On sequential functionals of type 3

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Abstract

We show that the extensional ordering of the sequential functionals of pure type 3, e.g. as defined via game semantics [2, 4], is not *cpo*-enriched. This shows that this model does not equal Milner's [9] fully abstract model for PCF.

1 Introduction

In [14], Scott defined the logic LCF, a logic for computable typed functionals. He constructed a"mathematical" model for LCF, a construction that led to the discovery of domain theory. By the *Scott model* we will mean the hierarchy of algebraic domains constructed in [14]. [14] remained a famous, unpublished manuscript for many years, but was later published as an historical document [15]. LCF has its root in Platek's thesis [11] and the work on continuous functionals initiated by Kleene [6] and Kreisel [7]. Platek observed that nondeterministic **OR**, and equivalently, the nondeterministic conditional $\supset_{nd}$ are not captured by this calculus. These objects will be present in Scott's model, but undefinable in his language for LCF.
In [12] Plotkin viewed LCF as a programming language PCF. PCF is essentially typed λ-calculus with constants for fixed point operators for each type. PCF is given with its λ-calculus operational semantics and a denotational semantics using Scott's model. In this paper we will let PCF be the special version where an elementary calculus for the algebra on the natural numbers and the booleans is integrated. In the technical part we have to assume some basic familiarity with PCF.
Adding a constant for the conditional $\supset_{nd}$ with a nondeterministic evaluation procedure, Plotkin showed that all finitary objects in Scott's model are definable. Plotkin also showed that there are computable objects that are not PCF-definable relative to $\supset_{nd}$, i.e. in the calculus known as PCF^+.

Given PCF or a similar system, two closed terms T_1 and T_2 of type σ will be called *observationally equivalent* if for all terms $M[x]$ of base type, where x is

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the only free variable and has type σ, either both $M[T_1]$ and $M[T_2]$ evaluate to the same normal form or neither evaluate to a normal form at all. A model for PCF or the similar system is called *fully abstract* if terms that are observationally equivalent will be interpreted as the same object in the model. Then Scott's model is not fully abstract for PCF, but for PCF^+.

Milner [9] constructed a typed hierarchy of algebraic domains that served as a fully abstract model for PCF. Moreover he showed that up to isomorphism there will be only one such model. Milner's model is constructed as a kind of term model. The natural question was then if there is a more conceptually well based construction of Milner's model. This not very precise problem was known as the *fully abstraction problem.*

Several attempts have been made to solve this problem. Most relevant for this paper are the game-semantical approaches in Abramsky, Jagadeesan and Malacaria [2] and independently in Hyland and Ong [4]. In both papers, the sequential functionals are constructed in two guises, as intensional and as extensional objects. The finitary and intensional sequential functionals are the functionals realized by finite strategies or dialogues. The finitary and extensional functionals will be the compacts in Milner's model. These are characterized via the extensional ordering of the intensional finitary functionals. The remaining problem, first mentioned in Hyland and Ong [4], but also pointed out by Plotkin in [13], was if the infinitary objects in Milner's model could be obtained directly from one of the infinitary sequential functionals, or if the extensional collapse of the sequential functionals from [2] or [4] is a proper subset of Milner's model.

The main result of this paper is the solution to this remaining problem. We will show via one example that the extensional collapse of the sequential functionals is not an algebraic domain, constructing an unbounded increasing sequence. This shows that Milner's model still is too rich and, though we will not give the proof, it will actually contain a computable object of type 3 that is not PCF-definable, even relative to a total function of type 1.

The technical level of this paper is rather low, and the reader will not need much background from domain theory or computability theory in general. Further background in domain theory can be obtained from many sources, e.g. from Abramsky and Jung [1], Stoltenberg-Hansen & al. [16] or Amadio and Curien [3]. The latter also provides a more than sufficient background on PCF.

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2 Preliminaries

Sometimes the objects in a typed hierarchy will be defined intensionally, e.g. via the algorithms for them. Then we may, by recursion on the type σ, define an equivalence relation $\approx_\sigma$ where

$$\approx_0 \text{ is the 'denotes the same object'-relation}$$

on the base type, and

$$F \approx_{\sigma\to\tau} G \Leftrightarrow \forall x \forall y (x \approx_\sigma y \to F(x) \approx_\tau G(y))$$

for higher types. This will induce an equivalence relation on the set of intensional objects. The equivalence classes at each type may be viewed as functionals of the same type, and it is this functional hierarchy we call *the extensional collapse*. In the cases considered here, the base type will contain an object representing the 'undefined', and will then be ordered in the standard way. We order the function spaces of the extensional collapse using the pointwise ordering inductively. This is what we call *the extensional ordering*.

