Push Forward of Holomorphic Forms

Einar Andreas Rødland

Cand. Scient. Thesis
July, 1992

Institute of Mathematics
University of Oslo, Norway

Contents

Preface

This is the thesis for my Cand. Scient. degree at the University of Oslo. My primary field of study is several complex variables and algebraic geometry. Initially the purpose was a thesis on residues in several complex variables; however, as the title states, it has been specialized to push forward of holomorphic forms by holomorphic maps. Of course, the two subjects are strongly related and I believe that several results on residues may be proved more easily by using push forward and a relation similar to $\mathrm{Res}_f\, \omega(z) = \mathrm{Res}_v\, f_*\omega(v)$.

The methods used are both analytic and algebraic. I believe most of the algebraic results on push forward may be generalized to maps $f : F^n \to F^n$, F an algebraicly closed field, and similarly for algebraic varieties, using equation 1.3 of definition 1.3 to give a formal definition of push forward. This, I have not looked into. My initial interest was push forward of forms; the results of interest in algebra is primarily those of chapter 3, and these came later. If I can find an algebraic proof of theorem 2.19, I believe the rest of the algebraic results would follow quite easily.

Some of my inspiration to study push forward of forms may have come from the following well known result proved by elementary residue calculations in one or several variables: as stated in [5] at page 667, if $f : U \to W$ is holomorphic, surjective, open, and finite, $h : W \to U$ holomorphic, then the **trace** $\sigma_h(w) = \sum_{f(z)=w} h(z)$ is holomorphic on W. I had seen this result in one variable and could easily generalize it to several. My idea was to generalize this from functions to forms; this is done in chapter 1. When noticing how I chose to define the push forward, the link to this formula seems clear. At the end of the chapter, I show that the push forward may be done by proper maps between manifolds or varieties.

The next chapter studies push forward in the algebraic case; on $\mathbf{C}^n$ and on varieties. I find that push forward is algebraic operation: push forward of an algebraic form by an algebraic map is algebraic. In fact, push forward equals the algebraically defined trace. Furthermore, I study the relation between the degree of the form and its push forward.

In chapter 3, I find that a polynomial h is uniquely represented by a sum $\sum_{j=1}^{m} a_j\ (g_j/c_j) \circ f$ where a_j is a polynomial basis, $m = \deg f$. The polynomials g_j may be found by the relation $g_j\ dv = f_*(b_j h\ dz)$ where b_j is a dual polynomial basis. In the generic case, I can choose the basis a_j so as to make it a basis for $\mathbf{C}[z]$ over $\mathbf{C}[f]$.

Chapter 4 deals with the relation between push forward and residues. Also, I generalize a result from an article [3] on residues, using methods from elsewhere in this thesis and some of its results.

I have used some known results. These are listed in the appendix with either proofs or references to the source of the result. The appendix also contains definitions of some concepts used in this thesis; this is for the sake of clearity, as some of these are defined differently by some authors.

Finally, I would like to call attention to the notation used throughout the thesis. All definitions, lemmas, etc. are ended with a black box: ■. Several chapters and sections contain a paragraph labeled "Notation"; the conditions set here apply to the entire chapter or section unless otherwise stated and, hence, these requirements applies even if they may not be repeated in lemmas and theorems.

Einar Andreas Rødland
Oslo, 1st July 1992

Chapter 1

Push Forward of Holomorphic Forms

The integral

$$\int_{|f_j|=r_j} \frac{h\ dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} \tag{1.1}$$

where $f_1, \ldots, f_n,\ h : \mathbf{C}^n \to \mathbf{C}$ are holomorphic, often occurs when studying residues. This integral may be studied by taking the push forward of the form $h\ dz$ by the map $f : \mathbf{C}^n \to \mathbf{C}^n$.

Notation 1.1 In this chapter, unless otherwise specified, I will use the following:

1. $U \subseteq \mathbf{C}^n$ is an open set.

2. $f : \overline{U} \to \mathbf{C}^n$ is holomorphic. On U, I will primarily use coordinates z; on the image, coordinates v. If f are polynomials, $U = \mathbf{C}^n$.

3. $\mathcal{J}_{j\,k} = (\partial f_j / \partial z_k)$ and $\Delta = \det \mathcal{J}$. I demand that $\Delta(z) \neq 0$; i.e., the image of f is of dimension n.

4. $h : \overline{U} \to \mathbf{C}$ is holomorphic, and $\omega = h\ dz = h\ dz_1 \wedge \cdots \wedge dz_n$.

5. I will often drop indices: $dz = dz_1 \wedge \cdots \wedge dz_n$, $|f - v| = r$ says that $|f_j - v_j| = r_j$ for all j, $|f - v| < r$ means that $|f_j - v_j| < r_j$, etc. For $\mu \in \mathbf{N}_0^n$, $|\mu| = \sum \mu_j$; $\mu + 1 = (\mu_1 + 1, \ldots, \mu_n + 1)$. For two n-tuples u_j and v_j, $u \cdot v = \sum u_j v_j$; $z^\mu = \prod z_j^{\mu_j}$.

■

1.1 Defining a Global Push Forward

The usual definition of push forward is by

$$\int f_*\phi \wedge \psi = \int \phi \wedge f^*\psi, \tag{1.2}$$

where ϕ and ψ are forms. This defines $f_*\phi$ as a current, but does not necessarily give a form. Under certain conditions $f_*\phi$ will become a form; i.e., it may be represented by a form.

I will, however, attack this problem from the other side. I will define push forward in a different way such that it is clear that it is a form, but where the push forward is not immediately defined everywhere. However, where it is defined, it will be equal to the push forward defined by equation 1.2.

Example 1.2 If $\Delta(z) \neq 0$ there is a neighborhood U of z such that f is biholomorphic on U. If we let $g = f^{-1}$ the Jacobi determinant of g is $\Delta_g = \Delta^{-1}$; hence, we get $f_*(z)(h(z)\, dz) = g^*(z)(h(z)\, dz) = h(z)\, dg = h(z)\Delta_g(v)\, dv = h(z)/\Delta(z)\, dv$ where $f(z) = v$. ■

Definition 1.3 If $\Delta(z) \neq 0$, there is a natural local push forward:

$$f_*(z)(h(z)\, dz) = \frac{h(z)\, dv}{\Delta(z)}.$$

I will define a **global push forward**:

$$f_*\omega(v) = f_*(h\, dz)(v) = \sum_{f(z)=v} f_*(z)(h(z)\, dz) = \sum_{f(z)=v} \frac{h(z)\, dv}{\Delta(z)}. \tag{1.3}$$

Let V be the set of all $v \in f(U) - f(\partial U)$ with $\# f^{-1}(v) < \infty$ such that there is a neighborhood O of v with $f^{-1}(O)$ bounded; V is open. Now, $f_*\omega$ is well defined for all $v \in V$ with $\Delta \neq 0$ on $f^{-1}(v)$; I will call this set $\hat{V}$. ■

Now, $f : U \to V$ will be holomorphic, surjective, open, and finite; such are the conditions for the trace formula as stated at page 667 of [5]. It will also be proper. I will show that no further conditions are needed for a similar formula for n-forms. This result will be stronger than the classical trace formula because of the division by the Jacobi determinant required in the equation 1.3.

The reason for specifying $v \notin f(\partial U)$ is to avoid cases like $U = \{z \in \mathbf{C} \mid \operatorname{Re} z > 0\}$, $f(z) = z^4$ and $v = 1$.

Lemma 1.4 With ω, f, and $f_*\omega$ as above, $f_*\omega = H\ dv$ where H is holomorphic on $\hat{V}$.

Proof: Let $v \in \hat{V}$, $\{p_1, \ldots, p_m\} = f^{-1}(v)$, and let U_j be neighborhoods of p_j such that $f|_{U_j} : U_j \to V$ is biholomorphic on the image. Let g_j be the inverse of $f|_{U_j}$ defined on a neighborhood of v. Now,

$$H(v) = \sum_{f(z)=v} \frac{h(z)}{\Delta(z)} = \sum_{j=1}^{m} \frac{h(g_j(v))}{\Delta(g_j(v))}$$

which is clearly holomorphic in a neighborhood of v. ∎

If $f_*\omega$ is known, the integral (1.1) may be written

$$\int_{|f|=r} \frac{\omega}{f_1 \cdots f_n} = \int_{|v|=r} \frac{f_*\omega}{v_1 \cdots v_n}.$$

1.2 Extending the Push Forward

It is obviously of interest to extend $f_*\omega$ beyond the region where it is immediately defined. In fact it is possible to extend it to all of V. In order to do this, I need a lemma saying that the integral of a holomorphic n-form is independent of path in some sense.

Lemma 1.5 The integral

$$\int_{|f-w|=r} \frac{\omega}{\prod_j f_j - v_j}$$

does not depend on the choice of r and w as long as $|v - w| < r$, and $\{z \mid |f(z) - w| = r\} \subset U$ is compact.

Proof: I will show that the integral equals $\int_{|f-v|=\rho} \omega / \prod f_j - v_j$ for any $\rho < r - |w - v|$. Let $A \subset \mathbf{C}^n \times \mathbf{R}$ be defined as

$$A = \Big\{ (z,t) \,\Big|\, |f(z) - w - t(v - w)| = r + t(\rho - r),\ 0 \le t \le 1 \Big\}.$$

There is a projection $\pi : A \to U$ by $\pi(z,t) = z$. This gives the pull back form $\pi^*(\omega)$ on A; $d\pi^*(\omega / \prod f_j - v_j) = \pi^* d(\omega / \prod f_j - v_j) = 0$. A satisfies the demands of the version of Stoke's theorem for varieties: see theorem A.24. Alternatively, A is semialgebraic — see the definition A.22 — and so we could use the version of Stoke's theorem for semianalytic sets: see theorem A.25. This makes

$$\begin{aligned}
\int_{|f-w|=r} \frac{\omega}{\prod_j f_j - v_j} - \int_{|f-v|=\rho} \frac{\omega}{\prod_j f_j - v_j} &= \int_{\partial A} \pi^* \left(\frac{\omega}{\prod_j f_j - v_j} \right) \\
&= \int_A d\pi^* \left(\frac{\omega}{\prod_j f_j - v_j} \right) \\
&= 0.
\end{aligned}$$

■

Theorem 1.6 The form $f_*\omega$ can be extended holomorphically to V, V as in definition 1.3.

Proof: Let $w \in \hat{V}$ and $r_j > 0$ be such that $\{v \in \mathbf{C}^n \mid |v - w| \leq r\} \subset \hat{V}$; as $\hat{V}$ is open, such an r exists. We get $f_*\omega(v) = H(v)\, dv$ with

$$H(v) = \frac{1}{(2\pi i)^n} \int |u - w| = r \frac{H(u)\, du}{\prod_j u_j - v_j} = \frac{1}{(2\pi i)^n} \int_{|f-w|=r} \frac{\omega}{\prod_j f_j - v_j}$$

for all v such that $|v - w| < r$. According to the above lemma, the integral is independent of the choice of r and w as long as $\{v \mid |v - w| = r\} \subset V$.

If we pick $v \in V \setminus \hat{V}$ and r and $w \in \hat{V}$ such that $\{v \mid |v - w| = r\} \subset V$, the above integral will define a value of $H(v)$. This extends H holomorphically. Hence, H may be defined on the entire V by this integral. ■

This theorem may also be proved by the method used in theorem 1.12; in this I do not need to apply theorems A.24 or A.25.

1.3 Push Forward of Holomorphic p-forms

So far, we have only pushed n-forms. There is, however, no reason why one could not define push forward for p-forms in general; in fact, the ordinary definition of $f_*\psi$ as a current does this. As before, I will concentrate on holomorphic forms.

Let ψ be a holomorphic p-form on $\mathbf{C}^n$, f as before. Wherever $\Delta \neq 0$, there is a local push forward $f_*(z)(\psi(z))$. This may be made into a global push forward as in definition 1.3.

Now, we may use the apparatus for push forward of holomorphic n-forms and the relation

$$\int_D f_*\psi \wedge \phi = \int_{f^{-1}(D)} \psi \wedge f^*\phi$$

to give an explicit expression for $f_*\psi$.

Lemma 1.7 Let $f : \overline{U} \to \mathbf{C}^n$ as in notation 1.1. If ψ is a holomorphic p-form on $\mathbf{C}^n$, the push forward of ψ by f is given by

$$f_*\psi = \sum_{|I|=p} P_I \; dv_I$$

where P_I is given by

$$P_I \; dv = (-1)^{[I,J]} f_*(\psi \wedge df_J)$$

where I, J, and $(-1)^{[I,J]}$ are such that $dv = (-1)^{[I,J]} \; dv_I \wedge dv_J$.

Let V be as in definition 1.3; then, $f_*\psi$ is well defined and holomorphic on V.

Proof: The form $\psi \wedge df_J$ is holomorphic. By theorem 1.6, its push forward is holomorphic; hence, so is P_I. ■

Due to this lemma, most results about push forward of holomorphic n-forms may be generalized to push forward of holomorphic p-forms.

1.4 Push Forward on Manifolds

It is, of course, tempting trying to do on varieties and manifolds what I have done in $\mathbf{C}^n$. However, in order to successfully define a global push forward, I find it necessary to put rather strong conditions on the maps and forms.

