau(x)$ for some $x \in \tau(\vec{X})$, we let

$$\varepsilon_\sigma(\gamma)(y) = \varepsilon_{V_i}(\gamma(x)).$$

Otherwise, choose n and λ such that $f_\tau(y) = \frac{1}{n+\lambda}$. Let

$$\begin{aligned}&\varepsilon_\sigma(\gamma)(y) = \\ &\varepsilon_{V_i}((1-\lambda) \cdot \textstyle\sum_{a \in \tau(\vec{X}_n)} \mu_{\tau,n}(y)(a) \cdot \gamma(\nu_{\tau,n}(a)) \\ &+\lambda \cdot \textstyle\sum_{b \in \tau(\vec{X}_{n+1})} \mu_{\tau,n+1}(y)(b) \cdot \gamma(\nu_{\tau,n+1}(b))\end{aligned}$$

The proof of continuity is a direct transcription of the corresponding argument in the proof of lemma 6.4
The definition of f_σ is a direct transcription of the definition of h in the same proof.
The probabilistic projection will consist of maps $\mu_{\sigma,n} : \sigma(\vec{U}) \to PD(\sigma(\vec{X}_n))$.
This construction, and the proof that it works, is an adjustment of the construction of the probabilistic selection and the proof that that one works, from the proof of Theorem 5.9.
We give some details:

Let $\gamma \in \tau(\vec{U}) \to U$ and recall that $\sigma(\vec{X}_n) = \tau(\vec{X}_n) \to X_{i,n}$.
Let $\phi \in \sigma(\vec{X}_n)$. We let

$$\mu_{\sigma,n}(\gamma)(\phi) = \prod_{a \in \tau(\vec{X}_n)} \mu_{X_{i,n}}(\gamma(\varepsilon_{V_i}(a))(\phi(a)).$$

We end the proof of the theorem by verifying that this construct satisfies the properties of a probabilistic projection.

Claim
Assume that $\gamma = \varepsilon_\sigma(g)$ for some $g \in \sigma(\vec{X})$. Let $\gamma = \lim_{n\to\infty} \gamma_n$ in $\sigma(\vec{U})$ and

for each $n \in \mathbb{N}$ let $\phi_n \in \sigma(\vec{X}_n)$ be such that $\mu_{\sigma,n}(\gamma_n)(\phi_n) > 0$.
Then $g_n = \lim_{n\to\infty} \nu_{\sigma,n}(\phi_n)$.

Proof of Claim
By the limit space characterization it is sufficient to show that

$$g(x) = \lim_{n\to\infty} \nu_{\sigma,n}(\phi_n)(x_n)$$

whenever $x = \lim_{n\to\infty} x_n$ in $\tau(\vec{X})$, so let this convergent sequence be given.
For each $n \in \mathbb{N}$, let $a_n \in \tau(\vec{X}_n)$ be such that $\mu_{\tau,n}(x_n)(a_n) > 0$.
Then $x = \lim_{n\to\infty} \nu_{\tau,n}(a_n)$ since $\{\mu_{\tau,n}\}_{n\in\mathbb{N}}$ is a probabilistic projection.
It follows that

$$\gamma(\varepsilon_\tau(x)) = \lim_{n\to\infty} \gamma_n(\varepsilon_\tau(\nu_{\tau,n}(a_n))).$$

Since $\mu_{\sigma,n}(\gamma_n)$ is a product-PD over the index set $\tau(\vec{X}_n)$ and $\mu_{\sigma,n}(\gamma_n)(\phi_n) > 0$, we must have that

$$\mu_{V_i,n}(\gamma_n(\varepsilon_\tau(\nu_{\tau,n}(a_n))))(\phi_n(a_n)) > 0.$$

Again we use the property of probabilistic projections and infer that

$$\gamma(\varepsilon_\tau(x)) = \lim_{n\to\infty} \varepsilon_{V_i}(\phi_n(a_n)).$$

But $g(x) = \varepsilon_{V_i}^{-1}(\gamma(\varepsilon_\tau(x)))$, so we get that

$$g(x) = \lim_{n\to\infty} \phi_n(a_n).$$

Now we will use that the a_n's were arbitrary. From the definition of $\nu_{\sigma,n}(\phi_n(x_n))$ and Lemma 5.8 we get that

$$g(x) = \lim_{n\to\infty} \nu_{\sigma,n}(x_n).$$

The Claim, and the Theorem, is proved.

8 Conclusions and further research

We have shown that the typed hierarchy of hereditarily continuous and total functionals over the Urysohn space U is rich enough to contain all typed hierarchies over separable metric spaces as topological sub-hierarchies. One problem is if this can be generalized to a situation where we do not consider the full space of continuous functions at types $\sigma = \tau \to \delta$, but just a zero-set oof functions.

All our spaces $\sigma(\vec{X})$ are homeomorphic to zero-subsets of spaces of the form $X \to U$ where $X \in \mathcal{Q}$, but we have not studied this class , denoted by $zero(\mathcal{Q} \to U)$ closer. P. K. Køber [10] has obtained some partial results related to strictly positive inductive definitions of topological spaces.

We have proved effective versions of the embedding theorem and we showed by an example that our approach via a probabilistic projection will not have a general effective version. Another problem will be if we may find alternative ways to construct effective embeddings. By effective we would mean that the embeddings can be represented by a computable object in the domain representations.

We know that U is homeomorphic to l_2, but we do not know if the l_2-structure on U is effective in the sense that there are computable

$$|| \cdot || : U \to \mathbb{R}_{\geq 0}$$

$$+ : U \times U \to U$$

$$\cdot : \mathbb{R} \times U \to U$$

representing the l_2 - structure, or any other Banach space structure on U. It may be of interest to equip U with some structure offering an internal computability theory, e.g. by identifying subsets representing $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$.

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