There are various equivalent approaches to the sequential functionals, the AJM-games in [2], the HON-games in [4], PCF_Ω (i.e. special PCF with extra oracles for all partial $f : \mathbb{N} \to \mathbb{N}$) and Kleene's $S1 - S9$ interpreted over the Scott model and relativized to all partial functions $f : \mathbb{N} \to \mathbb{N}$. In all cases, we will be interested in the extensional typed hierarchy.

The two game-theoretical models are models for PCF_Ω, and by Theorem 8.1 in [4], each extensional functional definable by a game will be PCF_Ω-definable. Thus in order to show that the extensional ordering of the sequential functionals is not an algebraically closed domain, it suffices to show that the set of PCF_Ω-definable functionals of pure type 3 contains an unbounded increasing sequence.

Ulrich Berger observed that modulo the fact that Kleene's $S1-S9$ are restricted to the pure types, interpreting $S1 - S9$ over the Scott model is equivalent to PCF. The argument is originally due to Platek [11], and may be found in Moldestad [10]. Berger showed that the argument from [11] and [10] is valid in the continuous case as well.
Thus an alternative approach to our main result would be to prove that the set of Scott-functionals of pure type 3 computable in $S1-S9$ relative to some partial function on $\mathbb{N}$ contains an increasing, unbounded sequence. This was actually the approach first taken, and for readers familiar with $S1 - S9$-computability this approach may be even more transparent. However, since the majority of the potential readers are likely to be more familiar with PCF than with $S1 - S9$, we choose PCF as our platform.

Remark 1 Using the characterization of sequential finitary functionals in e.g. Hyland and Ong [4] it is easy to show that the least upper bound of an increasing sequence of finitary sequential functionals of type 2 is itself sequential.

In this paper we do not need the precise definition of finitary sequential objects. The functionals we construct will be extensional. The type-3 functionals explicitly defined are proved to be sequential by providing an algorithm for them. The functionals F of pure type 2 that we construct will satisfy one of two properties:

1. There is a finite partial function τ from $\mathbb{N}$ to $\mathbb{N}$ such that $F(f)$ terminates if and only if $\tau \sqsubseteq f$.

2. There are two inconsistent finite partial functions σ and τ from $\mathbb{N}$ to $\mathbb{N}$ such that $F(f)$ terminates if and only if $\sigma \sqsubseteq f$ or $\tau \sqsubseteq f$.

In both cases F is trivially sequential.

The only technical property of sequential functionals of type 3 that we will need is

Lemma 1 *Let $F_1, \ldots, F_n$ be finitary sequential functionals of type 2, and let $a_1, \ldots, a_n$ be natural numbers.*
Assume that there is a sequential functional Φ such that $\Phi(F_i) = a_i$ for $i = 1, \ldots, n$.
Then the set of sequential Φ' of type 3 such that $\Phi'(F_i) = a_i$ for $i = 1, \ldots, n$ will have a minimal element in the extensional ordering.

Proof
The extensional collapse of the finitary sequential functionals of a fixed type forms the compacts in Milner's model, and then this reflects a property of algebraic domains.

In the final section we will show that our construction also shows that the minimal extensional model for PCF_Ω^+, i.e. PCF_Ω extended with a constant for $\supset_{nd}$, is not algebraically closed. Mainly because it simplifies the notation of the proof we will use the following definition:

Definition 1

$\supset_{nd} (0, y, z) = y.$

$\supset_{nd} (1, y, z) = z.$

$\supset_{nd} (x, y, y) = y.$

We will also briefly discuss a problem related to the continuous existential quantifier $\exists_\omega$ defined by

Definition 2

$\exists_\omega(f) = 0$ if $f(\bot) = ff$

$\exists_\omega(f) = 1$ if $f(n) = tt$ for some n.

where tt and ff are the two truth values.