Notation 1.8 Let $f : M \to N$ be proper, M and N complex manifolds of dimension n or Zariski-open subsets of (complex-analytic) varieties; as long as the map is proper and dominating, anything will do. ■

Lemma 1.9 If $\psi = p \; dz$ is a holomorphic n-form,

$$\frac{(-1)^{\frac{n(n+1)}{2}}}{(2i)^n} \; \psi \wedge \overline{\psi} = |p|^2 \bigwedge_{j=1}^{n} dx_j \wedge dy_j$$

is a real form.

Proof: We have

$$\begin{aligned} \psi \wedge \overline{\psi} &= |p|^2 \; dz \wedge \overline{dz} \\ &= |p|^2 \; (-1)^{\frac{n(n+1)}{2}} \bigwedge_{j=1}^{n} d\overline{z}_j \wedge dz_j \\ &= |p|^2 \; (-1)^{\frac{n(n+1)}{2}} \; (2i)^n \bigwedge_{j=1}^{n} dx_j \wedge dy_j . \end{aligned}$$

■

Lemma 1.10 Let $D \subset N$ be an open set and let ϕ be an n-form which is holomorphic on $D \setminus V$, V a complex-analytic set in D. If $\int_D \phi \wedge \overline{\phi}$ is bounded, then ϕ may be extended holomorphically to all of D.

Proof: If $v \in V \cap D$ is a smooth point of V: i.e., there is an open neighborhood $U \subset D$ of v and a biholomorphic map $\alpha : U \to \mathbf{C}^n$ such that $a_1 = 0$ on V. The Laurent series is $\alpha_* \phi(w) = \sum_\mu a_\mu w^\mu \, dw$, $\mu \in \mathbf{Z} \times \mathbf{N}_0^{n-1}$. Use the identity

$$\int_{|w|\le r} \alpha_*(\phi \wedge \overline{\phi}) = \int_{|w|\le r} \sum_{\mu,\nu} a_\mu \overline{a_\nu} w^\mu \, \overline{w}^\nu dw \wedge \overline{dw} = \int_{|w|\le r} \sum_{\mu} |a_\mu w^\mu|^2 \, dw \wedge \overline{dw}$$

for $\{|w| \le r\} \subset \alpha(U)$. This integral is finite only if $a_\mu = 0$ whenever $\mu_1 < 0$; hence, ϕ must be holomorphic on U.

This argument makes ϕ holomorphic except on the set of singular points of V. By lemma A.20, this set has codimension ≥ 2; hence, by lemma A.21, the form may be extended holomorphically to all of U. ■

Lemma 1.11 For $\omega = h \; dz$ a holomorphic n-form, $f_*\omega = H \; dv$, and $f_*(\omega \wedge \overline{\omega}) = G \; dv \wedge \overline{dv}$, we have $|H|^2 \le n|G|$ wherever they are both defined. As a consequence,

$$0 \le \frac{(-1)^{\frac{n(n+1)}{2}}}{(2i)^n} \int_D f_*\omega \wedge \overline{f_*\omega} \le \frac{(-1)^{\frac{n(n+1)}{2}}}{(2i)^n} \int_D f_*(\omega \wedge \overline{\omega})$$

for any region D.

Proof: Use the Cauchy–Schwarz inequality: for $a_1, \ldots, a_k \in \mathbf{C}$, we get $|\sum_{j=1}^k a_j|^2 \le n \sum_{j=1}^k |a_j|^2$.

$$\begin{aligned} |H|^2 &= \left| \sum_{f(z)=v} \frac{h(z)}{\Delta(z)} \right|^2 \\ &\le n \sum_{f(z)=v} \left| \frac{h(z)}{\Delta(z)} \right|^2 \\ &= n \, |G|. \end{aligned}$$

This argument works wherever f is finite: i.e., almost everywhere.

The second part follows directly from this and the fact that both forms are real and non-negative. The fact that they may be defined only almost everywhere does not affect the integrals as such sets have measure zero. ■

Theorem 1.12 If ω is a holomorphic n-form on M, the push forward $f_*\omega$ is holomorphic on N.

Proof: The usual way of defining the push forward is to relate it to the pull back by $\int f_*\omega \wedge \psi = \int \omega \wedge f^*\psi$. Wherever f is finite, this is equivalent to definition 1.3. By this definition, $f_*\omega$ becomes a current; I wish to show that it is not only a current, but a form.

Outside a region of codimension one (e.g., that defined locally by $\Delta = 0$), f is finite; hence, on this region, $f_*\omega$ is a form.

By the above lemmas, and the fact that f is proper,

$$\begin{aligned} \frac{(-1)^{\frac{n(n+1)}{2}}}{(2i)^n} \int_K f_*\omega \wedge \overline{f_*\omega} &\leq \frac{(-1)^{\frac{n(n+1)}{2}}}{(2i)^n} \int_K f_*(\omega \wedge \overline{\omega}) \\ &= \frac{(-1)^{\frac{n(n+1)}{2}}}{(2i)^n} \int_{f^{-1}(K)} \omega \wedge \overline{\omega} \\ &< \infty \end{aligned}$$

for compact sets $K \subset N$. If we let V be the image under f of the algebraic set $\Delta = 0$, lemma 1.10 gives us that $f_*\omega$ is holomorphic on the interior of K.

As any point in N has a compact neighborhood, $f_*\omega$ must be a form on N. ■

Chapter 2

Push Forward by Algebraic Maps

I am very interested in studying push forward by algebraic mappings. This is partially due to the fact that one may expect to be able to make stronger predictions for polynomial and rational maps then for general holomorphic functions; also, through polynomials, we are in touch with algebraic problems (see chapter 3).

2.1 Push Forward on Varieties

Notation 2.1 Let X and Y be varieties, $f : X \to Y$ a rational dominating map, and $K(X)$ and $K(Y)$ the fields of rational functions on X and Y. ∎

The main theorem of this section will be:

Theorem 2.2 If h is rational, ω a rational p-form, $f_*\omega$ will also be a rational p-form. If ω is a polynomial p-form and f is proper, then $f_*\omega$, too, is a polynomial p-form. ∎

It is necessary to demand properness in order that push forward of a polynomial form be polynomial; it is not sufficient that the map be finite: i.e., that $\# f^{-1}(q) < \infty$.

Example 2.3 Let $f : \mathbf{C}^2 \to \mathbf{C}^2$ by $f(x, y) = (x, xy^2+y)$. Solving $f(x, y) = (u, v)$ makes $x = u$ and $uy^2 + y = v$ which has at most two solutions. We find that $f_*(y) = -1/u$ (push forward of a function, not a form). ∎

Later, in lemma 2.18, I will prove that if f is defined by homogeneous polynomials, finiteness will imply properness. In the more general case there is a simple criterion for when f is proper. If the number of zeroes of $f - v$ is constant when counted with multiplicities, f is proper. This is due to the fact that the number of zeroes of $f - v$ in $\mathbf{CP}^n$ is $\prod \deg f_j$. Hence, the condition is that the number of zeroes of $f - v$ at the hyperplane at infinity, is independent of v; if $z_j \to \infty$ and $f(z_j) \to v$, that would not be the case.

I have proved the main part of the theorem in two different ways: one analytic and one algebraic. The original proof is the analytic, but the algebraic is far simpler.

2.1.1 Analytic Proof

Lemma 2.4 With $f : \mathbf{C}^n \to \mathbf{C}^n$ as above, there is a polynomial p such that the restriction $f : \mathbf{C}^n \setminus \{p \circ f = 0\} \to \mathbf{C} \setminus \{p = 0\}$ is proper.

Proof: Let $X \subset \mathbf{CP}^n \times \mathbf{CP}^n$ be the closure of $\{(z, f(z)) \mid z \in \mathbf{C}^n\}$. This is an algebraic set. By theorem A.6, the projection $Y = \pi_2(X \cap (H \times \mathbf{CP}^n))$, $H \subset \mathbf{CP}^n$ the hyperplane at infinity, is an algebraic set as well. The set Y has dimension $\leq n - 1$: it does not fill $\mathbf{CP}^n$; hence, there is a non-zero polynomial p which vanishes on Y.

Let $K \subset \mathbf{C}^n \setminus \{p = 0\} \subset \mathbf{CP}^n$ be a compact set. The pull back of K to $X \subset \mathbf{CP}^n$ is closed and does not meet the hyperplane $H \times \mathbf{CP}^n$, hence, it is bounded in $\mathbf{C}^n$; this makes the pull back compact. As this applies to all compacts that do not meet $\{p = 0\}$, the restriction of f described above must be proper. ■

Lemma 2.5 Let $Y \subset \mathbf{C}^q$ be an n-dimensional affine algebraic variety, $f : \mathbf{C}^n \to Y$ a dominating polynomial map, and $g : \mathbf{C}^n \to \mathbf{C}$ a polynomial. There are polynomials $p : \mathbf{C}^q \to \mathbf{C}$ and $q : \mathbf{C}^n \to \mathbf{C}$ such that $p(f(z)) = g(z)q(z) \not\equiv 0$.

Proof: Let $X \subset \mathbf{CP}^n \times \mathbf{CP}^q$ be the closure of $\{(z, v) \in \mathbf{C}^n \times \mathbf{C}^q \mid v \in Y, g(z) = 0\}$. By theorem A.6, the projection of X onto $\mathbf{CP}^q$ is an algebraic set. Let Z be the intersection between $Y \subset \mathbf{C}^q \subset \mathbf{CP}^q$ and the projection of X; Z is an algebraic set in Y. As the dimension of Z is less than that of Y, there is a polynomial r that vanishes on Z but not on Y. As $r \circ f = 0$ wherever $g = 0$, by corollary A.11, $g | (r \circ f)^k$ for some k. Use $p = r^k$.

Let me also sketch an alternative proof. For V an algebraic set of dimension d, let $A_k(V)$ be the polynomials on V of degree $\leq k$; $\dim_{\mathbf{C}} A_k(V) = O(k^d)$. If $V \to W$ by a map, we get a map $A(W) \to A(V)$. If $\dim V < \dim W$, $A(W)$ cannot be mapped injectively into $A(V)$ as $A(W)$ is 'larger' than $A(V)$. Let p be a non-zero polynomial in $A(W)$ which maps to zero in $A(V)$. ∎

Corollary 2.6 If $f : \mathbf{C}^n \to \mathbf{C}^n$ is a polynomial map with $\Delta \not\equiv 0$ and h is a polynomial, there are polynomials p and q such that $p(f(z)) = h(z)q(z)$. ∎

Lemma 2.7 Let $Y \subset \mathbf{C}^q$ be an algebraic set and $h : Y \to \mathbf{C}$ a holomorphic function such that $h(v) = O(\|v\|^k)$. Then, h is polynomial (on Y): i.e., may be defined by a polynomial.

Proof: Let $\tilde{Y}$ be the closure of Y in $\mathbf{CP}^q$ and let V_j denote the hyperplanes in $\mathbf{CP}^k$ corresponding to $u_j = 0$, $u_0, \ldots, u_q$ the coordinates of $\mathbf{C}^{q+1} \setminus \{0\} \to \mathbf{CP}^q$, $v_j = u_j/u_0$. On $\tilde{Y} \setminus V_j$,

$$\frac{h(v)}{v_j^k} = h\left(\frac{u_1}{u_0}, \ldots, \frac{u_q}{u_0}\right)\left(\frac{u_0}{u_j}\right)^k = O\left(\frac{\|u\|^k}{|u_j|^k}\right)$$

is holomorphic; even on V_0. Hence, h is meromorphic on $\tilde{Y}$. By lemma A.19, a meromorphic function on an algebraic set in some projective space is rational, and hence, h must be rational. As h is rational on $\tilde{Y}$ and holomorphic on Y, it must be polynomial on Y. ∎

Lemma 2.8 Let $Y \subset \mathbf{C}^q$ be an affine algebraic variety of dimension n, $f : \mathbf{C}^n \to Y$ a dominating polynomial map, and $\omega = h\, dz$ a rational n-form on $\mathbf{C}^n$. Then, $f_*\omega$ is a rational n-form on Y.

Proof: We may assume that Y is not contained in any linear subspace on $\mathbf{C}^q$; otherwise, we just reduce the problem to $Y \subset \mathbf{C}^{q'}$ for some $q' < q$.

Let $\pi : \mathbf{C}^q \to \mathbf{C}^n$ be a linear projection. Assume that $\pi \circ f : \mathbf{C}^n \to \mathbf{C}^n$ is such that $\Delta = \Delta_{\pi\circ f} \not\equiv 0$. By lemma 2.4 there is a polynomial R such that the restriction $\pi \circ f : \mathbf{C}^n \setminus \{p \circ f = 0\} \to \mathbf{C}^n \setminus \{p = 0\}$ is proper. Pick non-zero polynomials P, Q, p, and q such that $P(f(z)) = \Delta(z)Q(z)$ and $p(f(z)) = h_2(z)q(z)$ where $h = h_1/h_2$; according to lemma 2.5 such polynomials exist.

