A Case of Mirror Symmetry Defined as Non-Complete Intersections and Significant Topology Change for Multiple Mirror Manifolds

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Dr. Scient. Thesis
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Whenever you can, count.

—Sir Francis Galton (1822–1911)

Pereant qui ante nos nostra dixerunt.

—Aelius Donatus (4th Century)

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Preface

The aim of this thesis is to verify that two given families of complex Calabi–Yau threefolds, pfaffian varieties in $\mathbf{P}^6$, defined as non-complete intersections, and an orbifolding thereof, are mirror manifolds; or, rather, to verify some of the predictions about mirror manifolds hold in the case of these two families.

The strategy is first to ensure that the manifolds are indeed Calabi–Yau manifolds. The second step is to check the Euler and Hodge numbers of the two manifolds. If we denote a mirror-pair by M and W, mirror symmetry states that $h^{p,q}(M) = h^{3-p,q}(W)$, which implies that $\chi(M) = -\chi(W)$. Finally, the mirror map is computed and the number of lines found. This was the initial program for the thesis: the check an example of mirror symmetry for a non-complete intersection. With the pace of developement in mirror symmetry, such examples of verifications are today of less interest than before; however, a more surprising and interesting discovery was made.

As will be seen, solving the Picard–Fuchs equation on the orbifold initially gives a solution incompatible with the assumed mirror. However, for the family of Calabi-Yau manifolds, there are two points in the moduli space of complex structures which give different mirror maps and, hence, different mirrors. The second of these give the desired mirror. The first is found to correspond to an intersection of $G(2,7) \subset \mathbf{P}^{20}$ with seven general hyperplanes. Hence, there appears to be a strong link between the pfaffian Calabi–Yau manifolds and the grassmannian sections.

A few words should be said about the use of appendices. In appendix A, some of the mathematical tools used are described. This appendix was written in the course of learning and will most likely be of no interest to the reader; it does not aim at being a complete tool-chest of any kind, and, mostly, references to the appendix has been replaced with references to the litterature. In appendix B, computer calculations are referred.

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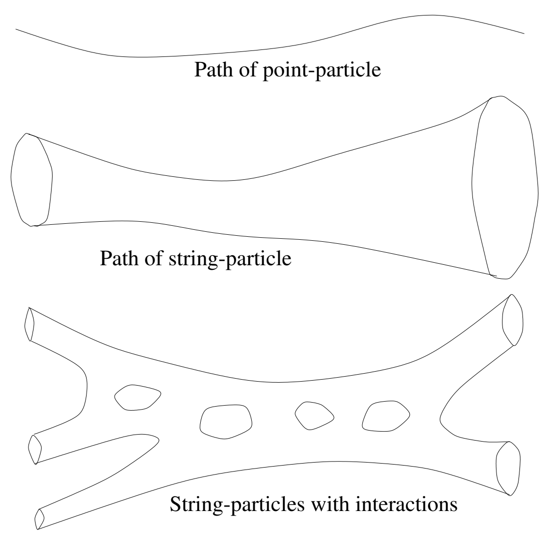
Chapter 0

Introduction

0.1 The Origin of Mirror Symmetry

Mirror symmetry originates from superstring theory. Superstring theory is a candidate for a unifying theory of quantum mechanics and general relativity (gravitation). The main new element to be introduced in superstring theory is the view of a particle, not as a point, but as a string: eg., S^1. I will skip all descriptions of the physical theory and the derivation of the consequences of mirror symmetry, instead refering the reader to the litterature—a good introduction for mathematicians and my primary resource has been [13], another presentation is given in [8], and a brief description of the physics leading to mirror symmetry may be found in [22]—briefly presenting the conclusions.



As time passes, a point-particle traces a curve in space-time. A string, however, traces a surface, Σ: the world sheet. Our interest will be centered around a particular case: where the world sheet S is a sphere. Giving it a complex structure and Kähler form, we may take this to be the standard complex projective line: $S = \mathbf{P}^1$.

In addition to the four dimensions of space-time, six additional dimensions are introduced in the form of a compact manifold. For the superstring theory to have the neccesary properties, real compact six-dimensional manifolds are used. At each point in space-time, this will be furnished with a complex structure and a Kähler structure making it into a complex Kähler threefold. Further restrictions arriving from physics make the manifold a Calabi–Yau manifold.

The physical properties are found using Feynman path integrals. These are integrals over all maps $\phi : \Sigma \to \mathcal{U}$ where $\mathcal{U}$ is the manifold representing the universe, locally $\mathbf{R}^4 \times M$ where $\mathbf{R}^4$ is the space-time and M is a Calabi–Yau manifold, and Σ is any surface. The physical field theory derived may be split into two components: one on $\mathbf{R}^4$ and one on M of which only that on M will be of interest to us, so we regard only $\phi : \Sigma \to M$. However, using the method of stationary phase, this integral reduces to an integral over instantons: the stationary phases, eg. those with (locally) minimal action. In our case, the instantons will merely be the holomorphic maps $\phi : \mathbf{P}^1 \to M$.

Central to the superstring theory, as to any quantum mechanical theory, is the correlation function. For a set of observables, for short denoted $\mathcal{O}_{p_i}$, which are combinations of a function or operator $\mathcal{O}_i$ on M and point p_i in $\Sigma = \mathbf{P}^1$, the correlation function is given by (after the simplifications refered to above)

$$\langle \mathcal{O}_{p_1} \ldots \mathcal{O}_{p_n} \rangle = \sum_{\nu \in \pi_2(M)} e^{-2\pi \int_{\mathbf{P}^1} \phi^*(\omega)} \int_{\mathcal{M}_\nu} \phi^*(\mathcal{O}_1)_{p_1} \cdots \phi^*(\mathcal{O}_n)_{p_n} D[\phi]. \qquad (1)$$

In particular, we will be concentrating on the three-point functions: those with $n = 3$. Others may be derived from these. As the correlation functions will not depend on choice of points $p_i \in \mathbf{P}^1$, these will not be included unless needed.

The Lagrangian, a function used in the Feynman path integrals which determine the physical properties, may be split into two components: one, the A-model, depending only on the Kähler structure; the other, the B-model, depending only on the complex structure.

For the A-model, we will be using forms in $H^{1,1}(M)$ as observables. The three-point correlation function for $v_1, v_2, v_3 \in H^{1,1}(M)$ then becomes

$$\langle v_1 v_2 v_3 \rangle = \int_M v_1 \wedge v_2 \wedge v_3 + \sum_{\nu \in H_2(M,\mathbf{Z})} N_\nu \frac{q^\nu}{1 - q^\nu} (\nu \cdot v_1)(\nu \cdot v_2)(\nu \cdot v_3) \qquad (2)$$

where N_ν is the number of rational curves homological to ν. It is worth noting that $\frac{q^\nu}{1-q^\nu} = \sum_{d=1}^{\infty} q^{d\nu}$ deals with multiple coverings of the curves. The variable q parametrizes the B-model moduli space: Kähler structures. If we let ω_i generate $H^{1,1}(M, \mathbf{Z})$, we may let $\omega = \sum_i t_i \omega_i$. This enters only as $e^{\nu \cdot \omega}$, hence, we may use $q_i = e^{t_i}$ as parameters instead.

For the B-model, we will be using forms $v_i \in H^{-1,1}(M) = H^1(M, \Theta_M)$ as observables. For these, the three-point correlation function simply becomes

$$\langle v_1 v_2 v_3 \rangle = \int_M (v_1 \wedge v_2 \wedge v_3 \cdot K_M) \wedge K_M \qquad (3)$$

where K_M is a global section of the canonical bundle: this depends on the complex structure. Also note that given K_M, we have $H^{-p,q}(M) \cong H^{3-p,q}(M)$.

Physically, there is no difference between the A-model and the B-model: they arise as different eigenspaces under the BRST-operator, Q and $\overline{Q}$. A change of sign in $\overline{Q}$ gives rise to an interchanging of the A-model and B-model. Hence, it is assumed that for any Calabi–Yau manifold, M, there is a mirror Calabi–Yau manifold, W, so that the A-model of M corresponds to the B-model of W, and vice versa. This is the (physical) mirror hypothesis.

0.2 How to Count Rational Curves using Mirror Manifolds

If we have a mirror pair, M and W, as above, there will be a correspondence between $H^{p,q}(M)$ and $H^{-p,q}(W)$ which respects the correlation function. Hence, if we know the correspondence between $H^{1,1}(M)$ and $H^{-1,1}(W)$ and can calculate the B-model correlation function on W, we know the A-model correlation function on M. As the A-model correlation function includes a sum over all rational curves, this will give us the number of curves in the various homology classes. This was first done in [3] for a special case; how to do this in more general is described in [13].

0.3 Topological Transitions

The Calabi–Yau manifold, M, lies at the foundation of the model. However, in changing the Kähler structure towards the edge of the Kähler cone—$\omega = \sum_i t_i \omega_i$ where $t_i > 0$, the edge being where one or more $t_i = 0$—the form at least partly changes sign from positive to negative. What often happens is that by performing a flop, a topological change, one may proceed: ie., the flop changes the sign of a component ω_i. Thus, the A-model moduli space may be composed from cells corresponding to the Kähler cones from various topologically different Calabi–Yau manifolds.

Though there are clear borders between the Kähler cones in the A-model moduli space, in the physical model these transitions are smooth. In particular, in the B-model of the corresponding mirror, the changes of complex structure are smooth. Hence, from a physical point of view, the Calabi–Yau manifolds are only different in so far as they produce different physical predictions. It is therefore natural to say that the A-model moduli space only parametrizes Calabi–Yau manifolds and their Kähler forms up to these topological changes; alternatively, that the Calabi–Yau manifolds with Kähler forms are merely representations of patches of the A-model moduli space.

From a more mathematical point of view, this identification of manifolds up to flops (and perhaps other topological changes) is most practical as different resolutions of various singularities often differ by a number of flops. Thus, different choices of resolutions will merely give rise to different parts of the A-model moduli space. Hence, we may conveniently work on the singular version of the Calabi–Yau manifold, which is often more practical;

and, at least partly, ignore the positivity condition of the Kähler form.

0.4 A Mathematical Turn

For mathematicians, mirror symmetry, as treated by physicists, has always appeared somewhat non-rigorous: not well-defined or properly proved. In particular, the path integral formalism and how this specializes into a sum over (rational) curves was somewhat unsatisfying: initially, it was not even fully clear what was the appropriate definition of 'number of rational curves'.

A proper formalization now exists in which the counting of rational curves is defined in terms of an appropriately compactified moduli space of stable maps ([20]). In this formalism, some of the consequences of mirror symmetry have been made formal and properly proved ([7]). These, however, do not make use of a mirror manifold.