3 An example

We will let σ and τ with indices denote finite partial functions on $\mathbb{N}$, f an arbitrary partial function on $\mathbb{N}$.
We let F (with indices) be sequential functionals of pure type 2, and Φ, Ψ with indices will denote sequential finitary functionals of pure type 3.
We will explicitly define finitary functionals F_0, F_1, ... where F_0 will be inconsistent with F_n for $n > 0$ while F_n and F_m will be consistent when $n > 0$ and $m > 0$. We will then prove the existence of sequential functionals Φ'_n such that $\Phi'_n(F_0) = 1$ while $\Phi'_n(F_i) = 0$ when $1 \leq i \leq n$.
The idea behind our example is to ensure that the computation of $\Phi'_n(F_0)$ always will require $n+1$ instances of functional applications independent of which algorithm for Φ'_n we choose. Thus there will be no way to uniformly bound them all by one sequential functional.

Definition 3 Let

$\tau_0(0) = 1$ and $\tau_0(n) = \perp$ for $n \neq 0$.

$F_0(f) = 0$ if $\tau_0 \sqsubseteq f$, $F_0(f) = \perp$ otherwise.

Now let $n \geq 1$. Let

$\tau_n(0) = 0$, $\tau_n(n) = 0$ and $\tau_n(i) = \perp$ otherwise,

$\sigma_n(i) = 0$ for $1 \leq i < n$, $\sigma_n(n) = 1$ and $\sigma_n(m) = \perp$ otherwise,

$F_n(f) = 1$ if $\tau_n \sqsubseteq f$ or $\sigma_n \sqsubseteq f$, $F_n(f) = \perp$ otherwise.

We see that τ_0 and σ_n are consistent when $1 \leq n$ so F_0 and F_n are inconsistent. If $1 \leq n$ and $1 \leq m$, then F_n and F_m are consistent. We will also make use of the observation that if $1 \leq n$, $1 \leq m$ and $n \neq m$, then σ_n and σ_m are inconsistent.

Lemma 2 *Each F_n is sequential.*

The proof is trivial.

Lemma 3 *For each n there is a sequential Φ_n such that*

$\Phi_n(F_0) = 1$

$\Phi_n(F_i) = 0$ *if* $1 \leq i \leq n$

Proof
We will use induction on n.
For $n = 0$ this is trivial, the constant functional $\Phi_0(F) = 1$ will do.
Now, let $n = m + 1$ and assume that Φ_m is constructed as required. Define $\Psi_n(F, i)$ by

$\Psi_n(F, 0) = \Phi_m(F)$,

$\Psi_n(F, i) = 0$ if $1 \leq i < n$,

$\Psi_n(F, n) = 1$ and

$\Psi_n(F, i) = \bot$ if $i > n$.

Then

$\tau_0 \sqsubseteq \lambda x.\Psi_n(F_0, x)$,

$\tau_i \sqsubseteq \lambda x.\Psi_n(F_i, x)$ for $1 \leq i < n$ and

$\sigma_n \sqsubseteq \lambda x.\Psi_n(F_n, x)$.

We let

$$\Phi_n(F) = 1 \dot{-} F(\lambda x.\Psi_n(F, x)).$$

We then verify the claim by computing:

$\Phi_n(F_0) = 1 \dot{-} F_0(\lambda x.\Psi_n(F_0, x)) = 1 \dot{-} F_0(\tau_0) = 1$.

$\Phi_n(F_i) = 1 \dot{-} F_i(\lambda x.\Psi_n(F_i, x)) = 1 \dot{-} F_i(\tau_i) = 0$ when $1 \leq i < n$.

$\Phi_n(F_n) = 1 \dot{-} F_n(\lambda x.\Psi_n(F_n, x)) = 1 \dot{-} F_n(\sigma_n) = 0$.

Remark 2 The functional

$$\lambda n.\lambda F.\Phi_n(F)$$

is actually PCF-definable.

We will use $\sqsubseteq$ for the extensional ordering of finitary sequential functionals also of types 2 and 3. By Lemma 1 we have that for each n there is a $\sqsubseteq$ - minimal Φ'_n such that $\Phi'_n(F_0) = 1$ and $\Phi'_n(F_i) = 0$ for $1 \leq i \leq n$.
Then $n \leq m \Rightarrow \Phi'_n \sqsubseteq \Phi'_m$. The aim is to show that this sequence has no upper bound in the extensional ordering of sequential functionals.