Define $E \subset \mathbf{R} \times \mathbf{R} \times \mathbf{C}^n \times \mathbf{C}^q$ by

$$E = \left\{ \left(\|v\| + \tfrac{1}{|P(v)R(\pi(v))|},\, \|z\|,\, z,\, v \right) \,\middle|\, P(v), R(\pi(v)) \neq 0,\, v = f(z) \right\}.$$

The set E is semialgebraic: see definition A.22. Let

$$l(x) = \sup\{y \mid (x, y, z, v) \in E\}.$$

Let $s(v) = \|v\| + 1/|P(v)R(v)|$ and $U_x = \{v \mid s(v) \leq x\}$. As $s(v) \to \infty$ whenever v goes towards infinity or $\{PR = 0\}$, U_x is closed and bounded; i.e., U_x is compact. (The risk was that there could have been a sequence of v's in U_x that converge to a point on $\{PR = 0\}$ or goes to infinity.) As U_x is compact and f is proper when restricted to $\{R \neq 0\}$, $f^{-1}(U_x)$ is compact. This makes z bounded for $s(v) \leq x$; hence, $x < \infty \Rightarrow l(x) < \infty$.

As $x < \infty \Rightarrow l(x) < \infty$, lemma A.23 says that $l(x) = Ax^a(1 + o(1))$, which, in this case, makes

$$\|z\| \leq A\left(1 + \|v\| + \tfrac{1}{|P(v)R(v)|}\right)^a.$$

Now, for large z,

$$\begin{aligned}
\left|\frac{h(z)}{\Delta(z)}\right| &= \left|\frac{h_1(z)\,q(z)\,Q(z)}{p(v)\,P(v)}\right| \\
&= O\left(\frac{\|z\|^k}{|p(v)\,P(v)|}\right) \quad \text{where } k = \deg h_1\,q\,Q \\
&= O\left(\frac{\left(1 + \|v\| + \frac{1}{|P(v)R(v)|}\right)^{ak}}{|p(v)P(v)|}\right).
\end{aligned}$$

This makes $|p(v)P(v)^{m+1}R(\pi(v))^m h(z)/\Delta(z)| = O(\|\pi(v)\|^l)$ for some l and, hence, $f_*\omega = H\,d\pi$ where $H(v)p(v)P(v)^{m+1}R(\pi(v))^m = O(\|\pi(v)\|^l)$.

The form $d\pi$ is a polynomial form on Y. Also, $H(v)p(v)P(v)^{m+1}R(\pi(v))^m = O(\|v\|^l)$ is holomorphic on Y; by lemma 2.7 it is polynomial. Then, H must be rational, making $f_*\omega$ a rational form. ■

Corollary 2.9 If $f : \mathbf{C}^n \to \mathbf{C}^n$ is polynomial and $\omega = h\,dz$ a rational n-form, then $f_*\omega$ is a rational n-form. ■

Lemma 2.10 Let $X \subset \mathbf{C}^{p+q}$ and $Y \subset \mathbf{C}^q$ be n-dimensional affine algebraic varieties such that $X = \{(x, y) \subset \mathbf{C}^{p+q} \mid y \in Y, P(x) = 0\}$, $P = (P_1, \ldots, P_r) : \mathbf{C}^{p+q} \to \mathbf{C}^r$ polynomials. Let ω be an n-form on X given by $\omega = \sum_{|I|=n} h_I\,dz_I$ where h_I are rational functions on $\mathbf{C}^{p+q}$. If π is the projection of X onto Y, $\pi_*\omega$ is a rational form on Y.

Proof: Let us now define $f : \mathbf{C}^{p+q} \to \mathbf{C}^{r+q}$ by $f(x, y) = (P(x, y), y)$. Now, $X = f^{-1}(\{0\} \times Y)$. By combining lemma 2.8 and lemma 1.7, we get that $f_*\omega$ is a rational n-form on the image of f; this form may be restricted to $\{0\} \times Y \cong Y$ giving $\pi_*\omega$. The form $f_*\omega$ may not be singular along $\{0\} \times Y$ as the projection of X on Y is finite almost everywhere. ■

Theorem 2.11 Let $X \subset \mathbf{CP}^p$ and $Y \subset \mathbf{CP}^q$ be n-dimensional algebraic varieties, $f : X \to Y$ a rational dominating map, ω a rational n-form on X; then, $f_*\omega$ is rational.

Proof: Let $Z \subset X \times Y$ define the rational map f and let π be the projection onto Y. The form ω may be pulled back to Z; this form I will denote ω and from now I will not mention X again.

Let H_1 and H_2 be hyperplanes in $\mathbf{CP}^p$ and $\mathbf{CP}^q$, $H = H_1 \times \mathbf{CP}^q \cup \mathbf{CP}^p \times H_2$, such that Z is not contained in H. By theorem A.6, $V = \pi(Z \cup H) \subset \mathbf{CP}^q$ is an algebraic set. Also, $Y \setminus V \subset \mathbf{CP}^q \setminus H_2 \cong \mathbf{C}^q$. Let g be a non-zero polynomial on $\mathbf{C}^q$ such that $g = 0$ on $V \setminus H_2$; as V is an algebraic set, such a g must exist. Now, $\hat{Y} = \{(y,t) \in \mathbf{C}^q \times \mathbf{C} \mid y/g(t) \in Y\}$ and $\hat{Z} = \{(z,t) \in \mathbf{C}^{p+q} \times \mathbf{C} \mid z/g(t) \in Z\}$ are affine varieties, and $\hat{Y} \cong Y \setminus \{g = 0\}$ and $\hat{Z} \cong Z \setminus \{g = 0\}$; hence, we may restrict the map $\pi : \hat{Z} \to \hat{Y}$.

Let $P_1, \dots, P_r$ be polynomials defining $\hat{Z}$ over $\hat{Y}$:

$$\hat{Z} = \left\{(x,y) \in \mathbf{C}^p \times \mathbf{C}^{q+1} \,\middle|\, y \in \hat{Y}, P(x,y) = 0\right\}.$$

These polynomials are simply polynomials generating the prime ideal

$$I(\hat{Z}) = \left\{p \in \mathbf{C}[x_1, \dots, x_p, y_1, \dots, y_q, t] \,\middle|\, p = 0 \text{ on } \hat{Z}\right\}$$

over the prime ideal

$$I(\hat{Y}) = \left\{p \in \mathbf{C}[y_1, \dots, y_q, t] \,\middle|\, p = 0 \text{ on } \hat{Y}\right\} \subset \mathbf{C}[x_1, \dots, x_p, y_1, \dots, y_q, t].$$

Now, we have the situation in lemma 2.10: the push forward $\pi_*\omega$ is a rational form on $\hat{Y}$ and, hence, on Y. ∎

2.1.2 Algebraic Proof

Definition 2.12 Let $F \subset K$ be fields, $[K : F] = m$, and let $a_1, \dots, a_m \in K$ be a basis for K over F. If $x \in K$, multiplication with x in K gives a linear transformation on K over F; let $A \in F^{m \times m}$ be the matrix such that $xa = A \cdot a$ where $a = [a_1, \dots, a_m]$. Define the **trace** of x to be $\operatorname{Tr}_{K/F} x = \operatorname{Tr} A = \sum_{j=1}^m A_{jj}$. The trace is independent of choice of basis, and, clearly, gives a map $K \to F$. ∎

Lemma 2.13 If $f : X \to Y$ is a regular dominating map, X and Y affine varieties, we get a map $f : K(Y) \to K(X)$. We may define the trace Tr :

$K(X) \to K(Y)$ as above. Now, the push forward of a function on X equals its trace: $f_*(h) = \mathrm{Tr}_{K(X)/K(Y)}\, h$ for all $h \in K(X)$.

Proof: As f is a dominating regular map, $K(X)$ is, by theorem A.13, a finite extension of $K(Y)$ of degree $m = \deg f$; hence, there is a basis a_j, and trace may be defined.

Let $h \in K(X)$ and let $ha = A \cdot a$, $A = [A_{ij}]_{ij}$ and $a = [a_j]_j$. Let $v \in Y$ be a point such that $f^{-1}(y) = \{z_1, \ldots, z_m\}$ are m distinct points. Now, we have $[h(z_k)a_i(z_k)]_{ik} = A \cdot [a_j(z_k)]_{jk} = [a_i(z_j)]_{ij} \cdot [\delta_{jk} h(z_k)]_{jk}$. This makes $\mathrm{Tr}\, h = \mathrm{Tr}\, A = \mathrm{Tr}([a_i(z_j)]_{ij} \cdot [\delta_{jk} h(z_j)]_{jk} \cdot [a_k(z_l)]_{kl}^{-1}) = \sum_j h(z_j) = f_*(h)$.

In order to do the above calculation, we must demand that $\det[a_i(z_j)]_{ij} \not\equiv 0$. Otherwise, we would have

$$\begin{aligned}
\det\Big[f_*(a_i a_j)\Big]_{ij} &= \det\left[\sum_k a_i(z_k) a_j(z_k)\right]_{ij} \\
&= \det\Big([a_i(z_k)]_{ik} \cdot [a_j(z_k)]_{kj}\Big) \\
&= \Big(\det[a_i(z_k)]_{ik}\Big)^2 \\
&= 0.
\end{aligned}$$

Now, we could find $g_j \in K(Y)$ such that for $h = \sum_j a_j g_j$, $f_*(a_i h) = \sum_j f_*(a_i a_j) g_j = 0$ for all i. But, as a_i is a basis for $K(X)$, $1/h = \sum_i a_i h_i$; now, $f_*(1) = f_*(h/h) = \sum_{i,j} h_i f_*(a_i a_j) g_j = 0$ which is obviously wrong. Hence, the assumption that $\det[a_i(z_j)]_{ij} \not\equiv 0$ must be wrong.

As $\det[a_i(z_j)]_{ij} \neq 0$ outside a subvariety, the equality $\mathrm{Tr}\, h = f_*(h)$ holds globally.

■

Lemma 2.14 Let $f : X \to Y$ be a rational dominating map; this gives a map $f : K(Y) \to K(X)$ and, hence, a trace $\mathrm{Tr} : K(X) \to K(Y)$. Trace equals the push forward of functions as in the above lemma.

Proof: Let $Z \subset X \times Y$ be the variety defining f, π the projection of Z onto Y. As in theorem 2.11, we may a polynomial g defining a variety on Y such that $\hat{Y} = \{(y,t) \in \mathbf{C}^q \times \mathbf{C} \mid y/g(t) \in Y\}$ and $\hat{Z} = \{(z,t) \in \mathbf{C}^{p+q} \times \mathbf{C} \mid z/g(t) \in Z\}$ are affine varieties, $\hat{Y} \cong Y \setminus \{g = 0\}$ and $\hat{Z} \cong Z \setminus \{g = 0\}$; the restricted map $\pi : \hat{Z} \to \hat{Y}$ is a regular dominating map, hence, by the above lemma, trace and push forward of functions are the same. ■

Lemma 2.15 Let $f : X \to Y$ be a rational dominating map, ω a rational n-form on X. Then, the push forward $f_*\omega$ is a rational n-form on Y.

Proof: Let ψ be a rational n-form on Y; by lemma A.5, such a form exists. Then, $\omega = h\, f^*\psi$ where h is a rational function on X. The push forward of ω is $f_*\omega = f_*(h\, f^*\psi) = f_*(h)\,\psi$. By the above lemma, $f_*(h) = \operatorname{Tr} h \in K(Y)$; i.e., $f_*(h)$ is rational. ■

2.1.3 Proof of the Main Theorem

Proof: If ω is a polynomial n-form and f is proper, the push forward is rational by the above theorems and holomorphic by theorem 1.12; hence, it is a polynomial form.

The push forward $f_*\omega = \sum_{|I|=p} P_I\, dv_I$. The form $\omega \wedge df_J$ is rational and, if ω is polynomial, polynomial; hence, by theorem 2.2, P_I, as they are expressed in lemma 1.7, are rational and, if ω is polynomial and f is proper, polynomial. ■

Corollary 2.16 If $f : \mathbf{C}^n \to \mathbf{C}^n$ is defined (almost everywhere) by rational functions such that $\Delta \not\equiv 0$ — i.e., f is a dominating rational map — and $\omega = h\, dz$ is a rational n-form, then $f_*\omega$ is a rational n-form. ■

Corollary 2.17 If $f : \mathbf{C}^n \to \mathbf{C}^n$ is defined by a proper polynomial map and ω is a polynomial form, then $f_*\omega$, too, is a polynomial form. ■

2.2 Push Forward by a Generic Polynomial Map

The Jacobi relation — see page 671 of [5] — states that if $\deg h < \sum_j \deg f_j - n$ and $f = v$ intersect transversally in $\prod_j \deg f_j$ points, $f_*(h\, dz)(v) = 0$. In a similar setting, there is a similar result giving a bound on the degree of the push forward.

The restriction may be formulated in a series of equivalent ways; this is done in the following lemma. A generic polynomial map $f = g + \hat{g} : \mathbf{C}^n \to \mathbf{C}^n$ where g is the highest order term, satisfies the conditions of the lemma. This is most easily seen from statement five: the common zeroes of f lie outside the hyperplane at infinity. Of course, by restricting ourselves by this condition, we leave out some special cases of 'difficult' polynomial maps, but, in return, we get very strong results: see theorems 2.19, 3.12, and 3.14.

Lemma 2.18 Let $g : \mathbf{C}^n \to \mathbf{C}^n$ where g_j are homogeneous polynomials. The following are equivalent:

1. $g : \mathbf{C}^n \to \mathbf{C}^n$ is finite.
2. The set in $\mathbf{CP}^{n-1}$ defined by $g = 0$ is empty.
3. $f = g + \hat{g}$ is proper for all $\hat{g}$ such that $\deg \hat{g}_j < \deg g_j$.
4. g is proper.
5. If $f = g + \hat{g}$ (as in 3), extend f to $\mathbf{CP}^n \supset \mathbf{C}^n$; $f_1, \dots, f_n$ have no common zero on the hyperplane at infinity.
6. If $f = g + \hat{g}$ (as in 3), the general number of solutions to $f(z) = v$ is $\deg f = \prod_j \deg f_j = \prod_j \deg g_j$.