Chapter 1

A Family of Calabi–Yau Manifolds

The aim of this section is to produce a pair of families of Calabi–Yau threefolds which are not complete intersections, and which are possible mirrors. This is done using the pfaffians of the principal 6×6-minors of a skew-symmetric 7×7-matrix whose terms are linear functions in variables $x_0, \ldots, x_6$. This will in general produce a threefold in $\mathbf{P}^6$.

The definition most frequently used by mathematicians is the following:

Definition 1.1 A *Calabi–Yau manifold* is a compact, complex Kähler manifold M with $h^{i,0}(M) = 0$ for $0 < i < \dim M$ and with trivial canonical bundle ω_M — $\omega_M \cong \mathcal{O}_M$ — or, equivalently, with vanishing first Chern class. ■

From the original setting of mirror symmetry, superstring theory, the Calabi–Yau manifold does not come with a fixed complex or Kähler structure. A physically more natural definition ([13]) is a real, 6-dimensional, compact, connected, orientable manifold which admits Riemannian metrics whose holonomy is contained in (or equal to) SU(3). For such manifolds there will always exist a complex structure such that the metric is Kähler. With the restriction $h^{2,0} = 0$, there is only a finite number of complex structures available; if the holonomy is SU(3), there are exactly two complex structures, one the conjugate of the other. Actually, a complex structure and the cohomology class of a Kähler form uniquely determines a Ricci-flat Kähler metric. This will produce a trivial canonical bundle.

1.1 Pfaffians

Lemma 1.2 Let N be a skew-symmetric 7×7-matrix, and let $N_{\{i\}}$ be the matrix with the i'th row and column removed. Then, there are polynomials, p_i, in the elements of N, called the *pfaffians*, so that $\det N_{\{i\}} = p_i^2$. If the signs of the p_i are chosen appropriately,

$N \cdot P = 0$, where $P = [p_i]$. Furthermore, the adjoint[1] of N is PP^{T}.

Proof: Let $N = [N_{ij}]$ be a general skew-symmetric 7×7-matrix: ie., with formal elements N_{ij}, $N_{ij} + N_{ji} = 0$. Of course, as $\det N = \det N^{\mathrm{T}} = \det(-N) = -\det N$, the rank must be less than 7; hence, over the polynomial ring, the rank of N is 6. The adjoint of N is a non-zero symmetric matrix. However, as $N \cdot \operatorname{adj} N = \det N = 0$, we get $\operatorname{rank} N + \operatorname{rank} \operatorname{adj} N \leq 7$. This forces $\operatorname{rank} \operatorname{adj} N = 1$.

Now, let $M = \operatorname{adj} N$. We may write $M_{ij} = p_i p_j$ where the p_i are square roots of M_{ii} with a proper choice of sign. From $p_i^2 = M_{ii}$ it is not immediately clear that p_i are polynomials, but as $p_i p_j$ are polynomials, and p_i^2 and p_j^2 have no common factor for $i \neq j$, we find that all p_i must be polynomials. Furthermore, $0 = N \cdot \operatorname{adj} N = N \cdot P \cdot P^{\mathrm{T}}$, and so $N \cdot P = 0$. ■

Actually, if $M = [M_{ij}]$ is a skew-symmetric $2n \times 2n$-matrix, the pfaffian can be defined by the sum

$$\operatorname{Pf}(M) = \sum_{[\sigma] \in S_{2n}/\sim} \prod_{i=1}^{n} (-1)^{\sigma} M_{\sigma_i \sigma_{n+i}} = \frac{1}{n! \cdot 2^n} \sum_{\sigma \in S_{2n}} \prod_{i=1}^{n} (-1)^{\sigma} M_{\sigma_i \sigma_{n+i}} \tag{1.1}$$

where the first sum is over equivalence classes of $2n$-permutations yielding the same product.

By abuse of notation, I will use $\operatorname{Pf}(N)$ to denote the vector of pfaffians of the principal $2n \times 2n$-minors of a $(2n+1) \times (2n+1)$-matrix N, as defined above.

There is an alternative approach regarding the skew-symmetric matrix N as an element $N = \frac{1}{2} \sum_{i,j \in \mathbf{Z}_7} N_{ij} e_i \wedge e_j \in E \wedge E$ for $E = \operatorname{span}(e_i)_{i \in \mathbf{Z}_7}$ a 7-dimensional vector space. We then get

$$\wedge^3 N = 3! \cdot \sum_{i \in \mathbf{Z}_7} \operatorname{Pf}(N_{\{i\}}) \bigwedge_{j \neq i} e_j. \tag{1.2}$$

Thus, $\wedge^3 N$ contains all the information of $\operatorname{Pf}(N)$.

1.2 Varieties defined by the Pfaffians

Definition 1.3 We define a variety $X_A \subset \mathbf{P}^6_A$ by $P_A = 0$, where $P_A = \operatorname{Pf}(N_A)$, and $N_A = \sum_{i \in \mathbf{Z}_7} A_i x_i$ with A_i skew-symmetric matrices and x_i coordinates on $\mathbf{P}^6$: ie., N_A is a skew-symmetric matrix whose terms are linear combinations of $x_0, \ldots, x_6$. We also demand that N_A have general rank 6 (ie., rank 6 over the polynomial ring $\mathbf{C}[x_i]$) and that for all values of x_i (not all zero), $\operatorname{rank} N(x) \geq 4$: this is the generic case. ■

[1] For invertible matrices, the adjoint is $\det N \cdot N^{-1}$; its elements are defined using the determinants after removing one line and one row and a proper choice of sign.

Do note that the definition is simply that X_A are the points where N_A has rank 4. (A skew-symmetric matrix always has even rank.) If we had allowed N_A to have rank 2 at points, we would have had degenerate cases to deal with.

In the setting of $N \in E \wedge E$, we may reformulate this slightly. Let $X \subset \mathbf{P}(E \wedge E)$ be the variety of $N \in E \wedge E$ with $\wedge^3 N = 0$. Let $A = \operatorname{span}(A_i)_{i \in \mathbf{Z}_7} \subset \mathbf{P}(E \wedge E)$ be the $\mathbf{P}^6$-hyperplane spanned by $A_i \in E \wedge E$. We may then define $X_A = X \cap A$. If we parametrize A by $\sum_i A_i x_i$, we get the original definition of X_A.

Lemma 1.4 The variety X_A is a locally complete intersection.

Proof: With the rank of N_A being at least 4, by permuting the rows/columns we may locally put it on the form

$$\left[\begin{array}{c|c} U & V \\ \hline -V^{\mathrm{T}} & W \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline (U^{-1}V)^{\mathrm{T}} & I \end{array}\right] \left[\begin{array}{c|c} U & 0 \\ \hline 0 & W + V^{\mathrm{T}}U^{-1}V \end{array}\right] \left[\begin{array}{c|c} I & U^{-1}V \\ \hline 0 & I \end{array}\right] \tag{1.3}$$

where U is an invertible principal 4×4-minor. Let M denote the matrix above at the right-hand centre with U and $M' = W + V^{\mathrm{T}}U^{-1}V$ on the diagonal; this is similar to N_A. Four of the pfaffians of M are now zero, the remaining being the elements of M' multiplied by the locally non-zero polynomial $\operatorname{Pf}(U)$:

$$\operatorname{Pf}(M) = \left[\begin{array}{c} 0 \\ \hline \operatorname{Pf}(U)\operatorname{Pf}(M') \end{array}\right] = \left[\begin{array}{c|c} I & U^{-1}V \\ \hline 0 & I \end{array}\right] P_A. \tag{1.4}$$

Looking at the part of $\mathbf{P}^6$ with $\operatorname{Pf}(U) \neq 0$, we find that X_A is defined locally by $M' = 0$: a codimension 3 complete intersection. ■

Lemma 1.5 With the variety $X_A \subset \mathbf{P}^6$ defined by $P_A = \operatorname{Pf}(N_A) = 0$ as above, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^6}(-7) \xrightarrow{P_A^{\mathrm{T}}} 7\mathcal{O}_{\mathbf{P}^6}(-4) \xrightarrow{N_A} 7\mathcal{O}_{\mathbf{P}^6}(-3) \xrightarrow{P_A} \mathcal{O}_{\mathbf{P}^6} \longrightarrow \mathcal{O}_{X_A} \longrightarrow 0 \tag{1.5}$$

which may also be written

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^6}(-7) \xrightarrow{P_A^{\mathrm{T}}} 7\mathcal{O}_{\mathbf{P}^6}(-4) \xrightarrow{N_A} 7\mathcal{O}_{\mathbf{P}^6}(-3) \xrightarrow{P_A} \mathcal{J}_{X_A} \longrightarrow 0. \tag{1.6}$$

In the notation of $X_A \subset \mathbf{P}(E \wedge E)$, $\mathcal{O} = \mathcal{O}_{\mathbf{P}(E\wedge E)}$, $P_A = \wedge^3 N$, we get the exact sequence

$$\begin{aligned} 0 \longrightarrow (\wedge^7 E^\vee)^2 \otimes \mathcal{O}(-7) &\xrightarrow{P_A} \wedge^7 E^\vee \otimes E^\vee \otimes \mathcal{O}(-4) \\ \xrightarrow{N_A} \wedge^7 E^\vee \otimes E \otimes \mathcal{O}(-3) &\xrightarrow{P_A} \mathcal{O} \longrightarrow \mathcal{O}_{X_A} \longrightarrow 0. \end{aligned} \tag{1.7}$$

Proof: From lemma 1.2, it follows that the compositions of two subsequent maps are zero.

Writing the matrix as in lemma 1.4 and looking locally at the open set with $\mathrm{Pf}(U) \neq 0$, we may again consider the skew-symmetric 3×3-matrix M'; its elements gives the pfaffians (up to product with $\mathrm{Pf}(U) \neq 0$).

The relations on the pfaffians of M', is exactly the matrix M'. Thus, the kernel of P_A is the image of N_A.

Similarly, as the kernel of M' is again its pfaffian vector, the kernel of N_A must be the image of P_A^{T}. ■

Lemma 1.6 The variety X_A is of dimension 3 and degree 14 in $\mathbf{P}^6$, has Hilbert series[2] $(1+3t+6t^2+3t^3+t^4)/(1-t)^4$ and Hilbert polynomial $7n(n^2+2)/3$.

Proof: All follows from the Hilbert series: $P_M(t) = \sum_{i=0}^{\infty} H^0(M(i))t^i$ for any module M. For any exact sequence $L.$, $\sum_k (-1)^k P_{L_k}(t) = 0$. Hence, using the resolution 1.5 and $P_{\mathcal{O}_{\mathbf{P}^6}}(t) = 1/(1-t)^7$, we find that $P_{\mathcal{O}_{X_A}}(t) = (1-7t^3+7t^4-t^7)/(1-t)^7 = (1+3t+6t^2+3t^3+t^4)/(1-t)^4$. The fourth power in the denominator indicates that the variety is dimension 4 in $\mathbf{C}^7$, and, hence, of dimension 3 in $\mathbf{P}^6$. Inserting $t=1$ in the numerator yields the degree, 14.