4 The proof

We are now going to prove the main result of the paper; we will show that there is no upper bound for $\{\Phi'_n\}_{n\in\mathbb{N}}$ in the Scott model that is definable in PCF_Ω.
Recall that PCF_Ω is a typed λ-calculus over the base types ι (for the natural numbers) and o (for the boolean values). The constants will be $O : \iota$ (for zero), $S : \iota \to \iota$ (for successor), $P : \iota \to \iota$ (for predecessor), $Z : \iota \to o$ (for 'zeroness'), the conditionals $\supset_b$: $o, b, b \to b$ for each base type b, the fixed point operators $Y_\sigma : (\sigma \to \sigma) \to \sigma$ for each type σ and $f : \iota \to \iota$ for each partial function $f : \mathbb{N} \to \mathbb{N}$.
We assume that the reader is familiar with the definition of terms in PCF_Ω and with the transition rules of the calculus.
Each term has an interpretation over the Scott model. For constants and closed terms of base type we will not distinguish notationwise between the constant or term and the interpretation in the Scott model. For instance we used f both for the function itself and for the constant representing it. We will also use e.g.

3 instead of $S(S(S0))$, and we will write $M = N$ when M and N are two closed terms of base type that reduce to the same normal form via the transition rules. Plotkin [12] showed that a closed term of base type is interpreted as a number or as a boolean value if and only if it is reduced to the normal form of the object via the transition rules. We let $\downarrow$ mean *terminates* and $\uparrow$ mean *does not terminate* in this sense.

We will use standard conventions for simplifying the syntax of terms. Then each therm of base type can be written in the form

$$M = M_0 N_1 \cdots N_k$$

where $k \geq 0$.
We will add $F_0, F_1, \ldots$ to the language of PCF_Ω. They will be special constants for themselves. We extend PCF_Ω with transition rules for these new constants that are abreviated as

$$\frac{\tau_0 \sqsubseteq N}{F_0 N \to 0}\ ,\ \frac{\tau_n \sqsubseteq N}{F_n N \to 1}\ ,\ \frac{\sigma_n \sqsubseteq N}{F_n N \to 1}$$

where $n > 0$. E.g. the rule

$$\frac{\tau_3 \sqsubseteq N}{F_3 N \to 1}$$

is an abreviation for

$$\frac{N0 \to 0\ ,\ N3 \to 0}{F_3 N \to 1}.$$

Plotkin's result refered to above is still valid. PCF_Ω extended with these constants and transition rules will have a deterministic top-down evaluation strategy. By *the length of an evaluation* we will mean the number of steps needed to obtain a normal form using this strategy. We use the sequentiality of each F_n when extending the strategy to cover the F_n's and the transition rules for them.

Throughout the rest of this paper, we will let z be a fixed variable of type $(\iota \to \iota) \to \iota$. When we write $M[F_i]$ we will always mean M with F_i substituted for z. In other cases we use M^x_N for the result of the substitution, always assuming that problems of substitutability are handled in the standard way.

We then have

Lemma 4 *Let M be a PCF_Ω-term of base type, with at most z free. If for infinitely many n, $M[F_n]\downarrow$, then*

$$\forall n \forall m (M[F_n]\downarrow \wedge M[F_m]\downarrow \Rightarrow M[F_n] = M[F_m]).$$

Proof
If $n, m > 0$ then F_n and F_m are consistent as elements of the Scott model, so this lemma is only challenging when $M[F_0]\downarrow$. Then we will use induction on the length of the evaluation of $M[F_0]$.

Let $M = M_0 N_1 \cdots N_k$. The proof will be by cases on the syntactical structure of M_0. M_0 will either be a constant, the variable z or a λ-abstraction. All cases except $M_0 = z$ will be trivial (we will follow an argument from Plotkin [12] here). The key to the argument is to show that the case $M_0 = z$ cannot possibly occur under our assumptions.

The induction base will be the case $k = 0$ and M_0 being the constant 0.
Then $M_0[F_n] = 0$ for all n.
If $M_0 = \lambda x.M_1$, we may use the induction hypothesis on the term

$$(M_1)^x_{N_1} N_2 \cdots N_k,$$

since $((M_1)^x_{N_1} N_2 \cdots N_k)[F_0]$ is what is obtained after one step of evaluating $M[F_0]$ according to our strategy.
Likewise, if $M_0 = Y_\sigma$, we may use the induction hypothesis on the term

$$N_1(Y_\sigma N_1)N_2 \cdots N_k.$$

Among the remaining constants in PCF_Ω, we will only consider the case $M_0 = \supset_\iota$, since the other cases are even simpler.
Then

$$M = \supset_\iota N_1 N_2 N_3.$$

By the assumption $N_1[F_n]\downarrow$ for infinitely many n including $n = 0$. By the induction hypothesis, all values are the same, e.g tt for 'true'. Then $N_2[F_n]\downarrow$ for infinitely many n including $n = 0$. Again all values are the same. As a consequence all values of $M[F_n]$ must be equal when defined.