Proof: I will show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$ and $2 \iff 5 \iff 6$:

$1 \Rightarrow 2:$ If $g(z) = 0$ for some $z \neq 0$, $g(\lambda z) = 0$ for all $\lambda \in \mathbf{C}$, and, hence, g is not finite.

$2 \Rightarrow 3:$ Assume that f is not proper; then there is an unbounded sequence z_j such that $f(z_j)$ is bounded. By picking a subsequence, you may demand that $|z_j| \to \infty$ and $f(z_j) \to v$. Let u be an accumulation point for $z_k/\|z_k\| \in S^{2n-1}$; as S^{2n-1} is compact, such a point must exist. We use that $g_j(z/\|z\|) = v_j/\|z\|^{d_j}$, $d_j = \deg g_j$, for all $z \in X$. Now,

$$g_j\left(\frac{z_k}{\|z_k\|}\right) = \frac{f_j(z_k) - \hat{g}_j(z_k)}{\|z_k\|^{d_j}} = \frac{v_j}{\|z_k\|^{d_j}} - \frac{\hat{g}_j(z_k)}{\|z_k\|^{d_j}} \to 0;$$

hence, $g(u) = 0$ with $u \neq 0$, giving a point in $\mathbf{CP}^{n-1}$ with $g = 0$.

$3 \Rightarrow 4:$ Let $\hat{g} = 0$.

$4 \Rightarrow 1:$ Lemma A.8.

$2 \iff 5:$ At the hyperplane at infinity $f = g$, hence, the equivalence of 2 and 5.

$5 \iff 6:$ Compactifying $\mathbf{C}^n$ we get $\mathbf{CP}^n$. Counted with multiplicities, the set of equations $f(z) = v$ has $\prod_j \deg f_j$ solutions in $\mathbf{CP}^n$; this follows from Bezout's theorem: theorem A.7. So, 6 holds iff and only iff all these solutions are in $\mathbf{C}^n \subset \mathbf{CP}^n$ which is statement 5.

■

Theorem 2.19 Let f_j and h be polynomials, $f_j = g_j + \hat{g}_j$ where g_j is homogeneous of degree d_j and $\deg \hat{g}_j < d_j$. Assume that g is proper. Then, $H(v) = \sum a_\mu v^\mu$ where $\mu \in \mathbf{N}_0^n$, $\mu \cdot d \leq \deg h - |d| + n$, and $a_\mu \in \mathbf{C}$: i.e., $\deg H \circ f \leq \deg h - \deg \Delta$.

Proof: We may assume that h is homogeneous of degree D; otherwise, we regard each homogeneous component separately. Now,

$$\begin{aligned}
\int_{|v_j|=r_j} \frac{H(v)\,dv}{v^{\mu+1}} &= \int_{|f_j(z)|=r_j} \frac{\omega}{f^{\mu+1}} \quad \text{let } z = \lambda u \text{ and } r_j = |\lambda|^{d_j} \rho_j \\
&= \int_{|f_j(\lambda u)|=|\lambda|^{d_j}\rho_j} \frac{h(\lambda u)\,\lambda^n\,du}{f(\lambda u)^{\mu+1}} \\
&= \int_{|g_j(u)+\hat{g}_j(\lambda u)/\lambda^{d_j}|=\rho_j} \frac{h(\lambda u)\,\lambda^n\,du}{f(\lambda u)^{\mu+1}}.
\end{aligned}$$

For some λ, define $A \subset \mathbf{C}^n \times \mathbf{R}$ by

$$A = \left\{(u,t) \,\middle|\, 0 \leq t \leq 1,\, |g_j(u) + t\,\hat{g}_j(\lambda u)/\lambda^{d_j}| = \rho_j\right\}.$$

It is necessary that $f_j(\lambda u)$ is non-zero for $(u,t) \in A$ for sufficiently large λ. Otherwise, the form would not be holomorphic on A.

Assume that there is a sequence (u_m, λ_m, t_m) and a k such that $|g_j(u_m)+t_m\hat{g}_j(\lambda_m u_m)/\lambda_m^{d_j}| = \rho_j$ for all j, $f_k(\lambda_m u_m) = 0$, and $\lambda_m \to \infty$.

If u_m were unbounded, there would be a subsequence such that $\|u_m\| \to \infty$ and $u_m/\|u_m\| \to \nu \in S^{2n-1}$. (This part of the proof resembles that of lemma 2.18.) Now,

$$\left| g_j\left(\frac{u_m}{\|u_m\|}\right) + \frac{t_m \hat{g}_j(\lambda_m u_m)}{\lambda_m^{d_j}\|u_m\|^{d_j}} \right| = \frac{\rho_j}{\|u_m\|^{d_j}} \to 0$$

for this sequence. As $t_m\hat{g}_j(\lambda_m u_m)/\lambda_m^{d_j}\|u_m\|^{d_j} \to 0$ in the above limit, we must have $g_j(u_m/\|u_m\|) \to 0$ making $g(\nu) = 0$; this is not possible.

Now, u_m is bounded and, hence, has an accumulation point u. The above conditions make $|g_j(u)| = \rho_j$ for all j and $g_k(u) = 0$ (using $g_k(u_m) = (f_k(\lambda_m u_m) - \hat{g}_k(\lambda_m u_m))/\lambda_m^{d_j}$). These are contradictory and, hence, such a sequence may not exist: for sufficiently large λ such u and t may not exist.

Now, $f_j \neq 0$ on A for all j for λ sufficiently large. Using Stoke's theorem — theorem A.24 or theorem A.25 — as in lemma 1.5, we get

$$\begin{aligned}
\int_{|v_j|=r_j} \frac{H(v)\,dv}{v^{\mu+1}} &= \int_{|g_j(u)+\hat{g}_j(\lambda u)/\lambda^{d_j}|=\rho_j} \frac{h(\lambda u)\,\lambda^n\,du}{f(\lambda u)^{\mu+1}} \\
&= \int_{|g_j(u)|=\rho_j} \frac{h(\lambda u)\,\lambda^n\,du}{f(\lambda u)^{\mu+1}} + \int_A d\left(\frac{h(\lambda u)\,\lambda^n\,du}{f(\lambda u)^{\mu+1}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{|g_j(u)|=\rho_j} \frac{\lambda^{D+n-|d|-\mu\cdot d}h(u)}{g(u)^{\mu+1}} \left(\frac{g(\lambda u)}{f(\lambda u)}\right)^{\mu+1} du \\
&= \lim_{\lambda\to\infty} \int_{|g_j(u)|=\rho_j} \frac{\lambda^{D+n-|d|-\mu\cdot d}h(u)}{g(u)^{\mu+1}} \left(\frac{g(\lambda u)}{f(\lambda u)}\right)^{\mu+1} du \\
&= 0 \quad \text{if } D+n-|d|-\mu\cdot d < 0.
\end{aligned}
$$

This makes $H(v) = \sum a_\mu v^\mu$ where $\mu \in \mathbf{N}_0^n$, $\mu \cdot d \leq \deg h - |d| + n$.

There is an alternative proof. It is simply applying the global residue theorem — theorem A.9 — to the form $\omega/f^{\mu+1}$ for μ such that $\mu \cdot d > \deg h - |d| + n$: this states that $\sum_p \operatorname{Res}_p \omega/f^{\mu+1} = 0$ for $p \in \mathbf{CP}^n$. Using different affine coordinates on $\mathbf{CP}^n$, we find that none of the residues lie at the hyperplane at infinity; this makes $\operatorname{Res}_0 f_*\omega/v^{\mu+1} = 0$, hence, the power series for H contains no v^μ-terms. ■

In the case of push forward of a function, not a form, we have the following result.

Corollary 2.20 Let f, g, and h be as in the above theorem. Then $f_*(h) = H$ where $\deg H \circ f \leq \deg h$.

Proof: As $H dv = f_*(h\Delta dz)$, the above theorem implies that $\deg H \leq \deg(h\Delta) - \deg \Delta = \deg h$. ■

Now, one might hope for a more general result: for general f, $H\,dv = f_*(h\,dz) \Rightarrow \deg H \circ f \leq \deg h - \deg \Delta$. This, however, is not true in general as the following example shows.

Example 2.21 Let $f : \mathbf{C}^2 \to \mathbf{C}^2$ by $f(x, y) = (x^2, x^3 + y)$; $\Delta = 2x$. If $f(x, y) = (u, v)$, we have an alternative solution: $f(-x, y + 2x^3) = (u, v)$. Let $\omega = xy\, dx \wedge dy$. This makes $f_*\omega = v\; du \wedge dv$. We find that $\deg H \circ f = 3$, whereas $\deg h - \deg \Delta = 1$. ■

This example shows that the demand in theorem 2.19 that the highest degree terms define a finite map, is needed.

Corollary 2.22 If $n = 1$, $U = f^{-1}(V) \subset \mathbf{C}$, and $\overline{U}$ is compact with piecewise C^1-boundary, then $f_*\omega(v) = H(v)\,dv$ where

$$
H(v) = \sum_{f(z)=v} \frac{h(z)}{f'(z)} = \frac{1}{2\pi i} \int_{\partial U} \frac{h(z)\,dz}{f(z) - v}.
$$

For h a polynomial, $\deg H \leq (\deg h + 1)/\deg f - 1$. ■

2.3 An Example of Push Forward by a Non-Finite Map

The calculations below show what push forward does for a particularly simple group of maps. To some extent, it gives a demonstration of push forward by a non-finite map.

Lemma 2.23 Let $f_j = z^{\alpha_j}$, $\alpha_j \in \mathbf{N}_0^n$; I will denote this by $f = z^\alpha$. For $\nu \in \mathbf{Z}^n$,

$$f_*(z^\nu\, dz) = \begin{cases} 0 & \text{if } \nu + 1 \notin \operatorname{Span}_{\mathbf{Z}} \{\alpha_1, \ldots, \alpha_n\} \\ v^\mu & \text{if } \nu + 1 = \sum \alpha_j + \mu \cdot \alpha \text{ where } \mu \in \mathbf{Z}^n \end{cases}$$

Proof: Let me first set the notation: for $\mu = \{\mu_{j\,k}\}$, $z^\mu = (z^{\mu_1}, \ldots, z^{\mu_n})$ where $z^{\mu_j} = \prod_k z_k^{\mu_{j\,k}}$. This makes $(z^\mu)^\nu = z^{\nu\cdot\mu}$ where $\nu \cdot \mu$ is the matrix product.

We first note that $\Delta(z) = \det\alpha \cdot z^{\sum \alpha_j - 1}$; hence, $\det\alpha \neq 0$. For $h(z) = z^\nu$ we get

$$H(v) = \sum_{z^\alpha = v} \frac{z^{\nu - \sum \alpha_j + 1}}{\det\alpha}.$$

If $u^\alpha = 1$, we get $f(u \times z) = f(z)$ where $u \times z = (u_1 z_1, \ldots, u_n z_n)$. For $\sigma \in \mathbf{Z}^n$,

$$\sum_{f(z)=v} z^\sigma = \sum_{f(u\times z)=v} (u \times z)^\sigma = \sum_{f(z)=v} u^\sigma z^\sigma;$$

hence, either $u^\sigma = 1$ for all u with $u^\alpha = 1$, or the sum equals zero.

Let us assume that $u^\sigma = 1$ for all u with $u^\alpha = 1$. Let $\beta = \det\alpha \cdot \alpha^{-1}$; $\beta \in \mathbf{Z}^{n\times n}$. Let $\mu = \sigma \cdot \beta / \det\alpha$; I wish to prove that $\mu \in \mathbf{Z}^n$. We have

$$z^{\det\alpha} = (z_1^{\det\alpha}, \ldots, z_n^{\det\alpha}) = 1 \Rightarrow (z^\beta)^\alpha = z^{\alpha\cdot\beta} = 1 \Rightarrow (z^\beta)^\sigma = z^{\sigma\cdot\beta} = 1;$$

hence, $\det\alpha | \sigma \cdot \beta$, and $\mu \in \mathbf{Z}^n$. Now, we get

$$z^\sigma = z^{\sigma\cdot\beta\cdot\alpha/\det\alpha} = z^{\mu\cdot\alpha} = (z^\alpha)^\mu.$$

Now, if $\sigma = \nu - \sum \alpha_j + 1$ is as above,

$$\sum_{z^\alpha = v} \frac{z^{\nu - \sum\alpha_j + 1}}{\det\alpha} = \sum_{z^\alpha = v} \frac{z^\sigma}{\det\alpha} = \sum_{z^\alpha = v} \frac{(z^\alpha)^\nu}{\det\alpha} = \sum_{z^\alpha = v} \frac{v^\mu}{\det\alpha} = v^\mu.$$

Here, I have used that the number of solutions to the equation $f(z) = z^\alpha = v$ is

$$\begin{aligned} \frac{1}{(2\pi i)^n} \int_{|f(z)|=r} \frac{df_1 \wedge \cdots \wedge df_n}{\prod (f_j - v_j)} &= \frac{1}{(2\pi i)^n} \int_{|f(z)|=r} \frac{\Delta\, dz}{\prod f_j} \\ &= \frac{1}{(2\pi i)^n} \int_{|f|=r} \frac{\det\alpha\; dz}{\prod_j z_j} \\ &= \det\alpha \end{aligned}$$

where $r > |v|$.