The Hilbert polynomial may be found substituting $t^p/(1-t)^q$ in the Hilbert series with $\binom{q-1+n-p}{q-1}$. ■

Lemma 1.7 With $X_A \subset \mathbf{P}^6$ defined by $P_A = 0$ as above, $h^{1,0} = h^{2,0} = 0$.

Proof: We may decompose the resolution 1.5 into short exact sequences:

$$\begin{array}{c} 0 \longrightarrow \mathrm{Im}\, P_A \longrightarrow \mathcal{O}_{\mathbf{P}^6} \longrightarrow \mathcal{O}_{X_A} \longrightarrow 0, \\ 0 \longrightarrow \mathrm{Im}\, N_A \longrightarrow 7\mathcal{O}_{\mathbf{P}^6}(-3) \longrightarrow \mathrm{Im}\, P_A \longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}_{\mathbf{P}^6}(-7) \longrightarrow 7\mathcal{O}_{\mathbf{P}^6}(-4) \longrightarrow \mathrm{Im}\, N_A \longrightarrow 0. \end{array} \tag{1.8}$$

The long exact cohomology-sequences, and the fact that $H^i(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(n)) = 0$ for all n if $0 < i < 6$ ([10], III.5.1), yields

$$\begin{aligned} H^i(X_A, \mathcal{O}_{X_A}) &\cong H^i(\mathbf{P}^6, \mathcal{O}_{X_A}) \cong H^{i+1}(\mathbf{P}^6, \mathrm{Im}\, P) \\ &\cong H^{i+2}(\mathbf{P}^6, \mathrm{Im}\, N_A) \cong H^{i+3}(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(-7)) = 0 \quad \text{for } i = 1,2. \end{aligned} \tag{1.9}$$

One may prove $H^3(X_A, \mathcal{O}_{X_A}) \cong \mathbf{C}$ in a similar way, but we will anyhow need a stronger result about the canonical bundle. ■

[2]The Hilbert series is defined as $\sum_{n=0}^{\infty} \dim H^0(\mathcal{O}(n))t^n$, whereas the Hilbert polynomial equals $\dim H^0(\mathcal{O}(n))$ for large n.

Lemma 1.8 With $X_A \subset \mathbf{P}^6$ defined by $P_A = 0$ as above, the dualizing sheaf $\omega^\circ_{X_A} \cong \mathcal{O}_{X_A}$.

Proof: We have $\omega_{\mathbf{P}^6} \cong \mathcal{O}_{\mathbf{P}^6}(-7)$ ([10], II.8.20.1). Using the resolution 1.5, we find that

$$\begin{aligned}
\omega^\circ_{X_A} &\cong \mathcal{E}xt^3_{\mathbf{P}^6}(\mathcal{O}_{X_A}, \omega_{\mathbf{P}^6}) \\
&\cong h^3(\mathcal{H}om(\text{Resolution}, \omega_{\mathbf{P}^6})) \quad ([10], \text{III.6.5}) \\
&\cong \mathcal{H}om(\mathcal{O}_{\mathbf{P}^6}(-7), \mathcal{O}_{\mathbf{P}^6}(-7))/(P^{\mathrm{T}})^*(\mathcal{H}om(7\mathcal{O}_{\mathbf{P}^6}(-4), \mathcal{O}_{\mathbf{P}^6}(-7))) \\
&\cong \mathcal{O}_{\mathbf{P}^6}/\{u \mapsto \phi(u \cdot P^{\mathrm{T}}) = \textstyle\sum_{i=0}^6 \phi_i(u) p_i \mid \phi : 7\mathcal{O}_{\mathbf{P}^6}(-4) \longrightarrow \mathcal{O}_{\mathbf{P}^6}(-7)\} \\
&\cong \mathcal{O}_{\mathbf{P}^6}/\mathcal{J}_{X_A} \\
&\cong \mathcal{O}_{X_A}.
\end{aligned} \tag{1.10}$$

■

If X_A is smooth, the dualizing sheaf $\omega^\circ_{X_A}$ is equal to the canonical bundle $\omega_{X_A} = \Omega^3_{X_A}$. In some of the cases we will find that X_A is not smooth. The singularities, however, will all have resolutions giving fibers of codimension 2, and, hence, $\omega^\circ_{X_A}$ may be identified with the canonical bundle on the resolved variety.

There will be a need for knowing the global section of ω_{X_A}. We know that $\omega^\circ_{X_A} \cong \omega_{\mathbf{P}^6} \otimes_{\mathcal{O}_{\mathbf{P}^6}} \wedge^3 \mathcal{N}_{X_A} \cong \mathcal{O}_{\mathbf{P}^6}(-7) \otimes_{\mathcal{O}_{\mathbf{P}^6}} \wedge^3 \mathcal{N}_{X_A}$ ([10], III.7.11). We may use this to construct such a form.

Lemma 1.9 For μ a permutation of $\mathbf{Z}_7$, the differential form

$$(-1)^\mu \frac{dp_{\mu_0} \wedge dp_{\mu_1} \wedge dp_{\mu_2}}{\mathrm{Pf}(N_{[\mu_3\mu_4\mu_5\mu_6]})} \tag{1.11}$$

defining a global section on $\mathcal{O}_{\mathbf{P}^6}(7) \otimes_{\mathcal{O}_{\mathbf{P}^6}} \wedge^3 \mathcal{N}^\vee_{X_A}$, does not depend on choice of μ. Here, $N_{[\mu_3\mu_4\mu_5\mu_6]}$ is the 4×4-submatrix of N_A consisting of rows and columns $\mu_3, \mu_4, \mu_5, \mu_6$. Furthermore, this form is non-vanishing (as a section of ${\omega^\circ_{X_A}}^\vee$)[3].

Proof: First, let me point out that, as forms in $\Omega^3_{\mathbf{P}^6}(9)|_{X_A}$,

$$dp_{\mu_0} \wedge dp_{\mu_1} \wedge dp_{\mu_2} = x_0^9 d\left(\frac{p_{\mu_0}}{x_0^3}\right) \wedge d\left(\frac{p_{\mu_1}}{x_0^3}\right) \wedge d\left(\frac{p_{\mu_2}}{x_0^3}\right); \tag{1.12}$$

hence, it is well-defined as a form. There are normally some restrictions when defining forms in a projective setting, as will be seen later.

The pfaffian $\mathrm{Pf}(N_{[\mu_3\mu_4\mu_5\mu_6]})$ is zero wherever $p_{\mu_0} = p_{\mu_1} = p_{\mu_2} = 0$ does not define a complete intersection, and then $dp_{\mu_0} \wedge dp_{\mu_1} \wedge dp_{\mu_2}$ must be zero. Hence, $\mathrm{Pf}(N_{[\mu_3\mu_4\mu_5\mu_6]})$ divides $dp_{\mu_0} \wedge dp_{\mu_1} \wedge dp_{\mu_2}$. Thus we know that the form is globally defined and without poles.

We know that ${\omega^\circ_{X_A}}^\vee$ is a trivial line-bundle, hence, a globally defined form must trivialize the sheaf. In particular, for different μ, the form must be equal up to multiplication with a non-zero constant. The constant may be found by explicit checking. ■

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[3]I wish to ignore problems pertaining to the existence of singular points. Specifying that I view the form as a section on ${\omega^\circ_{X_A}}^\vee$ rather than as a 3-form, does the trick.

In order to get a global section on $\omega^\circ_{X_A}$, we may take the 'inverse' of this form: ie., if ψ is the above form, we take the dual form to be ϕ such that $\langle\psi,\phi\rangle = 1$. This gives us the 3-form

$$\Omega = (-1)^\mu(-1)^\nu \frac{\mathrm{Pf}(N_{[\mu_3\mu_4\mu_5\mu_6]})}{\frac{\partial(p_{\mu_0},p_{\mu_1},p_{\mu_2})}{\partial(x_{\nu_0},x_{\nu_1},x_{\nu_2})}} d^3(x_{\nu_3},x_{\nu_4},x_{\nu_5},x_{\nu_6}) \tag{1.13}$$

which is a global section on $\omega^\circ_{X_A}$, where, for ν a permutation of $0,\ldots,6$, we denote by $d^3(x_{\nu_3},x_{\nu_4},x_{\nu_5},x_{\nu_6})$ the global section on $\Omega^3_{\mathbf{P}^6}(4)$ defined by

$$\begin{aligned} d^3(x_{\nu_3},x_{\nu_4},x_{\nu_5},x_{\nu_6}) &= x^3_{\nu_3}\cdot d\left(\frac{x_{\nu_4}}{x_{\nu_3}}\right)\wedge d\left(\frac{x_{\nu_5}}{x_{\nu_3}}\right)\wedge d\left(\frac{x_{\nu_6}}{x_{\nu_3}}\right) \\ &= x_{\nu_3}dx_{\nu_4}\wedge dx_{\nu_5}\wedge dx_{\nu_6} - x_{\nu_4}dx_{\nu_3}\wedge dx_{\nu_5}\wedge dx_{\nu_6} \\ &\quad + x_{\nu_5}dx_{\nu_3}\wedge dx_{\nu_4}\wedge dx_{\nu_6} - x_{\nu_6}dx_{\nu_3}\wedge dx_{\nu_4}\wedge dx_{\nu_5}. \end{aligned} \tag{1.14}$$

Lemma 1.10 The form Ω as defined above gives a non-vanishing global section of $\omega^\circ_{X_A}$, and, furthermore, is independent of choice of μ and ν. Ie., $\Omega \in H^0(X_A,\omega^\circ_{X_A})$ such that $f \mapsto f\Omega$ gives an isomorphism $\mathcal{O}_{X_A} \cong \omega^\circ_{X_A}$.

Proof: Setting $x_0 = 1$, we find the wedge product of the two forms, Ω and 1.11, to be $dx_1\wedge\cdots\wedge dx_6$. Similarly applies to other $x_i = 1$. ■

At this point, the cases of primary interest are the smooth X_A.

Lemma 1.11 The generic X_A is smooth: ie., for A outside a subvariety of the parameter space of positive codimension, X_A is smooth.

Proof: To prove this, we first prove that the $X \subset \mathbf{P}(E\wedge E)$ as given by $X = \{N \in E\wedge E \mid \wedge^3 N = 0\}$ has a singular locus of sufficiently low dimension, and then use Bertini's theorem (eg., follows from proof of theorem 8.18 in [10]) which states that a general hyperplane section will be smooth.