The remaining case is when $M_0 = z$, i.e. $M = zN$ where N will be of type $\iota \to \iota$ and may contain z as a free variable, but no other free variables.
Since $M[F_0]\downarrow$, we must have that $F_0(N[F_0])\downarrow$.
In the denotational semantics, this means that $\tau_0 \sqsubseteq N[F_0]$, so we must have that $(N0)[F_0] \to 1$.
If $M[F_n]\downarrow$ for some $n > 0$, we must have that $\sigma_n \sqsubseteq N[F_n]$ or $\tau_n \sqsubseteq N[F_n]$ in the denotational semantics. Since σ_n and σ_m are inconsistent when $n \neq m$, while F_n and F_m are consistent, we must have that $\tau_n \sqsubseteq N[F_n]$ for infinitely many n.
Back to the operational semantics, this implies that

$$(N0)[F_n] \to 0$$

for these n's, since $\tau_n(0) = 0$.
Clearly the evaluation of $(N0)[F_0]$ is shorter than that of $(zN)[F_0]$ (by our transition rule for F_0), so by the induction hypothesis we cannot have that $(N0)[F_0] \to 1$ and $(N0)[F_n] \to 0$ for infinitely many n. Thus this case will never occur.
This ends the proof of the lemma.

An immediate consequence of the lemma is:

Theorem 1 *There exists a sequence of finitary sequential functionals of type 3 that is increasing in the extensional ordering, but that has no upper limit in the set of sequential functionals.*

5 Discussion

Theorem 1 has the obvious consequence

Corollary 1 *The extensional collapse of the sequential functionals and Milner's model do not coincide at pure types* ≥ 3.

We proved this corollary for type 3. For higher types we use the fact that there are sequential embedding-projection pairs (π_n, ν_n) between type 3 and type(n) for $n > 3$, so we just embed the sequence $\{\Phi'_k\}_{k\in\mathbb{N}}$ into the sequential functionals of type n. If Ψ is a sequential upper bound for $\{\pi_n(\Phi'_k)\}_{k\in\mathbb{N}}$, then $\nu_n(\Psi)$ will be a sequential upper bound for $\{\Phi'_k\}_{k\in N}$, something that does not exist.
In fact, if t is a pure type and t' is a mixed type of at least the same level as t, we may find a sequential embedding-projection pair between the objects of type t and the objects of type t', so the result extends trivially from pure type 3 to all types of level ≥ 3.

If we view each F_n as an element of the original Scott hierarchy, there will be a functional Φ such that $\Phi(F_0) = 1$ while $\Phi(F_n) = 0$ when $n \geq 1$. We actually know that the minimal such Φ is PCF^{++}-definable, i.e. definable in PCF extended with the non-deterministic conditional and continuous existential quantifier. However, just the non-deterministic conditional $\supset_{nd}$ will not suffice:

Theorem 2 *There exists an increasing sequence of sequentially defined finitary objects of pure type 3 in the Scott model such that the least upper bound is not definable in* PCF^+_Ω.

Using the construction in Section 3 and the notation from Section 4, the theorem follows from the following

Lemma 5 *Let M be a term in PCF^+_Ω. Suppose that for infinitely many n, $M[F_n]\downarrow$.*
Then all values $M[F_n]$ are compatible.

Proof
This is again only nontrivial in the case when $M[F_0]\downarrow$, since otherwise we may use that all relevant F_n's are consistent. Thus we will assume that $M[F_0]\downarrow$ and use induction on the length of the derivation of $M[F_0]$ as before. The proof is split into cases according to the syntactical structure of M. All cases dealt with in the proof of Lemma 4 are dealt with in the same way here.
The only new case then is where $M = \supset_{nd} N_1 N_2 N_3$. In this argument, i will denote an element in $\{2, 3\}$ and $\bar{i}$ will denote the other element. For the sake of simplicity, we may as well assume that if $N_1[F]\downarrow$, then $N_1[F] \in \{0, 1\}$.
We may use the induction hypothesis for N_i for $i = 1, 2, 3$.
The proof is split into three cases:

Case 1
$N_1[F_n]\downarrow$ for infinitely many n including $n = 0$.