The condition $\sigma \in \mathrm{Span}_{\mathbf{Z}} \{\alpha_1, \ldots, \alpha_n\}$ is equivalent to $\sigma = \mu \cdot \alpha$ for some $\mu \in \mathbf{Z}^n$ which is again, by the above calculation, equivalent to $u^\alpha = 1 \Rightarrow u^\sigma = 1$. If this condition is not satisfied, we cannot get $u^\alpha = 1 \Rightarrow u^\sigma = 1$, and then the sum must be zero. ■

Chapter 3

A Bezout-Like Identity

If $g \in \mathbf{C}(f)$ and ω is a form, $f_*(g\omega) = gf_*(\omega)$. If there are ω_j such that for any ω, $\omega = \sum g_j\omega_j$ for some $g_j \in \mathbf{C}(f)$, we would have $f_*\omega = \sum_j g_j f_*\omega_j$. If we could find a dual basis ν_j such that $f_*(\nu_i\omega_j) = 0$ whenever $i \neq j$, we could in fact calculate the g_j in the representation of ω.

The identity $\omega = \sum_j g_j\omega_j$ is of course a Bezout identity. Initially, all we know about the g_j is the existence. However, if they are given by some push forward, we find the g_j given by an explicit construction. Of course, calculating the push forward may be difficult of even practically impossible, but knowing a way of constructing them is an enormously advantageous position to meerly knowing their existence; for example, finding a bound on the degree is far easier.

Notation 3.1 In this chapter, $f : \mathbf{C}^n \to \mathbf{C}^n$ is a proper map defined by polynomials. Let $m = \deg f$ and pick $q \in \mathbf{C}^n$ such that $f^{-1}(q) = \{p_1, \ldots, p_m\}$ are m distinct points. ■

3.1 Finding a Basis for C(z) over C(f)

Lemma 3.2 Let $V \subset D \subset \mathbf{C}^q$ be an analytic subvariety, D an open set, $a_1, \ldots, a_k : f^{-1}(D) \to \mathbf{C}$ holomorphic and linearly independent over $\mathbf{C}$ on $f^{-1}(v)$ for some $v \in V$. If $g_1, \ldots, g_k : D \to \mathbf{C}$ are holomorphic, and $\sum_{j=1}^k a_j(z)\, g_j(f(z)) = 0$ on $f^{-1}(V)$, then $g = 0$ on V.

Proof: There is a neighborhood $U \subset V$ of v such that a_j are linearly independent over $\mathbf{C}$ on $f^{-1}(u)$ for every $u \in U$. As $\sum_j a_j(z)g_j(f(z)) = 0$ for $z \in f^{-1}(U)$ and the a_j are linearly independent, g_j must be zero on U. As the g_j are zero on an open subset of V, they must be zero on the entire V. ■

Lemma 3.3 If $a_1, \ldots, a_m$ are rational, finite on $f^{-1}(q)$, and linearly independent over $\mathbf{C}$ on $f^{-1}(q)$, then they give a basis for the field of rational functions $\mathbf{C}(z)$ over $\mathbf{C}(f)$.

Proof: Assume that $\sum_j a_j g_j = 0$, $g_j \in \mathbf{C}(f)$. By multiplying all g's by a common denominator, we may take them to be polynomials. Now, according to lemma 3.2, $\sum_j a_j g_j = 0$ would imply $g = 0$, hence, a_j are linearly independent over $\mathbf{C}(f)$. The dimension of $\mathbf{C}(z)$ over $\mathbf{C}(f)$ is m: see corollary A.14; hence, $a_1, \ldots, a_m$ form a basis. ■

In lemma 3.10, this result is proved for $a_1, \ldots, a_m \in \mathbf{C}[z]$ generating $\mathbf{C}[z]/(f)$ without the demand that $f = 0$ be m distinct points.

Lemma 3.4 Let a_j be a basis for $\mathbf{C}(z)$ over $\mathbf{C}(f)$ and $h : \mathbf{C}^n \to \mathbf{C}$ be a polynomial. If $f_*(a_i h \, dz) = 0$ for all i, then $h = 0$.

Proof: Assume that $h \neq 0$. As a_i is a basis for $\mathbf{C}(z)$, $\Delta/h = \sum_j a_j g_j$ for some $g_j \in \mathbf{C}(f)$. Now, $f_*(\Delta \, dz) = m \, dv$, whereas

$$f_* \left(h \sum_j a_j g_j \, dz \right) = \sum_j g_j f_*(h a_j \, dz) = 0;$$

this is clearly contradictory. ■

Lemma 3.5 Let a_j be a basis for $\mathbf{C}(z)$ over $\mathbf{C}(f)$. There are rational functions, or, if you wish, polynomials b_j such that

$$f_*\Big(a_i(z) \, b_j(z) \, dz\Big) = \begin{cases} 0 & \text{if } i \neq j \\ c_i(v) \, dv & \text{if } i = j \end{cases}$$

where c_i are rational and non-zero. Such b_j will necessarily form a basis for $\mathbf{C}(z)$ over $\mathbf{C}(f)$.

If a_i are polynomials and $f_*(a_i a_j \, dz) = A_{ij} \, dv$, b_j may be chosen polynomial and such that $f_*(a_i b_i \, dz) = \det A \, dv$. Hence, if $\det A \in \mathbf{C}$, one may pick b_j such that $c_j = 1$.

Proof: If $\det A = 0$, there would have to be $g \in \mathbf{C}(f)^m$, not all zero, such that $\sum_j A_{ij} \, g_j = 0$. For $h = \sum_j a_j g_j$, this would mean that $\sum_j f_*(a_i a_j g_j \, dz) = f_*(a_i h \, dz) = 0$ for all i. Now, according to lemma 3.4, $h = 0$; furthermore, as a_i are linearly independent over $\mathbf{C}(f)$, $g_j = 0$.

Let $B = A^{-1}$; B, too, is a matrix over $\mathbf{C}(f)$. Let $b_j = \sum_k a_k B_{kj}$. Now,

$$\begin{aligned} f_*(a_i b_j\ dz) &= f_*\left(\sum_k a_i a_k B_{kj}\ dz\right) \\ &= \sum_k f_*(a_i a_k\ dz)\ B_{kj} \\ &= \sum_k A_{ik} B_{kj} \\ &= \delta_{ij}. \end{aligned}$$

Thus, we have our b_j. To make b_j polynomial, multiply by a suitable polynomial in f; according to corollary 2.6 such a polynomial exists. That c_j is rational follows from theorem 2.2.

If b_j did not form a basis of $\mathbf{C}(z)$, $\sum_j b_j g_j = 0$ for some $g_j \in \mathbf{C}(f)$ not all zero. This would make $\sum_j f_*(a_i b_j g_j\ dz) = c_i g_i\ dv$ equal zero, and, hence, $g_i = 0$ for all i.

In the second part, let $B = \det A \cdot A^{-1}$; this will be a matrix over $\mathbf{C}[f]$. Set $b_j = \sum_k a_k B_{kj}$ and calculate as above; the result is $f_*(a_i b_j\ dz) = \delta_{ij} \det A$. ∎

There is, of course, a good indicator for when a_j are linearly independent on $f^{-1}(v)$: if $f^{-1}(v) = \{u_1, \ldots, u_m\}$ (multiplicities counted), $\det(a_j(u_k))_{jk}$ is zero if and only if a_j are linearly dependent. Now, the ordering of the u_j will affect the sign of the determinant, and, hence, it is not necessarily a well defined function in v. In order to correct this, take the square: let

$$D(v) = \left(\det\Big(a_j(u_k)\Big)_{jk}\right)^2.$$

Now, there is a relation between D and $\det A$ where A_{ij} are as in the above lemma: if $f^{-1}(v) = \{u_1, \ldots, u_m\}$,

$$\begin{aligned} \det A(v) &= \det\Big(A_{ij}(v)\Big)_{ij} \\ &= \det\left(\sum_k \frac{a_i(u_k) a_j(u_k)}{\Delta(u_k)}\right)_{ij} \\ &= \det\left(\sum_{k,l} a_i(u_k) \cdot \frac{\delta_{kl}}{\Delta(u_k)} \cdot a_j(u_k)\right)_{ij} \\ &= \det\Big(a_i(u_k)\Big)_{ik} \det\left(\frac{\delta_{kl}}{\Delta(u_k)}\right)_{kl} \det\Big(a_j(u_k)\Big)_{kj} \\ &= \frac{D(v)}{\prod_k \Delta(u_k)} \end{aligned}$$

Theorem 3.6 Let a_j, b_j, and h be as in lemma 3.5 and $f_*(b_j\,h\,dz) = H_j(v)\,dv$. We have

$$h(z) = \sum_{j=1}^{m} \frac{a_j(z)\,H_j(f(z))}{c_j(f(z))}.$$

Proof: Let $\hat{h}$ denote the right side of the above equation. We find that the push forward $f_*(b_j(h-\hat{h})\,dz) = 0$ for all j; hence, according to lemma 3.4, $h = \hat{h}$. ∎

This result is not new. A similar theorem may be found in [1]: theorem 5.17 stating that $\mathbf{C}[z] \subset \sum_{j=1}^{m} u_j \mathbf{C}[f]$, $u_j \in \mathbf{C}(z)$. (It does not state that m is the degree of $\mathbf{C}(z)$ over $\mathbf{C}(f)$, but it follows from the proof.) In order to use the theorem, we must show that $\mathbf{C}[f]$ is integrally closed in $\mathbf{C}(f)$, $\mathbf{C}(z)$ is a finite separable extension of $\mathbf{C}(f)$, and $\mathbf{C}[z]$ is the integral closure of $\mathbf{C}[f]$ in $\mathbf{C}(z)$. As part of this, one needs that $\mathbf{C}[z]$ is integral over $\mathbf{C}[f]$; at page 669 of [5] this is proved: $p \in \mathbf{C}[z]$ is a root of $H(x) = \prod_{f(z)=v}(x - p(z)) \in \mathbf{C}[v, x]$. The rest is quite simple.

3.2 Finding a Basis for C[z] over C[f]

As stated above, there is a basis for $\mathbf{C}(z)$ over $\mathbf{C}(f)$. It would, of course, be of great interest to find a basis, or at least a set of generators for $\mathbf{C}[z]$ over $\mathbf{C}[f]$.

Hypothesis 3.7 The basis a_j may be chosen such that $c_j = 1$ in theorem 3.6; or at least such that $c_j | H_j$. This would make every polynomial h into a sum $h = \sum_j a_j g_j$ where $g_j \in \mathbf{C}[f]$. ∎

Now, the set $G = \{z^\mu \mid |\mu| < m\}$ where $m = \deg f$ generates $\mathbf{C}[z]$ over $\mathbf{C}[f]$. To show this, we must prove that any z^μ where $|\mu| = m$ may be written as a linear combination of elements of G over $\mathbf{C}[f]$. Define a polynomial over $\mathbf{C}[f]$ by $P_t(x) = \prod_{f(z)=v}(x - t \cdot z)$, $t = (t_1, \ldots, t_n)$, $t \cdot z = t_1 z_1 + \cdots + t_n z_n$; $P_t(t \cdot z) = 0$. The highest order term of $P_t(x)$ is x^m making

$$P_t(t \cdot z) = \sum_{|\mu|=m} \binom{m}{\mu} t^\mu z^\mu + \text{lower degree terms in } z;$$

hence, the identity $\partial^m P_t / \partial t^\mu = 0$ gives z^μ as a linear combination of lower degree monomials over $\mathbf{C}[f]$.

This, however, is not a very strong result, and far from as strong as the hypothesis.

By lemma 3.3, a set of polynomials $a_1, \dots, a_m \in \mathbf{C}[z]$ which is a basis of $\mathbf{C}[z]/(f - q) = \mathbf{C}[z]/(f_1 - q_1, \dots, f_n - q_n)$ over $\mathbf{C}$ is a basis for $\mathbf{C}(z)$ over $\mathbf{C}(f)$. I will find a basis for $\mathbf{C}[z]/(g)$ of minimal degree and show that it must also be a basis for $\mathbf{C}[z]$ over $\mathbf{C}[f]$ where $f = g + \hat{g}$.

However, in order to do this, I will add some extra restrictions:

Notation 3.8 I will assume that $f = g + \hat{g}$ where g_j is homogeneous of degree d_j, $\deg \hat{g}_j < d_j$, and g is itself a proper map. Now, $m = \deg f = \deg g = \prod d_j$. ■

My first step is finding a basis for $\mathbf{C}[z]/(g)$.

Lemma 3.9 If g is as above, there is a set $a_1, \dots, a_m \in \mathbf{C}[z]$ of homogeneous polynomials such that a_j form a basis of $\mathbf{C}[z]/(g)$ over $\mathbf{C}$ and the number of a_j of degree k equals the number of $\mu \in \mathbf{N}_0^n$ such that $|\mu| = k$ and $\mu_j < d_j$.