If at a given point $x \in X$, $N_{\{ijk\}}$, the matrix N with rows and columns i, j and k removed, has rank 4, then the jacobian

$$\frac{\partial(p_i,p_j,p_k)}{\partial(x_{jk},x_{ki},x_{ij})} = \det\begin{bmatrix} \frac{\partial p_i}{\partial x_{jk}} & 0 & 0 \\ 0 & \frac{\partial p_j}{\partial x_{ki}} & 0 \\ 0 & 0 & \frac{\partial p_k}{\partial x_{ij}} \end{bmatrix} = \mathrm{Pf}(N_{\{ijk\}})^3 \neq 0, \tag{1.15}$$

making X smooth at that point. Conversely, X is singular at any point at which N has rank 2. The rank 2 locus of N in $\mathbf{P}(E\wedge E)$ has codimension 10. Hence, the singular locus of X has codimension 10 in $\mathbf{P}(E\wedge E)$: codimension 7 in X.

With X_A of dimension 3, the expected dimension of the singular locus inherited from X is the dimension of X_A minus the codimension of singular locus in X: $3-7<0$. By Bertini's theorem, a generic intersection $X_A = X\cap A$, not meeting the singular locus of X, is smooth. ■

To summarize, we now have:

Proposition 1.12 The generic X_A is a smooth Calabi–Yau threefold with a global section of the canonical bundle Ω as given by 1.13.

Proof: For X_A smooth, the canonical bundle is isomorphic to the dualizing sheaf. Hence, the canonical bundle is trivial the form defined in 1.13 gives a global section of the canonical bundle. ■

1.3 Cohomology of the Smooth Varieties

Let us first look at the smooth varieties X_A. The singular varieties will constitute a variety in the parameter space of positive codimension, so the general X_A is smooth. I wish to determine the cohomology of these varieties. We already know $h^{p,0}(X_A) = h^p(\mathcal{O}_{X_A})$, but in order to determine the remaining part, some further information is needed.

Lemma 1.13 For the sheaf $\mathcal{J}^2_{X_A}$ we have the resolution

$$0 \longrightarrow 21\mathcal{O}_{\mathbf{P}^6}(-8) \xrightarrow{N\cdot} 48\mathcal{O}_{\mathbf{P}^6}(-7) \xrightarrow{\Phi} 28\mathcal{O}_{\mathbf{P}^6}(-6) \xrightarrow{P^{\otimes 2}} \mathcal{J}^2_{X_A} \longrightarrow 0, \tag{1.16}$$

where the elements are regarded as 7×7-matrices that are respectively skew-symmetric matrices, general matrices modulo the identity matrix (or with zero trace), and symmetric matrices. The three maps are $N\cdot : A \mapsto NA - I/7 \cdot \mathrm{trace}(NA)$ (if viewed modulo the identity, the last term may be dropped), $\Phi : B \mapsto BN + N^{\mathrm{T}}B^{\mathrm{T}}$, and $P^{\otimes 2} : C \mapsto P^{\mathrm{T}}CP$.

Proof: All the compositions are clearly zero, hence, it remains to prove that the kernels are contained in the images. The last map, $P^{\otimes 2}$, is clearly surjective as $\mathcal{J}^2_{X_A}$ is generated by $p_i p_j, i \leq j$.

Following the same lines as lemma 1.5, we may locally look at a skew-symmetric 3×3-matrix M' and its pfaffians which are simply its elements. With $m_i = M'_{\{i\}}$ being the elements (and pfaffians) of M', the relations on $m_{ij} = m_i m_j$ are clearly no other than $m_{ij} = m_{ji}$ and $m_{ij} m_k = m_{jk} m_j$: this gives exactly the relations making the sequence exact at $28\mathcal{O}_{\mathbf{P}^6}(-6)$.

Consider the map $48\mathcal{O}_{\mathbf{P}^6}(-7) \longrightarrow 28\mathcal{O}_{\mathbf{P}^6}(-6)$ by $B \mapsto BN + N^{\mathrm{T}}B^{\mathrm{T}}$. We then have

$$\begin{aligned} \Phi(B) = BN - NB^{\mathrm{T}} = 0 \quad &\Rightarrow \quad BN = NB^{\mathrm{T}} \\ &\Rightarrow \quad NB^{\mathrm{T}}P = 0 \\ &\Rightarrow \quad B^{\mathrm{T}}P = fP \quad \text{for some } f\text{, by 1.5} \\ &\Rightarrow \quad (B^{\mathrm{T}} - fI)P = 0 \\ &\Rightarrow \quad B^{\mathrm{T}} - fI = UN \quad \text{for a matrix } U\text{, by 1.5} \\ &\Rightarrow \quad B = -NU^{\mathrm{T}} + fI. \end{aligned} \tag{1.17}$$

The final $B = -NU^{\mathrm{T}} + fI$ equals $-NU^{\mathrm{T}}$ modulo I; alternatively, we might just remove the trace by making $B = -NU^{\mathrm{T}} + I/t \cdot \mathrm{trace}(NU^{\mathrm{T}})$. Thus, we have proved that the sequence is exact at $48\mathcal{O}_{\mathbf{P}^6}(-7)$.

Consider the map $21\mathcal{O}_{\mathbf{P}^6}(-8) \longrightarrow 48\mathcal{O}_{\mathbf{P}^6}(-7)$ by $A \mapsto NA - I/7 \cdot \mathrm{trace}(NA)$. The image is zero if and only if $NA = fI$. However, as N (and A) are skew-symmetric, they have rank at most 6, so $f = 0$ for fI to have rank less than 7. So, for A to map to zero, we must have $NA = 0$. However, using the exact sequence 1.5, we have that $NA = 0 \Rightarrow A = PU^{\mathrm{T}}$ for some vector U. However, $A = -A^{\mathrm{T}} = -UP^{\mathrm{T}}$, so $NA = 0 \Rightarrow NU = 0 \Rightarrow U = gP \Rightarrow A = gPP^{\mathrm{T}}$. However, for $A = gPP^{\mathrm{T}}$ to be skew-symmetric, g must be zero, making $A = 0$. Hence, the map is injective. ■

Proposition 1.14 For X_A smooth, $h^{1,1} = \dim H^1(\Omega_{X_A}) = 1$ and $h^{1,2} = \dim H^2(\Omega_{X_A}) = 50$.

Proof: There are three facts that I will use in what follows: that $H^*(\mathcal{O}_{\mathbf{P}^6}(-r)) = 0$ for $0 < r < 7$; if we have a resolution $0 \to A_n \to \cdots \to A_0 \to I$ where $H^*(A_i) = 0$ for $i < n$, then $H^p(I) \cong H^{p+n}(A_n)$; $h^6(\mathcal{O}_{\mathbf{P}^6}(-r-7)) = h^0(\mathcal{O}_{\mathbf{P}^6}(r)) = \binom{r+6}{6}$.

Using this result on the resolution of $\mathcal{O}_{X_A}(-1)$ (twist the entire sequence 1.5 by -1), we get $h^p(\mathcal{O}_{X_A}(-1)) = h^{p+3}(\mathcal{O}_{\mathbf{P}^6}(-8))$ which is 7 for $p = 3$, otherwise zero. Using these results and the cohomology of $\mathcal{O}_{X_A}$ on the long exact sequence of

$$0 \longrightarrow \Omega_{\mathbf{P}^6}|_{X_A} \longrightarrow 7\mathcal{O}_{X_A}(-1) \longrightarrow \mathcal{O}_{X_A} \longrightarrow 0, \tag{1.18}$$

we find that $h^0(\Omega_{\mathbf{P}^6}|_{X_A}) = h^2(\Omega_{\mathbf{P}^6}|_{X_A}) = 0$, $h^1(\Omega_{\mathbf{P}^6}|_{X_A}) = h^0(\mathcal{O}_{X_A}) = 1$, and $h^3(\Omega_{\mathbf{P}^6}|_{X_A}) = h^3(7\mathcal{O}_{X_A}(-1)) - h^3(\mathcal{O}_{X_A}) = 48$.

For $\mathcal{J}_{X_A}$, the above results and the resolution 1.6 gives $h^p(\mathcal{J}_{X_A}) = h^{p+3}(\mathcal{O}_{\mathbf{P}^6}(-7))$ which is 1 for $p = 3$, otherwise zero. For $\mathcal{J}^2_{X_A}$, the resolution 1.16 splits into two short exact sequences $0 \longrightarrow 21\mathcal{O}_{\mathbf{P}^6}(-8) \longrightarrow 48\mathcal{O}_{\mathbf{P}^6}(-7) \longrightarrow \mathcal{K} \longrightarrow 0$ and $0 \longrightarrow \mathcal{K} \longrightarrow 28\mathcal{O}_{\mathbf{P}^6}(-6) \longrightarrow \mathcal{J}^2_{X_A} \longrightarrow 0$. From the second, we get $h^p(\mathcal{J}^2_{X_A}) = h^{p+1}(\mathcal{K})$. From the first, the only non-zero part of the long exact sequence is

$$0 \longrightarrow H^5(\mathcal{K}) \longrightarrow 21H^6(\mathcal{O}_{\mathbf{P}^6}(-8)) \longrightarrow 48H^6(\mathcal{O}_{\mathbf{P}^6}(-7)) \longrightarrow H^6(\mathcal{K}) \longrightarrow 0. \tag{1.19}$$

This makes $h^4(\mathcal{J}^2_{X_A}) - h^5(\mathcal{J}_{X_A}) = h^5(\mathcal{K}) - h^6(\mathcal{K}) = 21h^6(\mathcal{O}_{\mathbf{P}^6}(-8)) - 48h^6(\mathcal{O}_{\mathbf{P}^6}(-7)) = 21 \cdot 7 - 48 = 99$.

As we have assumed the variety to be smooth, we have a short exact sequence

$$0 \longrightarrow \mathcal{J}^2_{X_A} \longrightarrow \mathcal{J}_{X_A} \longrightarrow \mathcal{N}^\vee_{X_A} \longrightarrow 0 \tag{1.20}$$

and another sequence

$$0 \longrightarrow \mathcal{N}^\vee_{X_A} \longrightarrow \Omega_{\mathbf{P}^6}|_{X_A} \longrightarrow \Omega_{X_A} \longrightarrow 0 \tag{1.21}$$

from which we may find the remaining Hodge numbers.