By the induction hypothesis, all $N_1[F_n]$'s are compatible. Let $i = 2 + N_1[F_0]$. Then $N_i[F_0]\downarrow$ and for infinitely many n, $N_i[F_n]\downarrow$. Thus by the induction hypothesis again, all $N_i[F_n]$'s are compatible. It follows that all $M[F_n]$'s are compatible.

Case 2
$N_1[F_0]\downarrow$, but $N_1[F_n]\uparrow$ for all but finitely many n.

By assumption we have that for infinitely many n, $N_2[F_n]\downarrow$, $N_3[F_n]\downarrow$ and $N_2[F_n] = N_3[F_n]$.
Let $i = 2 + N_1[F_0]$. Then $N_i[F_0]\downarrow$ and by the induction hypothesis, all $N_i[F_n]$'s are compatible.
Now, let $n \geq 1$ and assume that $M[F_n]\downarrow$.
If $N_1[F_n] = N_1[F_0]$ or $N_1[F_n]\uparrow$, we have that

$$M[F_n] = N_i[F_n] = N_i[F_0] = M[F_0].$$

If $N_1[F_n]\downarrow$ and $N_1[F_n] \neq N_1[F_0]$, we have that $M(F_n) = N_{\bar{i}}[F_n]$. Since we are in Case 2, there will be another m (actually infinitely many) such that $N_i[F_m] = N_{\bar{i}}[F_m]$ and both are defined. Since F_n and F_m are consistent we have that

$$M[F_n] = N_{\bar{i}}[F_n] = N_{\bar{i}}[F_m] = N_i[F_m] = N_i[F_0] = M[F_0].$$

Case 3
$N_1[F_0]\uparrow$.

Then $N_2[F_0] = N_3[F_0]$, both terminates and

$$M[F_0] = N_2[F_0] = N_3[F_0].$$

For infinitely many n, at least one of $N_2[F_n]$ and $N_3[F_n]$ will terminate, so pick i such that infinitely many $N_i[F_n]$ terminate. Then, by the induction hypothesis, all values $N_i[F_n]$ are compatible. There will be two sub-cases:

3.1: For infinitely many n, $M[F_n]\downarrow$ via the termination of $N_1[F_n]$.

Since all relevant F_n's are consistent, let a be the common value of $N_1[F_n]$ when terminating. Then we may use $i = a + 2$ above. We then have that $M[F_n] = N_i[F_n]$ for all n, and by the induction hypothesis, all these values are compatible.

3.2: $N_1[F_n]\uparrow$ for all but finitely many n.

Then for infinitely many n,

$$N_2[F_n]\downarrow \wedge N_3[F_n]\downarrow \wedge N_2[F_n] = N_3[F_n].$$

By the induction hypothesis it follows that the values $N_i[F_n]$ are compatible for all n and both i, so all values of $M[F_n]$ are compatible.

This ends the proof of the lemma.

There is one minor extra comment to make:
Let Φ be the least upper bound of the Φ'_n's in the Scott model.
$\exists_\omega$ will not be PCF-definable in Φ. The reason is that if this were the case, $\exists_\omega$ would itself be the least upper bound of sequential objects, which it clearly is not. However, $\exists_\omega$ is the least upper bound of objects PCF-definable in $\supset_{nd}$, (as is every other object), so this simple argument does not show that $\exists_\omega$ is not definable in Φ and $\supset_{nd}$. In fact, trying out some standard methods, we have not been able to prove that $\exists_\omega$ is not definable in $\supset_{nd}$ and Φ. Based on this lack of success, we offer

Conjecture 1 *There is a sequence of finitary, sequential functionals of type 3 increasing in the extenstional ordering such that $\exists_\omega$ is PCF-definable in $\supset_{nd}$ and the least upper bound of the sequence.*

The reason why we think that this conjecture reflects an interesting problem, is because we may express this conjecture in more informal terms as follows:
The process of taking the completion of a model for PCF_Ω and then extend the completion to a model for PCF^+_Ω commutes with first extending the original model to a model for PCF^+_Ω and then to a completion.
If $\exists_\omega$ is not PCF^+_Ω - definable in Φ, we will have found an indication of a degree-structure of the computable objects in the Scott model relative to PCF^+, or of all objects relative to PCF^+_Ω. An alternative problem will be to ask if there exists such a degree structure at all.

Remark 3 The counterexample in Section 2 is the main achievement of the paper. The way we found the example was by first solving the problem in a positive way for two special cases and then realizing why the proofs in these special cases could not be merged into one universal proof. The fact that infinitely many F_s's, but not all, are pairwise consistent is crucial to the construction.

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