Proof: For $A = \oplus A_k$ a homogeneous ideal in $\mathbf{C}[z]/I$, let $D_k A = \dim_{\mathbf{C}} A_k$ and $P_A(x) = \sum D_k A\, x^k$. We have $P_{\mathbf{C}[z]}(x) = 1/(1-x)^n$ where $1/(1-x) = 1 + x + x^2 + \cdots$; if $A \subset B$ are homogeneous ideals, $P_{B/A} = P_B - P_A$ as $B/A = \oplus B_k/A_k$ makes $\dim(B/A)_k = \dim B_k/A_k = \dim B_k - \dim A_k$.

Now, let $A^p = \mathbf{C}[z]/(g_1, \dots, g_p) = A^{p-1}/(g_p)$.

The function g_p is not a zero divisor in A^{p-1}; in fact, $g_1, \dots, g_n$ is a regular sequence as $g^{-1}(0) = \{0\}$: see corollary A.16.

In A^{p-1}, the degree of g_p is d_p: i.e., there is no h such that $g_p = h$ in A^{p-1} and $\deg h < d_p$. If such a h existed, the set of common zeroes $g_1 = \cdots = g_{p-1} = h = g_{p+1} = \cdots = g_n = 0$ would have at most $\deg h \cdot \prod_{j \neq p} d_j$ elements, which are too few.

Now, we find that $A^{p-1} \xrightarrow{\cdot g_p} (g_p)A^{p-1}$ is a bijection. I will show that an element of degree k maps to one of degree $k + d_j$ and, hence, that $P_{(f_p)A^{p-1}}(x) = x^{d_p}\, P_{A^{p-1}}(x)$. Let us assume that $g_p h - h' = \sum_{j=1}^{p-1} g_j h_j$; if we consider only the homogeneous terms of degree $d_p + \deg h$, we get $g_p h = \sum_{j=1}^{p-1} g_j h_j$ making g_p a null divisor which, as we have already proved, it is not.

Now, we have

$$P_{A^p}(x) = P_{A^{p-1}/(g_p)A^{p-1}}(x) = P_{A^{p-1}}(x) - P_{(g_p)A^{p-1}}(x) = (1 - x^{d_p})\, P_{A^{p-1}}(x).$$

Using $P_{\mathbf{C}[z]}(x) = P_{A^0}(x) = 1/(1-x)^n$, we get

$$P_{\mathbf{C}[z]/(g)}(x) = \frac{\prod (1 - x^{d_j})}{(1-x)^n} = \prod (1 + x + \cdots + x^{d_j - 1}) = \sum \alpha_k x^k.$$

This makes $D_k(\mathbf{C}[z]/(g)) = \alpha_k$; pick a_j such that a_j are homogeneous polynomials of which there are α_k polynomials of degree k and such that they span the vector space $\mathbf{C}[z]/(g)$ over $\mathbf{C}$: if we write $\mathbf{C}[z]/(g) = \oplus B_k/A_k$ where A_k and B_k are homogeneous of degree k, we know that $\dim_{\mathbf{C}} B_k/A_k = \alpha_k$ and, hence, we may pick α_k elements in B_k which form a basis for B_k/A_k.

The second part of the lemma is easily seen from the definition of α_k.

It should also be noticed that this basis contains exactly $m = \deg g$ elements as $\sum \alpha_k = P_{\mathbf{C}[z]/(g)}(1) = \prod d_j = m$. ■

By the following lemma, the above basis will be a basis for $\mathbf{C}(z)$ over $\mathbf{C}(g)$. This result resembles lemma 3.3 but does not contain the demand that $f = q$ be m distinct points.

Lemma 3.10 If $a_1, \ldots, a_m \in \mathbf{C}[z]$ are homogeneous polynomials generating for $\mathbf{C}[z]/(f)$ over $\mathbf{C}$, then they also form a basis for $\mathbf{C}(z)$ over $\mathbf{C}(f)$.

Proof: Let $(f) \subset \mathbf{C}[f]$ be the maximal ideal generated by f, $A = \mathbf{C}[f]_{(f)}$ the localization of $\mathbf{C}[f]$ at (f) and define the A-module $M = \mathbf{C}[z]_{(f)}$.

I will show that $M/(f)M \cong \mathbf{C}[z]/(f)$. We have $\mathbf{C}[z]/(f) \to M/(f)M$. An element of $M/(f)M$ may be represented by $p/q \in M$ where $p \in \mathbf{C}[z]$ and $q \in \mathbf{C}[f]$ with $q = 1$ in $A/(f)A \cong \mathbf{C}$: i.e., $q = Q \circ f$ where $Q(0) = 1$. Now, $p = p/q$ in $M/(f)M$ as $q - 1 \in (f)A$ and $p - p/q = p(q-1)/q$; hence, the image of p in $\mathbf{C}[z]/(f)$ maps to $p/q \in M/(f)M$.

Now, a_j is a basis for $M/(f)M$. By Nakayama's lemma — lemma A.17 — a_j generates M over A. I will show that a_j also generates $\mathbf{C}(z)$ over $\mathbf{C}(f)$.

Any element of $\mathbf{C}(z)$ may be written p/q where $p \in \mathbf{C}[z]$ and $q \in \mathbf{C}[f]$. As $p \in \mathbf{C}[z] \subset M = \mathbf{C}[z]_{(f)}$, $p = \sum_j a_j p_j$ where $p_j \in A = \mathbf{C}[f]_{(f)} \subset \mathbf{C}(f)$. Hence, $p/q = \sum a_j p_j / q$.

As the number of generators equals the dimension of $\mathbf{C}(z)$ over $\mathbf{C}(f)$, it is a basis. ■

If $\dim_{\mathbf{C}} \mathbf{C}[z]/(f) = m$, as I believe it to be, the a_j above will be a basis for $\mathbf{C}[z]/(f)$.

Next, I will show that a basis over $\mathbf{C}[z]/(g)$ containing m polynomials of minimal degree — i.e., as in lemma 3.9 — will also be a basis for $\mathbf{C}[z]$ over $\mathbf{C}[g]$; and, furthermore, that it will also be a basis for $\mathbf{C}[z]$ over $\mathbf{C}[f]$. In order to do that I will need the following lemma.

Lemma 3.11 Let $a_1, \ldots, a_m \in \mathbf{C}[z]$ be homogeneous polynomials forming a basis for $\mathbf{C}[z]$ over $\mathbf{C}[g]$. If $f = g + \hat{g}$ as above, a_j will also be a basis for $\mathbf{C}[z]$ over $\mathbf{C}[f]$.

Proof: Let $h \in \mathbf{C}[z]$ be a homogeneous polynomial and $h = \sum a_j \, h_j \circ g$, $h_j \in \mathbf{C}[v]$. Now, $h - \sum a_j \, h_j \circ f$ will be of a lower degree than h.

Repeating this process (for the highest degree term) reduces the degree by at least one every step. It terminates, leaving h expressed as a linear combination of a_j over $\mathbf{C}[f]$. ■

Theorem 3.12 With $f = g + \hat{g}$ as above, pick a basis $a_1, \ldots, a_m$ for $\mathbf{C}[z]/(g)$ as done in lemma 3.9. If $A_{ij} = f_*(a_i a_j \, dz)$, $\det A$ is a non-zero constant, hence, by lemma 3.5, there is a dual basis $b_j \in \mathbf{C}[z]$ given by $b = A^{-1} \cdot a$ such that $f_*(a_i b_j \, dz) = \delta_{ij} \, dv$.

Proof: By lemma 3.9, the number α_k of polynomials a_j of degree k is given. Let the a_j be ordered by increasing degree.

By theorem 2.19, $f_*(h dz) = 0$ if $\deg h < |d| - n$ and is a constant if $\deg h = |d| - n$. If we group the a_j by their order and similarly group the components of $A = [A_{ij}]$, we get

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix} = \begin{bmatrix} M_{00} & \cdots & M_{0\,|d|-n} \\ \vdots & \ddots & \vdots \\ M_{|d|-n\,0} & \cdots & M_{|d|-n\,|d|-n} \end{bmatrix}$$

where M_{st} is the $\alpha_s \times \alpha_t$ matrix produced by the $f_*(a_i a_j \, dz)$ where $\deg a_i = s$ and $\deg a_j = t$. Now, notice that $\alpha_k = \alpha_{|d|-n-k}$; also, note that $M_{st} = 0$ if $s + t < |d| - n$, and that M_{st} is a complex valued square matrix if $s + t = |d| - n$. Hence, $\det A = \pm \prod_{s+t=|d|-n} \det M_{st} \in \mathbf{C}$ is a constant. But, why cannot $\det A = 0$?

If $f = g$, we find that as a_j are linearly independent over $\mathbf{C}(g)$, $\det A_g$ cannot be identically zero; hence, a_j is a basis for $\mathbf{C}[z]$ over $\mathbf{C}[g]$. By lemma 3.11, it will also be a basis for $\mathbf{C}[z]$ over $\mathbf{C}[f]$; hence, $\det A \neq 0$.

Finally, apply the last part of lemma 3.5 to the case $\det A \in \mathbf{C}$. ■

Now, I have proved the hypothesis for the generic map f: where the zeroes of f lie off the hyperplane at infinity. Still, I believe the hypothesis to be correct more generally. However, I have neither been able to prove it nor to disprove it.

From the above proof, we may obtain some important information on the degree of the dual basis. This is essential for finding bounds on the degrees of H_i in $h \sum a_i \, H_i \circ f$.

Lemma 3.13 Let f, a_j, and b_j be as above. Then, $\deg a_i b_i = |d| - n$ or, equivalently, $\deg b_i = \deg a_{m-i}$, the a_i ordered by increasing degree.

Proof: In order to prove this, I will have to take a closer look at A and its inverse. We split A into components M_{st} consisting of the $f_*(a_i a_j\, dz)$ with $\deg a_i = s$ and $\deg a_j = t$; by theorem 2.19, we have (for each element of the matrix) $\deg M_{st} \circ f \leq s + t + n - |d|$.

We start our inverting of A by 'fixing' the diagonal: let S be the matrix $S = [S_{st}]$ where $S_{st} = 0$ if $s + t \neq |d| - n$ and $S_{st} = M_{ts}^{-1}$ if $s + t = |d| - n$. This makes $A \cdot S = [N_{st}]$ where $\deg N_{st} \circ f \leq s - t$ and on the diagonal $N_{ss} = I$ (the identity matrix of the right size).

Let $P_{st} = 0$ whenever $s < t$ and $P_{ss} = I$; for $s > t$, let

$$P_{st} = \sum_{u_1 = s > u_2 > \cdots > u_\nu = t} (-1)^\nu N_{u_1 u_2} \cdots N_{u_{\nu-1} u_\nu}.$$

Now, $A \cdot S \cdot P = [N_{su}] \cdot [P_{ut}] = I$; i.e., $P = (A \cdot S)^{-1}$ or $A^{-1} = S \cdot P$.

Now, as $\deg N_{st} \circ f \leq s - t$, $\deg P_{st} \circ f \leq s - t$; just look at the definition. Multiplying P by S changes the ordering in one of the indexes but does not increase the degree; hence, $b = a \cdot S \cdot P$ makes $\deg b_i = \deg a_{m-i}$ or $\deg a_i b_i = |d| - n$. ∎

Theorem 3.14 With f and a_j as above and $h \in \mathbf{C}[z]$, $h = \sum a_j\, H_j \circ f$ where $H_i \in \mathbf{C}[v]$ is defined by $H_i\, dv = f_*(h b_i\, dz)$. Furthermore, $\deg H_i \circ f \leq \deg h - \deg a_i$.

Proof: As $H_i dv = f_*(h b_i dz)$, by theorem 2.19 and the above lemma, $\deg H_i \circ f \leq \deg h + \deg b_i + n - |d| = \deg h - \deg a_i$. ∎

This may be applied to the special case of Bezout identities.

Corollary 3.15 If $h \in (f_1, \ldots, f_n) \subset \mathbf{C}[z]$, $h = \sum_j f_j h_j$ where $\deg h_j \leq \deg h - \deg f_j$.

Proof: Let $h = \sum_i a_i\, H_i \circ f$; as $h \in (f)$, $H_i(0) = 0$, hence, we may write $H_i(v) = \sum_j v_j H_{ij}(v)$; $\deg H_{ij} \circ f = \deg H_i \circ f - \deg f_j \leq \deg h - \deg a_i - \deg f_j$. Let $h_j = \sum_i a_i\, H_{ij} \circ f$; now, $h = \sum f_j h_j$ and $\deg h_j \leq \deg h - \deg f_j$. ∎

3.3 Calculating Push Forward

This section will not contain lemmas or theorems. In working with this thesis, I have made a series of observations which I have not been able to prove or even formulate rigorously. Yet, I do believe these observations to be of interest.

Notation 3.16 Let $f = g + \hat{g} : \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map with no common zeroes at the hyperplane at infinity: i.e., as in the above section. Let $d_j = \deg f_j = \deg g_j$, $m = \deg f = \prod d_j$. ∎

From the previous section, we know that there is a basis a_j for $\mathbf{C}[z]$ over $\mathbf{C}[f]$ with α_k elements of degree k, $\sum \alpha_k x^k = \prod(1 + x + \cdots + x^{d_j - 1})$. Finding such a basis is normally quite simple. First, note that we may take the basis to contain monomials only: i.e., $a_j = z^{\mu_j}$.