First, note that as $\mathcal{N}^\vee_{X_A}$ is a sheaf on X_A, $h^p(\mathcal{N}^\vee_{X_A}) = 0$ for $p > 3 = \dim X_A$. Entering this into the long exact sequences of the resolution 1.20, we get $h^5(\mathcal{J}^2_{X_A}) = 0$ as both $h^4(\mathcal{N}^\vee_{X_A}) = 0$ and $h^4(\mathcal{J}_{X_A}) = 0$. Hence, we have $h^4(\mathcal{J}^2_{X_A}) = 99$. Now, having both $h^*(\mathcal{J}_{X_A})$ and $h^*(\mathcal{J}^2_{X_A})$, the long exact sequence of 1.20 gives $h^2(\mathcal{N}^\vee_{X_A}) = 0$ and $h^3(\mathcal{N}^\vee_{X_A}) = 98$.

Finally, with full knowledge of $h^*(\mathcal{N}^\vee_{X_A})$ and $h^*(\Omega_{\mathbf{P}^6}|_{X_A})$, we can use the long exact sequence of 1.21 to get $h^1(\Omega_{X_A}) = 1$ and $h^2(\Omega_{X_A}) = 50$. ■

As the canonical bundle is trivial, $h^{2,1}(X_A) = h^1(\Omega^2_{X_A}) = h^1(\Theta_{X_A})$ which is the dimension of the moduli space of complex structures (or deformations of the complex structure) of the manifold.

The X_A are parametrized by A which contains 7 skew-symmetric 7×7-matrices: $7 \cdot 21$ parameters. The transformation $N_A \mapsto U^T N_A U$ where U is a non-degenerate complex matrix, gives 49-parameter classes that produce the same variety; furthermore, rotation in $\mathbf{P}^6$ reduces the dimension of the parameter space of varieties by another 48, leaving 50. This corresponds to the dimension of the moduli space of complex structures.

1.4 Subfamilies of Varieties and a Group Action

So far, we have only taken an interest in the general variety in the X_A-family. Generally, this variety is smooth; however, there are a number of subfamilies that are reducible or have singularities.

Viewing X_A as a variety in $\mathbf{P}(E \wedge E)$, we construct an action on $\mathbf{P}(E \wedge E)$ by $\sigma : e_i \mapsto e_{i+1}$ and $\tau : e_i \mapsto e_i w^i$ where w is a 7th root of unity, $(e_i)_{i \in \mathbf{Z}_7}$ a basis of E. The commutator of σ and τ is a constant, so as an action on $\mathbf{P}(E \wedge E)$, these two maps form an abelian 7×7 groups which I denote G.

Note that the action of G restricts to X. However, the action does not restrict to a general 7-hyperplane A, so it does not restrict to X_A for general A. The 7-hyperplanes A which are fixed under G may all be described as $A_Y = \operatorname{span}\{\sum_{i+j=k} y_{i-j} e_i \wedge e_j\}_{k \in \mathbf{Z}_7}$ where $y_i + y_{-i} = 0$; in matrix notation, $N_Y = [x_{i+j} y_{i-j}]_{i,j \in \mathbf{Z}_7}$. This subfamily of X_A parametrized by $Y = [y_1 : y_2 : y_3]$, I denote X_Y. Unlike the general variety X_A, the varieties X_Y are not smooth: they have 49 double-points at $[x_i]_{i \in \mathbf{Z}_7} \in \{g([y_i]_{i \in \mathbf{Z}_7}) \mid g \in G\}$. Notice that σ maps x_i to x_{i+2}, and τ maps x_i to $w^{2i} x_i$.

In order to study the parameter space of X_Y for symmetries, we consider the normalizer, F, of G in $\mathrm{GL}(E)$: ie., $F = \{u \in \mathrm{GL}(E) \mid uGu^{-1} = G\}$. If $U \in \mathrm{GL}(E)$, then U acts on X by restriction from $\mathbf{P}(E \wedge E)$. The maps of F will then act on the parameter space of G-fixed 7-planes, A_Y, and, hence, give maps on the Y-parameter space identifying isomorphic varieties in the family.

We find that the maps of G on $A_Y \cong \mathbf{P}^6$ (with coordinates x_i as above) have 56 fixed-points: 7 fixed-points for each of the 8 subgroups of G. We may pick one of these subgroups, $H \subset G$, without loss of generality due to the action of the normalizer. The group H then has 7 fixed-points in $\mathbf{P}^6$. We then find that the $Y \in \mathbf{P}^2$ which put these fixed-points on X_Y correspond to three lines in $\mathbf{P}^2$, of which we may pick one line without loss of generality: $y_3 = 0$. We set $y = y_2/y_1 \in \mathbf{P}^1$, and denote the matrix N_y and the variety X_y. In addition to the 49 double-points, we now also have 7 fixed-points which are also double-points. For some special values of y, we may get extra singularities; for $y = 0$ and $y = \infty$, X_y decomposes into 14 $\mathbf{P}^3$s ($y = 0$ and ∞).

Inspired by previous mirror constructions by orbifolding, it is hypothesized that X_A and a minimal resolution of X_y/H are mirror manifolds.

Let us go through the above in some greater detail.

Lemma 1.15 The group $G = (\tau, \sigma)$ acting on $\mathbf{P}^6$ by $\sigma(x_i) = x_{i+2}$ and $\tau(x_i) = x_i w^{2i}$ where $w = e^{2\pi i/7}$, is a 7×7 abelian group. The group also acts on the varieties X_Y and X_y. There are 56 fixed-points in $\mathbf{P}^6$ (for some element of G): 7 for each of its 8 subgroups. For each such subgroup in G, there is a set of three lines in the parameter space $\mathbf{P}^2$ of Y that do not meet in one point, such that the fixed-points under the subgroup lies in X_Y.

There is a group F, the normalizer of G in the group general linear matrices, containing G which acts on $\mathbf{P}^6 \times \mathbf{P}^2$ such that each $f \in F$ gives maps $f_Y : \mathbf{P}^2 \longrightarrow \mathbf{P}^2$ and $f_X : \mathbf{P}^6 \longrightarrow \mathbf{P}^6$ such that $f_X : X_Y \longrightarrow X_{f_Y(Y)}$. Using F, the 8 sets of line-triplets may be interchanged, and, furthermore, the three lines may be interchanged. Hence, we may choose one of these subgroups and one of the three lines in the line-triplet without loss of generality: eg., the subgroup $H = (\tau)$ and the line $y_3 = 0$ on the line-triplet $y_1 y_2 y_3 = 0$, leaving σ to act transitively on X_y.

Finally, having chosen our special line in the parameter space $\mathbf{P}^2$, there is a subgroup of F which leaves this line fixed, mapping the parameter y to $w^k y$, thus identifying X_y with $X_{w^k y}$. This makes $\phi = y^7$ the natural parameter on the 1-parameter space, $X_\psi \cong X_y$.

Proof: Represent τ and σ by 7×7-matrices such that for $x = [x_i]$ a vector, the maps are given by $x \mapsto \tau x$ and $x \mapsto \sigma x$. If we denote the matrix $N_{x,y} = [x_{i+j} y_{i-j}]$, we get $N_{x,y} = \tau^{\mathrm{T}^{-1}} N_{\tau x, y} \tau^{-1}$ and $N_{x,y} = \sigma^{\mathrm{T}^3} N_{\sigma x, y} \sigma^3$. As matrices, $\tau\sigma = w^4 \sigma\tau$, making them commute as actions on $\mathbf{P}^6$. Hence, we have a 7×7 abelian group acting on $\mathbf{P}^6$ whose actions preserve the varieties X_Y and X_y.

Let me define maps on $\mathbf{P}^6$ by linear maps on $\mathbf{C}^7$: $\alpha_k : x_i \mapsto x_{ki}$, $\beta_k : x_i \mapsto w^{ki^2} x_i$, and $\gamma_k : x_i \mapsto \sum_j w^{ijk} x_j/\sqrt{7}$, where $i, j, k \in \mathbf{Z}_7$ are integers modulo 7. Later, we will regard these maps as maps on $\mathbf{P}^6$ and on $\mathbf{P}^2 \subset \mathbf{P}^6$ by $[y_1 : y_2 : y_3] \mapsto [0 : y_1 : y_2 : y_3 : -y_3 : -y_2 : -y_1]$, but let us for the moment just regard them as 7×7-matrices. These maps have (matrix) relations $\alpha_n \alpha_m = \alpha_{nm}$, $\beta_n \beta_m = \beta_{n+m}$, $\gamma_n \gamma_m = \alpha_{-mn^{-1}}$, $\beta_n \alpha_m = \alpha_m \beta_{nm^2}$, $\alpha_m \gamma_n = \gamma_{nm^{-1}}$, $\gamma_n \alpha_m = \gamma_{nm}$, $\alpha_n \sigma = \sigma^n \alpha_n$, $\tau \alpha_n = \alpha_n \tau^n$, $\beta_n \sigma = w^{-n} \sigma \tau^{2n} \beta_n$, $\beta_n \tau = \tau \beta_n$,

$\gamma_n \sigma = \tau^n \gamma_n$, $\gamma_n \tau = \sigma^{-n^{-1}} \gamma_n$.

If we let $N_{x,y} = [x_{i+j} y_{i-j}]$ (we do not need $y_i + y_{-i} = 0$ for this), we have the relations $(\beta_{5m}\alpha_n)^{\mathrm{T}} N_{\beta_m \alpha_n x, \beta_m \alpha_n y} \beta_{5m} \alpha_n$ and $N_{x,y} = \gamma_{5n}^{\mathrm{T}} N_{\gamma_n x, \gamma_n y} \gamma_{5n}$. These, together with the action of G, define the group F acting on $N_{x,y}$.

Regard the maps as projective maps: ie., modulo multiplication with a constant. We denote the subgroups $H_n = (\sigma\tau^n) \subset G$ and $H_\infty = (\tau)$. As each group is generated by a single element which may be represented by a 7×7-matrix, fixed-points correspond to eigenvectors, of which there are seven. Obviously, τ has 7 fixed-points: the points $[x_i]$ where all $x_i = 0$ except one. Using $\beta_{n/2}\gamma_{-1}\tau = \sigma\tau^n \beta_{n/2}\gamma_{-1}$, we find that τ acts on X_Y as $\sigma\tau^n$ does on $X_{\beta_{n/2}\gamma_{-1}Y}$. Hence, selecting $H = H_\infty$ would be equivalent to any other choice of subgroups of G.

The fixed-points of H lie in X_Y if and only if $y_1 y_2 y_3 = 0$. The map α_k acting on $N_{x,y}$ giving the equivalence of X_Y and $X_{\alpha_k Y}$, maps y_i to y_{ki} (using $y_i + y_{-i} = 0$, i modulo 7). Thus, y_3 is equivalent to any other of the lines in $y_1 y_2 y_3 = 0$. Furthermore, the effect of β_k on $y = [0 : 1 : y : 0 : 0 : -y : -1] \in \mathbf{P}^6$ is that of $y \mapsto w^{3k} y$. Thus, $\beta_{5k} : y \mapsto w^k y$.

Fixed-points under H are the points $[x_i]$ such that all x_i except one are zero. This yields 7 fixed-points under H. As σ leaves none of these fixed, these points are not fixed-points for any of the other subgroups of G. Thus, the number of fixed-points (for some element of G) is 56. ■

Proposition 1.16 In general, the variety X_A is smooth. The variety X_Y, however, contains 49 double-points: the images of $[0 : y_1 : y_2 : y_3 : -y_3 : -y_2 : -y_1]$ under G. In the case of X_y, in addition to the 49 double-points, the 7 fixed-points under the group action are also double-points. In general, these are all the singularities of X_y, however, X_y gains another 7 double-points if fixed-points from a subset of G other than H lie on X_Y: this occurs if Y lies on the intersection of two lines from two different line-triplets, ie., there are 21 such X_y. Furthermore, if Y lies on the intersection between two lines from the same line-triplet (if $y = 0$ or ∞), the variety degenerates into 14 $\mathbf{P}^3$s.

Proof: The parameter spaces of X_A, X_Y, and X_y are all projective spaces. In X_Y there are 49 double-points as identified in the appendix; in X_y the seven fixed-points are also identified as double-points. To prove that the general case is no worse than this, it suffices to find one example in each parameter space which has no more than these singularities.

In appendix B, Gröbner basis calculations are performed for two examples, one in each of the parameter spaces X_Y and X_y, having no more than the given singularities. Hence, the X_A are generally smooth (lemma reflem:smooth), the X_Y generally have no more than the 49 given double-points, and the X_y generally have no more than the given $49 + 7$ double-points.

Of course, there are several subfamilies for which there are other singularities. In particular, if an X_y is chosen so as to make y lie on a line-triplet in $\mathbf{P}^2$ corresponding to a subgroup of $H' \subset G$ other than H, then the fixed-points under H' will also be double-points, adding seven more double-points.

If $y = 0$, X_y is defined by equations $x_0x_3x_4 = x_2x_5x_6 = x_0x_1x_4 = x_2x_3x_6 = x_1x_4x_5 = x_0x_3x_6 = x_1x_2x_5 = 0$. This defines 14 $\mathbf{P}^3$s. ■

1.5 Orbifolding and Resolution of Singularities

From here and onwards, we consider only the generic variety in each of the families: ie., we consider smooth varieties in the family X_A, varieties in X_Y with 49 double-points, and varieties X_y with $49 + 7$ double-points. Varieties with other singularities than these or which degenerates are ignored.

The family X_A consists of smooth Calabi–Yau manifolds whose cohomology is known. The mirror candidate is a minimal resolution of X_y/H where H is a 7-subgroup of G with 7 fixed-points on X_y.

Lemma 1.17 There exists minimal resolutions, ie. one which leaves the dualizing sheaf trivial, of X_Y and X_y which respect the group action H (or G): ie., so that H (or G) acts on the resolved manifolds.

Proof: Analytically, we may do these resolutions locally, blowing up along a smooth surface passing through the double point. This replaces the double-point with a copy of $\mathbf{P}^1$. For each double-point there are two options: these differ by a flop. As the changes are codimension 2, there is no change in the dualizing sheaf; the holomorphic 3-form Ω extends holomorphically across these exeptional lines.

On X_Y, the group acts transitively, mapping one double-point onto another. Hence, in choosing a resolution of one, this determines the resolutions of the other. Alternatively, resolve X_Y/H (or X_Y/G), which has only one double-point; then $\widetilde{X_Y/H} \times_{X_Y/H} X_Y$ (or $\widetilde{X_Y/G} \times_{X_Y/G} X_Y$) will give a resolution of X_Y on which H (or G) acts.

On X_y, the 49 double-points inherited from the X_Y family may be resolved as for X_Y. For each of the fixed-points under H, when resoving the double-point, H will act on the exeptional line (possibly trivially). The actual effect of H on the exeption lines is determined in lemma 1.18. To ensure that G acts on the resolution, the choice of resolution of one of these double-points will determine the rest through the action of the group G as was the case for the other 49 double-points as elements of G not in H act transitively on X_y.

A concrete resolution of the double-points on X_y is given by blowing up along the smooth surfaces $S_i = \{x_i = x_{i+3} = x_{i-3} = y^2 x_{i+1} x_{i-1} - x_{i+2} x_{i-2} = 0\}$ in any order. This also ensures that the resolution is projective, though that is not needed. ■

Lemma 1.18 After resolving the double points as above, the action τ has 14 fixed points in $\widetilde{X}_y$: two on each of the $\mathbf{P}^1$s of the initial fixed points.

Proof: This is most easily observed by looking at each double point locally.

If we look at the fixed-point at $[1:0:0:0:0:0:0]$, the singularity is locally given by $x_2 = x_5 = 0$ (to the first order) and (to the second order) by $y^3 x_1 x_6 + x_3 x_4 = 0$. There are two options (which differ by a flop) for blowing up: either (to the first order) $[z_0 : z_1] = [x_1 : x_3] = [x_4 : x_6]$ or $[z_0 : z_1] = [x_1 : x : 4] = [x_3 : x_6]$. The effect of τ, being $x_i \mapsto w^i x_i$, is $[z_0 : z_1] \mapsto [z_0 : w^2 z_1]$ and $[z_0 : z_1] \mapsto [z_0 : w^3 z_1]$ respectively. In either case, there are two fixed points.[4] ■

Corollary 1.19 Let $\widetilde{X}_y$ (resp. $\widetilde{X}_Y$) be a[5] minimal resolution of X_y (resp. X_Y) as given above. This is smooth, so the dualizing sheaf isomorphic to the canonical bundle which must then be trivial. Hence, they are Calabi–Yau manifolds.

Proof: Most of this follows from the previous two lemmas. In addition, we need that $h^{0,1}$ and $h^{0,2}$ both remain zero, which is a known result ([4], [19]). ■

Lemma 1.20 The manifolds $\widetilde{X}_Y$ and $\widetilde{X}_y$ are Calabi–Yau manifolds. Furthermore, for $\widetilde{X}_Y$ we have $h^{1,1}(\widetilde{X}_Y) = h^{2,2}(\widetilde{X}_Y) = h^{1,2}(\widetilde{X}_Y) = h^{2,1}(\widetilde{X}_Y) = 2$; for $\widetilde{X}_y$ we have $h^{1,1}(\widetilde{X}_y) = h^{2,2}(\widetilde{X}_y) = 8$ and $h^{1,2}(\widetilde{X}_y) = h^{2,1}(\widetilde{X}_y) = 1$.

Proof: Passing from the general X_A to the special family X_Y, 49 double-points arise; passing on to X_y, another 7 double-points arise. These double-points arise by S^3s collapsing to points ([4]). The S^3s are cycles in H_3 and the change in cohomology in passing from the general X_A to X_Y or X_y is determined by the number of relations on the vanishing S^3-cycles: if there are s vanishing S^3-cycles and d relations, then the dimension of H_3 is reduced by $s - d$ and the dimension of H_4 is increased by d. Consequently, the minimal resolution of the double-points will reduce the dimension of H_3 by another $s - d$ and increase the dimension of H_2 by d. Thus, the problem of determining the cohomology is reduced to determining the number of independent relations on the vanishing S^3-cycles.

[4]Being a linear map on $\mathbf{P}^1$, the alternatives were two fixed-points (generic case), one fixed-point (would not give a finite group though), or leaving the entire $\mathbf{P}^1$ fixed.

[5]This is unique up to flops.

One approach to determining the change in cohomology is to determine the number of relations on the cycles ([19]) directly: eg., by actually finding the relations. In this, the effect of the group symmetry may be used to reduce the amount of calculations demanded. Here, another approach will be used, finding the number of relations from the size of the deformation space.

We consider the deformation space, $\mathrm{Def}\, X = H^1(X, \Theta_X)$. As we are dealing with Calabi–Yau manifolds with double-points, the moduli space is smooth ([18], [15]), and for minimal resolutions, $\widetilde{X}$, there is an inclusion $\mathrm{Def}\, \widetilde{X} \hookrightarrow \mathrm{Def}\, X$ ([5]). Hence, we know that $\dim \mathrm{Def}\, X_Y = \dim \mathrm{Def}\, X_A = 50$. The group G acts on X_Y and, therefore, induces an action on $\mathrm{Def}\, X_Y$.

The inclusion $\mathrm{Def}\, \widetilde{X_Y} \hookrightarrow \mathrm{Def}\, X_Y$ factors through the deformations of X_Y which retain the 49 double-points: ie., $\mathrm{Def}\, \widetilde{X_Y} \cong \mathrm{Def}(X_Y; s_1, \ldots, s_{49}) \subset \mathrm{Def}\, X_Y$ where $\mathrm{Def}(X_Y; \{s_i\})$ denotes deformations of X_Y and marked points s_i such that there are singularities at these points. As the generic variety X_A is smooth, the general deformation of X_Y will smooth all the double-points; hence, the inclusion $\mathrm{Def}(X_Y; \{s_i\}) \subset \mathrm{Def}\, X_Y$ is a proper inclusion. We may notice that the group G, which acts transitively on the generic X_Y, maps each of these double-points to any other. Hence, the action of G on $\mathrm{Def}\, X_Y$ restricts to $\mathrm{Def}(X_Y; \{s_i\}) \cong \mathrm{Def}\, \widetilde{X_Y}$.

These deformation spaces, $\mathrm{Def}\, X_Y$ and $\mathrm{Def}\, \widetilde{X_Y}$, form local systems on the parameter space: ie., we have vector bundles $V \subset W$ on $\mathbf{P}^2$ whose fibres over $Y \in \mathbf{P}^2$ are $\mathrm{Def}\, \widetilde{X_Y}$ and $\mathrm{Def}\, X_Y$ respectively. The group G acts on these local systems, hence, we may decompose them into (τ, σ)-eigenspaces: $V = \oplus_{i,j \in \mathbf{Z}_7} V^{(i,j)} \subset W = \oplus_{i,j \in \mathbf{Z}_7} W^{(i,j)}$ where (i,j) denotes eigenvalues (w^i, w^j). The dimensions of all these spaces and eigenspaces must necessarily be constant.

Now, in addition to the group G, we have the normalizer of G which induces maps on the parameter space, $\mathbf{P}^2$, which permute the eigenspaces: for $(i,j) \neq (0,0)$, each eigenspace may be mapped to any other by some map. Hence, for $(i,j) \neq (0,0)$, the dimensions $\dim V^{(i,j)}$ must all be the same, and the dimensions $\dim W^{(i,j)}$ must all be the same.