Example 3.17 Let $f : \mathbf{C}^3 \to \mathbf{C}^3$ be defined by $(x, y, z) \mapsto (x^2 - y, y^2 - xz, z^2 - x) = (u, v, w)$. First, we note that $x^2 = u + y$ and $z^2 = w + x$; these may be used in reducing the degree of any monomial containing x^2 or y^2. I will use $y^2 = v + xz$ in a similar way, though it does not reduce the degree.

The basis must contain α_k polynomials of degree k where $\sum \alpha_k x^k = (1 + x)^3 = 1 + 3x + 3x^2 + x^3$. First, we pick $a_1 = 1$. The rules of eliminating terms described in the above paragraph does not apply to any monomial of degree 1, hence, $a_2 = x, a_3 = y$, and $a_4 = z$. Using the rules on degree 2 leaves three monomials: $a_5 = xy$, $a_6 = xz$, and $a_7 = yz$. At degree 3, we nee only one term; the rules of prohibits x^2, y^2, and z^2 as factors, leaving $a_8 = xyz$. ∎

I have not succeeded in finding a more rigorous algorithm for the above calculations. However, I have never encountered any significant problems.

I have made the observation that the basis may not only be chosen to be monomial, but that one may demand that if z^μ is in the basis and $\nu \le \mu$, then z^ν, too, is in the basis. In larger and more complex examples, this rule will make the calculations much simpler.

In the above example, the basis was of the form $\{z^\mu \mid \mu \le \sigma\}$ for some σ. This may not be the case: e.g., take $(x, y) \mapsto (x^2 + y^2, xy)$. However, I do believe that in the generic case, such a basis may be chosen; and, in fact, that by performing a linear transformation (on the z) first, such a basis may be found.

In any basis, there will be only one element of degree $|d| - n$; I will order the basis a_j by increasing degree so as to make that element the last: a_m. By either theorem 2.19 or the Jacobi identity, $f_*(a_j\, dz) = 0$ for all $j < m$.

If we can find a representation of any polynomial h as a sum $h = \sum a_j\, H_j \circ f$, the push forward of $h\, dz$ may be calculated. In order to find such a representation, we use the same rules as in the above example.

Example 3.18 Let f be as in the above example and use the basis found there. Let $h = x^3yz$. Using $x^2 = u - y$, we get $h = (u+y)xyz$. Repeating the process using the same rules for reducing a polynomial as in the example, we get

$$\begin{aligned} x^3yz &= (u+y)xyz \\ &= uxyz + (v+xz)xz \\ &= uxyz + vxz + (u+y)(w+x) \\ &= uxyz + vxz + uw + ux + wy + xy \end{aligned}$$

■

In order to calculate the push forward of $\sum a_j\, H_j \circ f\, dz$, we must know $f_*(a_m\, dz)$. We may reduce the Jacobi determinant of f to a sum $\Delta = \sum a_j\, D_j \circ f$. This makes

$$f_*(\Delta\, dz) = D_m f_*(a_m\, dz) = m\, dv.$$

Example 3.19 With f as in the above examples, we find that $\Delta = 8xyz +$ lower degree terms. Hence, $f_*(a_8\, dz) = f_*(\Delta\, dz)/D_8 = m/8\, dv = dv$.

Now, we know all we need to know in order to calculate the push forward of $\omega = h\, dz = x^3yz\, dz$. We know that $x^3yz = h = \sum a_j\, H_j \circ f$ where $H_m = u$, hence, $f_*(x^3yz\, dz) = u\, dv$. ■

Chapter 4

Push Forward and Residues

If ω is a holomorphic n-form with polar divisor $D = (f_1) + \cdots + (f_n)$, we may define the residue of ω along D to be

$$\operatorname{Res}_f \omega = \frac{1}{(2\pi i)^n} \int_{|f|=r} \omega.$$

In [5], page 649, they use this definition but restrict it to a neighborhood of an isolated common zero of $f = 0$, denoting it $\operatorname{Res}_{\{0\}} \omega$; this would make $\operatorname{Res}_f \omega = \sum_P \operatorname{Res}_P \omega$.

In fact, the push forward itself is a residue: if ω is an n-form, the push forward $H\, dv$ may de defined by

$$H(v) = \operatorname{Res}_{f-v} \frac{\omega}{\prod f_j}.$$

4.1 Integral Representations and Push Forward

There is an identity

$$\int_D f_* \omega \wedge \phi = \int_{f^{-1}(D)} \omega \wedge f^* \phi;$$

of course, an example of this is the use of the Cauchy representation to extend the definition of $f_* \omega$ in theorem 1.6.

Given any integral representation

$$f_* \omega(v) = \int f_* \omega(\nu) \wedge R(v, \nu)$$

we may pull this back, getting

$$f_* \omega(v) = \int \omega(\zeta) \wedge R(v, f(\zeta)).$$

Let us look at the special case of integral representation of functions: let r be a function such that $h(0) = \int h(\zeta) r(\zeta)\, d\overline{\zeta} \wedge d\zeta$ for holomorphic h. Let $f : \overline{U} \to \overline{V}$ where U and V are connected open sets containing origo, V as in definition 1.3. If $\operatorname{Supp} r \subset V$, $f^{-1}(0) = \{0\}$ where $f(0) = 0$ is a zero of multiplicity m, then

$$\begin{aligned} h(0) &= \frac{1}{m} f_* h(0) \\ &= \frac{1}{m} \int f_* h(\nu)\, r(\nu)\, d\overline{\nu} \wedge d\nu \\ &= \frac{1}{m} \int h(\zeta)\, r\big(f(\zeta)\big) \left|\Delta(\zeta)\right|^2 d\overline{\zeta} \wedge d\zeta; \end{aligned}$$

a simple change of coordinates makes

$$h(z) = \frac{1}{m} \int h(\zeta)\, r\big(f(\zeta - z)\big) \left|\Delta(\zeta - z)\right|^2 d\overline{\zeta} \wedge d\zeta.$$

4.2 A Representation

Let $f : \mathbf{C}^n \to \mathbf{C}^n$ be holomorphic, of finite type, and $f(0) = 0$. According to [3], on a neighborhood of 0, $S = \{z \in \mathbf{C}^n \mid f_1(z) = \cdots = f_{n-1}(z) = 0\}$ is the union of a finite number of irreducible curves parametrized by holomorphic maps $\Phi_i : U_i \to \mathbf{C}^n$ where $0 \in U_i \subset \mathbf{C}$ and $\Phi_i(0) = 0$. If $\omega = h\, dz$ and $f_*\omega = H\, dv$, we get

$$\operatorname{Res}_f \frac{\omega}{\prod f_j} = H(0) = \sum_i \operatorname{Res}_0 \frac{h \circ \Phi_i \cdot d(f_n \circ \Phi_i)}{\Delta \circ \Phi_i \cdot f_n \circ \Phi_i}.$$

(He has used a somewhat different wording.)

This result may be generalized, but not, however, without problems. The main hindrance is that for $f : \mathbf{C}^n \to \mathbf{C}^p$, $X = f^{-1}(0)$ may be non-parametrizable.

Theorem 4.1 Let $F = (f, g) : \overline{U} \to \mathbf{C}^p \times \mathbf{C}^q$ be holomorphic, $p + q = n$, $U \in \mathbf{C}^n$ open, F proper; in short, F satisfies the demands of notation 1.1. Let V be as in definition 1.3; we demand that $0 \in V$. Let $X = g^{-1}(0)$, $X = \cup X_\mu$ where X_μ are the irreducible components, ω a holomorphic p-form, and $F_*\omega = \sum_I H_I\, dv_I$. Then,

$$\operatorname{Res}_F \frac{\omega \wedge dg}{\prod F_j} = H_{(1,\ldots,p)}(0) = \frac{1}{(2\pi i)^p} \sum_\mu \deg_{X_\mu} g \int_{X_\mu \cap \{|f| = r\}} \frac{\omega}{\prod f_j}$$

$v = (v', v'') \in \mathbf{C}^p \times \mathbf{C}^q$, for $r > 0$ such that $\{z \mid |g(z)| = r, f(z) = 0\} \subset V$. It would, of course, be tempting to denote this identity

$$\operatorname{Res}_F \frac{\omega \wedge dg}{\prod F_j} = \sum_\mu \deg_{X_\mu} g\ \operatorname{Res}_{X_\mu, f} \frac{\omega}{\prod f_j}.$$

If there is a local parametrization of X_μ by $\Phi_\mu : U_\mu \subset \mathbf{C}^p \to X_\mu \subset \mathbf{C}^n$, $\Phi_\mu(0) = 0$, we get

$$\int_{X_\mu \cap \{|f|=r\}} \frac{\omega}{\prod f_j} = \int_{|\zeta|=\rho} \frac{\Phi_\mu^* \omega}{\prod \zeta_j}$$

and, hence,

$$\mathrm{Res}_F \frac{\omega \wedge dg}{\prod F_j} = \sum_\mu \deg_{X_\mu} g \ \mathrm{Res}_\zeta \frac{\Phi_\mu^* \omega}{\prod \zeta_j}.$$

Proof: Due to lemma 1.7 we know that $F_* \omega$ is well defined and holomorphic.

As in lemma 1.7, $H_{(1,\ldots,p)}\, dv = F_*(\omega \wedge dg)$. By the comment at the beginning of this chapter, $H_{(1,\ldots,p)}(v) = \mathrm{Res}_{F-v}(\omega / \prod F_j)$.

By restricting the form $F_* \omega$ to the plane $v'' = 0$ ($\iff$ $v_{p+1} = \cdots = v_{p+q} = 0$), all terms in $\sum_I H_I\, dv_I$ cancel except $H_{(1,\ldots,p)}\, dv'$: this I will denote $H'\, dv'$. The Cauchy integral representation of H' becomes

$$H'(0) = \frac{1}{(2\pi i)^p} \int_{|v'|=r} \frac{H'\, dv'}{\prod_j dv'_j} = \frac{1}{(2\pi i)^p} \int_{\substack{|v'|=r \\ v''=0}} \frac{F_* \omega}{\prod_j dv'_j}.$$

I want to relate this to the form ω; not $F_* \omega$. The problem is that Δ may be identically zero on X_μ; otherwise, there is no problem: F is locally biholomorphic almost everywhere. In order to avoid this problem, we perturb: we may solve for $g = u$ where $\Delta \not\equiv 0$ on any connected component of $g = u$, and let $u \to 0$.

$$\begin{aligned}
\int_{\substack{|v'|=r \\ v''=0}} \frac{F_* \omega}{\prod_j v'_j} &= \lim_{u \to 0} \int_{\substack{|v'|=r \\ v''=u}} \frac{F_* \omega}{\prod_j v'_j} \\
&= \lim_{u \to 0} \int_{\substack{|f|=r \\ g=u}} \frac{\omega}{\prod_j f_j} \\
&= \sum_\mu \deg_{X_\mu} g \int_{X_\mu \cap \{|f|=r\}} \frac{\omega}{\prod_j f_j}.
\end{aligned}$$

Here, $\deg_{X_\mu} g$ denotes the local multiplicity of g along X_μ: i.e., the multiplicity of F in a neighborhood of z.

The second part comes simply by pulling back ω by Φ_μ. ∎

If $\phi = \omega \wedge dg$, we may write $\omega = \phi / dg$. If we are given ϕ, we may still define ω to be ϕ / dg, but this is not uniquely defined; however, it is uniquely defined as a form on $X = \{g = 0\}$ provided $dg \neq 0$ on X. Hence, we may write the result of the theorem

$$\mathrm{Res}_F \frac{\phi}{\prod F_j} = \sum_\mu \deg_{X_\mu} g \ \mathrm{Res}_{X_\mu, f} \frac{\phi / dg}{\prod f_j}$$

or, in the parametrized case,

$$\mathrm{Res}_F \frac{\phi}{\prod F_j} = \sum_\mu \deg_{X_\mu} g \ \mathrm{Res}_\zeta \frac{\Phi_\mu^*(\phi/dg)}{\prod \zeta_j}.$$

Appendix A

Results Used

These are known results that I have used in the text. Most of these are simply copied from a book or an article; in one case I have only use a very special case of a theorem and, for simplicity, stated this result for this special case. In one case I have found it simpler to provide a proof that to look up a good reference.

A.1 Some Definitions

I have chosen to add definitions of some of the concepts used. One of the reasons for doing this is of course completeness; however, several concepts may be given slightly differing definitions by various authors. The purpose of the definitions is not rigor but to establish a notation, and so, they may be a bit brief.

Definition A.1 An **algebraic set** is the zero-set of a collection of polynomials; if it is contained in a projective space, the polynomials must be homogeneous. Similarly, an **analytic set** is the zero-set of a collection of analytic — unless otherwise specified, complex-analytic — functions.

A algebraic set is **reducible** if it may be written as the union of two (affine) varieties where neither are the variety itself; otherwise, it is **irreducible**.

An irreducible algebraic set $V \subset \mathbf{CP}^q$ is called a **variety** or sometimes a **projective variety**; an irreducible algebraic set $V \subset V^q$ is called an **affine variety**.

■

Please note, that in many books, such as [5], a variety is not defined to be irreducible: it is meerly an algebraic set.

Definition A.2 The **Zariski topology** on an algebraic set is defined by V closed $\iff$ V an algebraic set. ■

Definition A.3 Let $V \subset \mathbf{C}^q$ be an algebraic set. A **polynomial** on V is a function $p : V \to \mathbf{C}$ defined by the restriction of a polynomial on $\mathbf{C}^q$. A **rational function** on V is the quotient of two polynomials p and q on V, $q \not\equiv 0$.