The eigenspace $W^{(0,0)}$ denotes the deformations of X_Y which respects the action of G: ie., retains the action. These were only the X_Y; hence, $W^{(0,0)}$ contains only the deformations along the $\mathbf{P}^2$ family, making $\dim W^{(0,0)} = 2$. These deformations along the $\mathbf{P}^2$ family also retains all the 49 double-points of the X_Y, hence, lie in V: so $W^{(0,0)} \subset V$. As we now have $V^{(0,0)} \subset W^{(0,0)} \subset V$, we necessarily have $V^{(0,0)} = W^{(0,0)}$.

With $\dim W = 50$ and $\dim W^{(0,0)} = 2$, for the other eigenspaces, all having the same dimension, we find $\dim W^{(i,j)} = 1$, $(i,j) \neq (0,0)$. Similarly, $\dim V < 50$ and $\dim V^{(0,0)} = 2$, so $\dim V^{(i,j)}$ must be zero for $(i,j) \neq (0,0)$; hence, $V = V^{(0,0)}$, $\dim V = 2$, which corresponds exactly to the deformations along the $\mathbf{P}^2$ family X_Y.

The family X_y is again a proper subfamily of X_Y with, in general, 7 extra double-points; hence, the deformation space of X_y retaining all the $49 + 7$ double-points has a lower dimension: it must then be of dimension one, corresponding to the $\mathbf{P}^1$ family X_y.

Again using that $h^{2,1}$ is the dimension of the deformation space, we get $h^{2,1}(\widetilde{X_Y}) = 2$ and $h^{2,1}(\widetilde{X_y}) = 1$. This tells us how many relations were on the vanishing S^3-cycles as stated in the first paragraph of this proof, and, thus, the change in cohomology. ■

Lemma 1.21 Let $\widetilde{X_y}$ be a resolution of X_y respecting the H-action as described in lemma 1.17. We divide this with the group $H = (\tau)$ to obtain the orbifold $\widetilde{X_y}/H$. There is a minimal resolution of this quotient, ie. one which leaves the canonical bundle trivial.

Proof: The existence of minimal resolutions of these quotient singularities is proved in [17]. Furthermore, Roan's article [16] gives a simple formula for the Euler characteristic,

$$\chi(\widetilde{V/H}) = \sum_{g,h \in H} \frac{\chi(V^g \cap V^h)}{|H|}, \tag{1.22}$$

which makes $\chi(\widetilde{X_y/H}) = 98$: the appropriate number given $\chi(X_A) = -98$.

In order to get at the Hodge numbers, I refer to [1]. By this, the effect of dividing by the group H and resolving the quotient singularities minimally[6] increases $h^{1,1}$ and $h^{2,2}$ each by 3 for each of the 14 fixed points, making $h^{1,1} = h^{2,2} = 50$. The other Hodge numbers remain unchanged. ■

We have now proved the following:

Proposition 1.22 There is a minimal resolution of X_y/H (for generic y for which X_y has 49 + 7 double-points) which we denote M_y. This is a Calabi–Yau threefold with $h^{1,1} = h^{2,2} = 50$ and $h^{1,2} = h^{2,1} = 1$. The global section of the canonical bundle is given by 1.13. ■

[6]Actually, the theorem used to do this depends not only on the size of the group, but on a particular representation of it in the group of special linear maps. However, these all give the same results.

Chapter 2

The Pfaffian Quotient and Curve Count on its Mirror

The methods used generally follow that described by D. R. Morrison: see [13] for a good overview. However, the Picard–Fuchs equation is obtained by a different method, more similar to that of [2].

In this chapter we will look at the family M_y of Calabi–Yau manifolds defined in the previous chapter: ie., the special 1-parameter family divided out with the 7-group and with singularities resolved in a minimal way. As the manifold depends only on y^7, we will parametrize the family by $\phi = y^7$ and denote the family M_ϕ.

Let Ω be the (3,0)-form on the family M_ϕ of Calabi–Yau manifolds. By the mirror symmetry, we are supposed to have a special point in the parameter space,[1] which in our case will be $\phi = 0$, and on a punctured neighborhood of this special point, continuous families $\gamma_0(\phi)$ and $\gamma_1(\phi)$ of 3-cycles on M_ϕ, such that:

1. The function $f_0(\phi) = \int_{\gamma_0} \Omega$ extends across $\phi = 0$.
2. The function $f_1(\phi) = \int_{\gamma_1} \Omega$ is such that $f_1(\phi)/f_0(\phi) = g(\phi) + \log \phi$ where $g(\phi)$ extends across $\phi = 0$.

The *natural coordinate* (or *mirror map*) is then given by[2] $t = f_1/f_0$. This corresponds to picking $\omega = t\omega_0$ as the Kähler form on the mirror. As this form enters only as $\int_\eta \omega$ with $\eta \in H_2(\mathbf{Z})$, it may be expressed in terms of $q(\phi) = e^{f_1/f_0} = \phi e^g$.

[1] This special point in the parameter space is supposed to correspond to the large radius limit of the mirror: ie., when the real part of the Kähler form on the mirror goes to infinity (or $-\infty$ depending on the sign used in the definition).

[2] I should point out that often a constant $2\pi i$ enters. This is a matter of definition: with what constant the Kähler form enters the correlation function on the mirror. The choice of constant will have no influence on the following.

The curve count on the mirror is arrived using the Mirror Symmetry assumption that the B-model Yukawa-coupling derived from the variation of complex structure (Hodge-structure) should be equal to the A-model Yukawa-coupling on the mirror which is expressed in terms of the corresponding Kähler structure given by the natural coordinate and the number of rational curves in any curve class (ie., of any given degree) by

$$\kappa_{ttt} = n_0 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}. \tag{2.1}$$

The Yukawa-coupling κ_{ttt} may be defined by[3]

$$\kappa_{ttt} = \int_{M_\phi} \tilde{\Omega} \wedge \nabla^3_{\frac{d}{dt}} \tilde{\Omega} \tag{2.2}$$

with ∇ the Gauss–Manin connection and $\tilde{\Omega} = \Omega/f_0$ the normalized (3,0)-form. (In the following, I will take $\nabla_u = \nabla_{\frac{d}{du}}$ for any parameter u.)

All of this can be determined from knowing the Picard–Fuchs equation. The Picard–Fuchs equation is a differential equation on the parameter space whose solutions are $\int_\gamma(\phi)\Omega$ for γ Gauss–Manin flat sections on $R_3\pi_*\mathbf{C}$: ie., $\gamma(\phi) = \sum_i u_i \nu_i(\phi)$ where $\nu_i(\phi) \in H_3(M_\phi, \mathbf{Z})$, $u_i \in \mathbf{C}$. As $h_3 = 4$, this equation has order 4: ie., for $f = \int_\gamma \Omega$, we have

$$\int_{\gamma(\phi)} \sum_{i=0}^{4} A_{\phi,i}(\phi)\nabla^i_\phi \Omega = \sum_{i=0}^{4} A_{\phi,i}(\phi) \left(\frac{d}{d\phi}\right)^i f(\phi) = 0 \tag{2.3}$$

for any ∇-flat $\gamma = \gamma(\phi)$: eg., for $\gamma \in \Gamma(R_3\pi_*\mathbf{C})$. Using a parameter other than ϕ will merely give other A-coefficients: eg., the logarithmic derivative $D_\phi = d/d\log(\phi) = \phi \cdot d/d\phi$ is often used as this will make $A_{\log\phi,i}$ without poles at $\phi = 0$.

First, I will find γ_0 and calculate f_0. From this, I will determine the Picard–Fuchs equation. Knowing the Picard–Fuchs equation, f_1 can be found as another special solution.[4] Furthermore, the Yukawa coupling, κ, satisfies the differential equation $\frac{d}{du}\log\kappa_{uuu} = -\tilde{A}_{u,3}/2\tilde{A}_{u,4}$ for any parameter u.[5]

2.1 Determining $f_0(\phi)$

Rather than working on M_ϕ, it is here convenient to work on X_ϕ: ie., the variety before dividing out with the group. This will have no effect on the calculations.

[3]The Yukawa coupling is actually a symmetric 3-tensor, and κ_{ttt} is short for $\kappa_{\frac{d}{dt}\frac{d}{dt}\frac{d}{dt}} = \kappa \cdot \frac{d}{dt} \otimes \frac{d}{dt} \otimes \frac{d}{dt}$ where the tangent vectors on the parameter space correspond to classes in $H^{-1,1}(M_\phi) = H^1(M_\phi, TM_\phi)$ describing the variation of the complex structure.

[4]Actually, the f_1 will only be found up to adding a multiple of f_0. This gives a constant, c_2, which must be determined later.

[5]Again, determining κ only up to a constant, c_1. Also, we will first express κ in terms of Ω using $A_{u,3}$ in the differential equation rather than the coefficient $\tilde{A}_{u,3}$ from the Picard–Fuchs equation of $\tilde{\Omega}$.

The first step in order to determine f_0 is determining a 3-cycle γ_0 that extends across $\phi = 0$. This is most easily done by looking at X_0: here, X_ϕ degenerates into 14 3-planes with the group acting on each plane. One of these 3-planes if $X_0 \subset \mathbf{P}^6$ is determined by $x_4 = x_5 = x_6 = 0$. The most natural 3-cycle on this plane is given by $|x_i/x_0| = \epsilon$ for $i = 1, 2, 3$. Denoting this cycle $\gamma_0(0)$, we may extend the definition to $\gamma_0(\phi)$ in a neighborhood of $\phi = 0$. This makes

$$f_0(\phi) = \int_{\gamma_0(\phi)} \Omega(\phi). \tag{2.4}$$

Rather than working with γ_0, it is more convenient to work with the 6-cycle Γ on $\mathbf{P}^6$ given by $|x_i/x_0| = \epsilon$ for $i = 1, 2, 3$ and $|x_i/x_0| = \delta$ for $i = 4, 5, 6$, and view Ω as the residue[6] of

$$\Psi = \Omega \wedge \frac{dp_{\nu_0}}{2\pi i p_{\nu_0}} \wedge \frac{dp_{\nu_1}}{2\pi i p_{\nu_1}} \wedge \frac{dp_{\nu_2}}{2\pi i p_{\nu_2}} = (-1)^\nu \frac{\mathrm{Pf}(N_{[\nu_3\nu_4\nu_5\nu_6]})}{(2\pi i)^3 p_{\nu_0} p_{\nu_1} p_{\nu_2}} \cdot x_0^7 \bigwedge_{i=1}^{6} d\left(\frac{x_i}{x_0}\right) \tag{2.5}$$

for any permutation ν of $\mathbf{Z}_7$. We now get

$$f_0(\phi) = \int_{\gamma_0(\phi)} \Omega(\phi) = \int_\Gamma \Psi(\phi) \tag{2.6}$$

where the last integral is over a cycle which is independent of ϕ.