A **rational function** on an algebraic set $V \subset \mathbf{CP}^q$ is the quotient of two homogeneous polynomials p and q of the same degree where q is not a zero divisor in the ring of polynomials on V. ■

Definition A.4 Let X and Y be affine varieties. A map $f : X \to Y$ is **regular** if it is defined by polynomials; i.e., if $X \subset \mathbf{C}^p$, $Y \subset \mathbf{C}^q$, f is the restriction of a polynomial map $\phi : \mathbf{C}^p \to \mathbf{C}^q$.

Let X and Y be varieties. A **rational map** $f : X \to Y$ is defined by a subvariety (i.e., irreducible algebraic set) $Z \subset X \times Y$ such that for a Zariski-open set of $X_0 \subset X$, the projection $Z \cap (X_0 \times Y) \to X_0$ is bijective.

A map is **dominating** if its image is Zariski-dense. ■

A.2 Algebraic Varieties

Lemma A.5 If $X \subset \mathbf{CP}^q$ is a Zariski-open subset of a variety of dimension n, there is a rational n-form on X which is not identically zero.

Proof: We may restrict ourselves to $X \subset \mathbf{C}^q$: a Zariski-open subset of the original X. The n-forms dz_I, $|I| = n$, may not all be zero on X. ■

In [8], theorem 2.23, the following result may be found:

Theorem A.6 Main Theorem of elimination theory. The projection $\pi_2 : \mathbf{CP}^n \times \mathbf{CP}^m \to \mathbf{CP}^m$ is closed, i.e., if $Z \subset \mathbf{CP}^n \times \mathbf{CP}^m$ is a closed algebraic set, then so is $\pi_2(Z)$. ■

A.2.1 Bezout's Theorem

This is a special case of theorem 5.16 in [8].

Theorem A.7 Let $f_1, \ldots, f_n$ be homogeneous polynomials on C^{n+1} such that their common zero in $\mathbf{CP}^n$ has dimension 0, then the number of common zeroes (counted with multiplicities) is $\prod_j \deg f_j$. ∎

A corollary to this is:

Corollary A.8 If $f : \mathbf{C}^n \to \mathbf{C}^n$ is polynomial and proper, then f is finite.

Proof: If $\{f = 0\}$ had dimension 1 or higher it would not be proper: it would intersect the hyperplane at infinity. Hence, $\{f = 0\}$ is, by the above theorem, a finite number of points; at most $\prod_j \deg f_j$ points (counted with multiplicities). ∎

A.2.2 Global Residue Theorem

This result is a special case of the residue theorem found in [5] at page 656:

Theorem A.9 Let $D_1, \ldots, D_n$ be effective divisors in $\mathbf{CP}^n$, $D = D_1 + \cdots + D_n$, ω a meromorphic n-form on $\mathbf{CP}^n$ with polar divisor D. Then, $\sum_P \operatorname{Res}_P \omega = 0$ where Res_P is the residue at P with respect to functions f_j with polar divisors D_j. ∎

A.3 Algebraic lemmas

This version of Hilberts nullstellensatz may be found as an exercise at page 85 of [1] or as theorem 1.3A in [2].

Theorem A.10 Let k be an algebraically closed field, let a be an ideal in $A = k[x_1, \ldots, x_n]$, and let $f \in A$ be a polynomial which vanishes at all points of $Z(a) = \{p \in k^n \mid g(a) = 0 \forall g \in a\}$. Then $f^r \in a$ for some integer $r > 0$. ∎

As a direct consequence, we have the following corollary.

Corollary A.11 Let $p, q \in \mathbf{C}[z]$ be such that $p = 0$ wherever $q = 0$. Then, $q|p^k$ for some $k \in \mathbf{N}$. ■

This lemma may be found on page 669 of [5].

Lemma A.12 The ring $\mathbf{C}[z]$ is integral over $\mathbf{C}[f]$.

Proof: Let $h \in \mathbf{C}[z]$; $p(x) = \prod_{f(z)=v}(x - h(z))$ is a polynomial over $\mathbf{C}[v]$ and $p(h) = 0$. ■

The following theorem is proposition 3.17 in [8]:

Theorem A.13 Let $f : X \to Y$ be a dominating regular map, X and Y affine varieties of dimension n; let $K(X)$ and $K(Y)$ denote the fields of rational functions on X and Y. Then there is a Zariski-open $Y_0 \subset Y$ such that $\deg f = \# f^{-1}(y) = [K(X) : K(Y)]$ for all $y \in Y_0$. ■

A corollary to this theorem is:

Corollary A.14 Let $\mathbf{C}(z) = \mathbf{C}(z_1 \ldots, z_n)$ and $\mathbf{C}(f) = \mathbf{C}(f_1, \ldots, f_n)$ denote the field of rational functions in z and f. If $f : \mathbf{C}^n \to \mathbf{C}^n$ is polynomial and finite of degree m, $[\mathbf{C}(z) : \mathbf{C}(f)] = m$: i.e., $\mathbf{C}(z)$ is a vector space of dimension m over $\mathbf{C}(f)$. ■

The following result may be found at page 660 of [5]. The corollary follows from this or may be deduced from theorem 8.21A in [2].

Lemma A.15 For $U \subset \mathbf{C}^n$ a sufficiently small neighborhood of the origin and $f : U \to \mathbf{C}^n$ a holomorphic mapping, then $f_1, \ldots, f_n$ is a regular sequence if and only if $f^{-1}(0) = \{0\}$. That f_j is a regular sequence is exactly that f_p is not a zero divisor in $\mathcal{O}/(f_1, \ldots, f_{p-1})$, $\mathcal{O}$ holomorphic functions on $\mathbf{C}^n$. ■

Corollary A.16 Let $g : \mathbf{C}^n \to \mathbf{C}^n$ be defined by polynomials and $g^{-1}(0) = \{0\}$. Then g_j is a regular sequence in the sense that g_p is not a zero divisor in $\mathbf{C}[z]/(g_1, \ldots, g_{p-1})$.

Proof: If h_p were a zero divisor in $\mathbf{C}[z]/(g_1, \ldots, g_{p-1})$, it would certainly be a zero divisor in $\mathcal{O}/(g_1, \ldots, g_{p-1})$; according to the above lemma, that is not the case. ■

The following version of Nakayama's lemma is proposition 2.8 in [1].

Lemma A.17 If A is a local ring with maximal ideal m, $k = A/m$, M is a finitely generated A-module. Let $a_1, \ldots, a_k \in M$. Then, if a_j generate M/mM over k, then they also generate M over A. ∎

A.4 When are Holomorphic Functions Algebraic?

This lemma is well known and easily proved.

Lemma A.18 A holomorphic function $g : \mathbf{C}^n \to \mathbf{C}$ with $|g(z)| = O(\|z\|^k)$, is a polynomial of degree $\leq k$.

Proof: If $|\mu| > k + n$, we get

$$\begin{aligned}
\int_{|z|=r} \frac{g(z)\,dz}{z^\mu} &= \lim_{t\to\infty} \int_{|z|=tr} \frac{g(z)\,dz}{z^\mu} \quad \text{let } z = tu \\
&= \lim_{t\to\infty} \int_{|u|=r} \frac{g(tu)\,t^n du}{t^{|\mu|} u^\mu} \\
&= 0
\end{aligned}$$

Hence, the power series expansion for g contains only terms of degree $\leq k$. ∎

The following result may be found in [5] at page 168.

Lemma A.19 Let $V \subset \mathbf{CP}^q$ be an algebraic set. Any meromorphic function on V is rational. (In the book, the word "variety" is used, not "algebraic set"; however, [5] does not demand that varieties be irreducible, and so, the change is meerly a 'translation'.) ∎

A.5 Removable Singularities

The following lemma may be found in [5] at page 21.

Lemma A.20 Let $V \subset M$, M a complex manifold, be an analytic set. The set of singular points of V is contained in an analytic set not equal to V. ∎

The following result is similar to the Levi extension theorem — page 396 of [5] — though I need a lemma for extending a holomorphic function whereas that is for extending meromorphic functions. I have chosen to give a new proof instead of specializing the theorem: both are quite simple.

Lemma A.21 Let $U \subset \mathbf{C}^n$ be an open set, $V \subset U$ an analytic variety of (complex) codimension ≥ 2. If ϕ is a holomorphic function or form on $U \setminus V$, then ϕ may be extended holomorphically to all of U.

Proof: Let g be a holomorphic function not identically zero such that $g = 0$ on V. There is a linear non-degenerate map $(\alpha, \pi) : U \to \mathbf{C} \times \mathbf{C}^{n-1}$ such that g is not identically zero on any fiber of π. Let $W = \pi(V)$; as V has dimension $n-2$ or less, so must W, making W of codimension 1 or higher.

Now, if h is holomorphic on $U \setminus V$, define

$$\hat{h}(z) = \frac{1}{2\pi i} \int_{\substack{\pi(\zeta)=\pi(z) \\ |g(\zeta)|=r}} \frac{h(\zeta)\, d\alpha(\zeta)}{\alpha(\zeta) - \alpha(z)}.$$

This function equals h outside $\pi^{-1}(W)$; furthermore, $\hat{h}$ is locally bounded and, hence, by Riemann's extension lemma, holomorphic across $\pi^{-1}(W)$, giving an extension of h.

If, instead, we should have a p-form $\phi = \sum_{|I|=p} h_I \, dz_I$, apply the above result to h_I. This gives the desired result. ■

A.6 Asymptotic Properties of Algebraic Functions

The following is from [7].

Definition A.22 A set in $\mathbf{R}^n$ is **semialgebraic** if it is a finite union of a finite intersection of sets defined by polynomial equations or inequalities: $\{P(x) = 0\}$, $\{P(x) > 0\}$ and $\{P(x) \geq 0\}$. ■

Lemma A.23 If E is a semialgebraic set in $\mathbf{R}^{2+n}$, and $f(x) = \sup\{y \mid (x, y, z) \in E \subset \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n\}$ is well defined and finite for large x, then either $f = 0$ for large x, or $f(x) = Ax^a(1 + o(1))$ when $x \to \infty$ where $A \neq 0$ and a is rational. ■

A.7 Stokes' theorem

At page 33 in [5] there is a version of Stokes' theorem on varieties for forms with compact support. This is not quite what I need, though almost. I have found two different versions which may be used. Possibly, by going through some extra trouble, I could have done without.

A.7.1 Stokes' theorem on Varieties

The following version of Stokes' theorem is from [4] (theorem II.4.2).

Theorem A.24 Suppose D is an oriented analytic domain on a (real analytic) subvariety V of some open set in $\mathbf{R}^n$: i.e., D is relatively compact in V, $D \subset V^k =$ set of points p in V at which the germ induced by V has dimension k, and ∂D is contained in some subvariety of some neighborhood of ∂D in V. Let α be a C^1 differential form of degree $k-1$ in $\mathbf{R}^n$. Define

$$\int_{\partial D} \alpha = \int_G \epsilon_D^k i^* \alpha,$$

where G is the set of points on ∂D at which ∂D is orientably embedded in V and $i : G \to \mathbf{R}^n$ is the natural injection. (ϵ_D^k is the 'multiplicity' of ∂D.) Then

$$\int_{\partial D} \alpha = \int_D d\alpha.$$

■

A.7.2 Stokes' theorem on Semianalytic sets

The following stronger version of Stokes' theorem is [6] (theorem II.B.2.1):

Theorem A.25 Let X be a paracompact real analytic manifold of dimension n, and M a locally closed semianalytic set in X of dimension p $(0 < p \leq n)$. (In our case it will even be semialgebraic). Then, the following diagram commutes:

$$\begin{array}{ccc} H_p(M;\mathbf{R}) & \xrightarrow{I(M)} & \mathcal{D}'_p(X) \\ {\scriptstyle \partial_{M,bM}}\downarrow & & \downarrow{\scriptstyle b} \\ H_{p-1}(bM;\mathbf{R}) & \xrightarrow{I(bM)} & \mathcal{D}'_{p-1}(X) \end{array}$$

where $\mathcal{D}'(X)$ are currents ($\mathcal{D}'_p(X)$ is the dual of p-forms with compact support — $\mathcal{D}_p(X)$), b is the border in $\mathcal{D}'(X)$, and $\partial_{M,bM}$ is the boundary in the exact sequence of homology

$$0 \to H_p(\overline{M};\mathbf{R}) \stackrel{j_{\overline{M},M}}{\longrightarrow} H_p(M;\mathbf{R}) \stackrel{\partial_{M,bM}}{\longrightarrow} H_{p-1}(bM;\mathbf{R}) \to \cdots$$

∎

Bibliography

[1] M. F. Atiyah & I. G. MacDonald, "Introduction to Commutative Algebra".

[2] R. Hartshorne, "Algebraic Geometry".

[3] G. Biernat, "La représentation paramétrique d'un résidu multidimensionnel", Rev. Roumaine math. pures appl., 36 (1991), 5–6.

[4] L. Bungart, "Integration on varieties", J. Math. Mech., 15, 1966.

[5] P. Griffiths & J. Harris, "Principles of Algebraic Geometry".

[6] M. E. Herrera, "Integration on a semianalytic set", Bull. Soc. math. France, 94, 1966.

[7] L. Hørmander, "The Analysis of linear PDO's, vol. II".

[8] D. Mumford, "Algebraic Geometry I: Complex Projective Varieties".