In order to make the numerator as simple as possible, choose $\nu_0, \nu_1, \nu_2 = 0, 3, 4$. This makes $\mathrm{Pf}(N_{[\nu_3\nu_4\nu_5\nu_6]}) = x_3 x_4$. Setting $x_0 = 1$ for simplicity (or writing x_i for x_i/x_0), the integral becomes

$$\int_\Gamma \frac{x_3 x_4}{(2\pi i)^3 p_0 p_3 p_4} \cdot \bigwedge_{i=1}^{6} \frac{dx_i}{x_i} = \int_\Gamma \frac{-1}{(2\pi i)^3 \cdot \prod_{i=0,3,4}(1 - \sum_{j=1}^{4} v_{i,j})} \cdot \bigwedge_{i=1}^{6} \frac{dx_i}{x_i} \tag{2.7}$$

where

$$[v_{i,j}] = \begin{bmatrix} \frac{x_2 x_5}{x_3 x_4} \cdot y & \frac{x_4 x_6}{x_3} \cdot y^2 & \frac{x_1 x_3}{x_4} \cdot y^2 & -\frac{x_1 x_6}{x_3 x_4} \cdot y^3 \\ \frac{x_1 x_4}{x_2 x_3} \cdot y & \frac{x_2}{x_3 x_6} \cdot y^2 & \frac{x_3 x_5}{x_2 x_6} \cdot y^2 & -\frac{x_5}{x_2 x_3} \cdot y^3 \\ \frac{x_3 x_6}{x_4 x_5} \cdot y & \frac{x_5}{x_1 x_4} \cdot y^2 & \frac{x_2 x_4}{x_1 x_5} \cdot y^2 & -\frac{x_2}{x_4 x_5} \cdot y^3 \end{bmatrix}. \tag{2.8}$$

We may expand the fraction on the right in terms of the $v_{i,j}$ and get

$$\int_\Gamma \Psi = \frac{-1}{(2\pi i)^3} \cdot \int_\Gamma \sum_{n \in \mathbf{N}_0^{4\times 3}} \prod_i \binom{n_i}{n_{i,1}, n_{i,2}, n_{i,3}, n_{i,4}} \cdot \prod_{i,j} v_{i,j}^{n_{i,j}} \cdot \bigwedge_{i=1}^{6} \frac{dx_i}{x_i}, \quad n_i = \sum_j n_{i,j}. \tag{2.9}$$

[6]It may seem that this residue is a residue on the entire $p_{\nu_0} = p_{\nu_1} = p_{\nu_2} = 0$, rather than just on the variety X_y, however, the pfaffian in the numerator below will be zero on the extra components. It should be pointed out, however, that Ω, which is defined uniquely on the variety, is not uniquely determined on the whole of $\mathbf{P}^6$: there are different choices of ν which give different extensions of Ω to $\mathbf{P}^6$, of which we use one particular.

The only terms that give a non-zero contribution are those with $v^n = \prod_{i,j} v_{i,j}^{n_{i,j}}$ independent of x_i. The ring of products of $v_{i,j}$ which do not contain x_i's are generated by (see A.3 for method)

$$\begin{aligned} r_1 &= v_{1,4}v_{2,3}v_{3,3} = -y^7 = -\phi, \\ r_2 &= v_{1,2}v_{2,3}v_{3,4} = -y^7 = -\phi, \\ r_3 &= v_{1,3}v_{2,4}v_{3,3} = -y^7 = -\phi, \\ r_4 &= v_{1,2}v_{2,2}v_{2,3}v_{3,1} = y^7 = \phi, \\ r_5 &= v_{1,3}v_{2,1}v_{3,2}v_{3,3} = y^7 = \phi, \\ r_6 &= v_{1,1}v_{2,1}v_{2,3}v_{3,1}v_{3,3} = y^7 = \phi. \end{aligned} \tag{2.10}$$

As $v_{1,4}$, $v_{3,4}$, $v_{2,4}$, $v_{2,2}$, $v_{3,2}$, and $v_{1,1}$ all are contained in only $r_1, \ldots, r_6$ respectively, there can be no relations between these generators. All v^n that are independent of x_i may thus be written $r^m = \prod_i r_i^{m_i}$.

Instead of evaluating the sum over v^n, we may now evaluate the sum over r^m including as weights the number of times the term $r^m = v^n$ occurs. This makes the integral, using the appropriate correspondence between m and n,

$$\begin{aligned} \frac{-1}{(2\pi i)^3}\int_\Gamma \Psi(\phi) &= \sum_{\substack{(m_i)\in\mathbf{N}_0^6 \\ m=\sum_i m_i}} (-1)^{m_1+m_2+m_3}\phi^m \prod_i \binom{n_i}{n_{i,1}, n_{i,2}, n_{i,3}, n_{i,4}} \\ &= \sum_{\substack{m_1,m_6,u_1,u_2\geq 0 \\ m=m_1+m_6+u_1+u_2}}^{\infty} (-1)^{m_1}\phi^m \cdot \frac{m!}{m_1!m_6!u_1!u_2!(m-u_1)!(m-u_2)!} \\ &\quad \cdot \sum_{m_2+m_4=u_1} (-1)^{m_2}\frac{(m+m_4+m_6)!}{m_2!m_4!(m_4+m_6)!} \\ &\quad \cdot \sum_{m_3+m_5=u_2} (-1)^{m_3}\frac{(m+m_5+m_6)!}{m_3!m_5!(m_5+m_6)!} \\ &= \sum_{\substack{m_1,m_6,u_1,u_2\geq 0 \\ m=m_1+m_6+u_1+u_2}}^{\infty} (-1)^{m_1}\phi^m \cdot \frac{m!}{m_1!m_6!u_1!u_2!(m-u_1)!(m-u_2)!} \\ &\quad \cdot \frac{(m+m_6)!}{(u_1+m_6)!}\binom{m}{u_1} \cdot \frac{(m+m_6)!}{(u_2+m_6)!}\binom{m}{u_2} \\ &= \sum_{\substack{m_1,m_6,u_1,u_2\geq 0 \\ m=m_1+m_6+u_1+u_2}}^{\infty} (-1)^{m_1}\phi^m \cdot \binom{m}{u_1}^2\binom{m}{u_2}^2\binom{m+m_6}{m}\binom{m+m_6}{m_1, u_1+m_6, u_2+m_6} \\ &= 1 + 5\phi + 109\phi^2 + 3317\phi^3 + 121501\phi^4 + 4954505\phi^5 + \cdots \end{aligned} \tag{2.11}$$

2.2 Determining the Picard–Fuchs equation

As $h^3 = 4$, we may pick a local basis[7] $\gamma_0(\phi)$, $\gamma_1(\phi)$, $\gamma_2(\phi)$, $\gamma_3(\phi)$ of $R_3\pi_*\mathbf{Z}$. This gives us functions $f_i(\phi) = \int_{\gamma_i(\phi)} \Omega(\phi)$. If we look at the vectors $V_i(\phi) = (\frac{d}{d\phi})^i[f_0, \ldots, f_3]$ for $i = 0, \ldots, 4$, there must be a linear relationship between them:

$$\sum_{i=0}^{4} a_i(\phi) V_i(\phi) = \sum_{i=0}^{4} a_i(\phi)(\frac{d}{d\phi})^i V_0(\phi) = 0. \tag{2.12}$$

These coefficients are unique up to multiplying with a function in ϕ.[8] This differential equation is the *Picard–Fuchs differential equation.* The $a_i(\phi)$ correspond to the $A_{\phi,i}(\phi)$ in the previous section; for another parameter, u, we will similarly get $A_{u,i}(u)$.

Initially, we do this locally. However, in our case we have a $\mathbf{P}^1$ parameter space and so the $A_{\phi,i}$ should expand to give rational functions which we may choose to be polynomials. To see this, we note that the Picard–Fuchs equation may be defined locally at any point of the parameter space which gives a non-singular manifold, hence, at all but a finite number of points in $\mathbf{P}^1$. If you choose the top coefficient $a_4 = 1$, the coefficients are unique and defined on all of $\mathbf{P}^1$ except a finite number of points, hence, rational.

We have some extra information about the coefficients: if we use the logarithmic coordinate $\log\phi$, the coefficients at our special point, $\phi = 0$, in the parameter space should yield $A_{\log\phi,i}(0) = 0$ for $i = 0, 1, 2, 3$.[9]

When the power series for $f_0(\phi)$ is known, we may try to write down the Picard–Fuchs differential equation in f_0 entering general polynomials for the A's. In this case, entering general terms $A_{\log\phi,i}(\phi)$ of degree 5 yields a solution:

$$\begin{aligned}\sum_{i=0}^{4} A_{\log\phi,i}\left(\phi\tfrac{d}{d\phi}\right)^i \;=\;& (1 - 57\phi - 289\phi^2 + \phi^3)(\phi-3)^2\left(\phi\tfrac{d}{d\phi}\right)^4 \\ & +4\phi(\phi-3)(85 + 867\phi - 149\phi^2 + \phi^3)\left(\phi\tfrac{d}{d\phi}\right)^3 \\ & +2\phi(-408 - 7597\phi + 2353\phi^2 - 239\phi^3 + 3\phi^4)\left(\phi\tfrac{d}{d\phi}\right)^2 \\ & +2\phi(-153 - 4773\phi + 675\phi^2 - 87\phi^3 + 2\phi^4)\left(\phi\tfrac{d}{d\phi}\right) \\ & +\phi(-45 - 2166\phi + 12\phi^2 - 26\phi^3 + \phi^4).\end{aligned} \tag{2.13}$$

This is the so called Picard–Fuchs operator.

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[7]Though I use γ, they need not be the same as the γ_0 and γ_1 of the previous section.

[8]In general, there is a possibility that the linear relationship might occur with fewer vectors, but in that case, $\nabla^3\Omega$ would not be able to have a $H^{0,3}$ component, and the Yukawa coupling would become identically zero.

[9]The reason for this is that our special point shall correspond to the large radius limit on the mirror, and that we should have maximally unipotent monodromy around this point.

2.3 Determining the Natural Coordinate

The next step is to determine f_1 and the natural coordinate. To do this, we use that f_1 is a solution to the Picard–Fuchs differential equation and that $f_1(\phi) = f_0(\phi) \cdot (g(\phi) + \log\phi)$ for some function g extending across $\phi = 0$. Entering a general power series in ϕ for g into the differential equation, we find

$$g(\phi) = \alpha + 14\phi + 287\phi^2 + \tfrac{26222}{3}\phi^3 + \tfrac{647143}{2}\phi^4 + \cdots \tag{2.14}$$

where α is a constant. We now get the natural coordinate

$$t = \frac{f_1(\phi)}{f_0(\phi)} = \log\phi + g(\phi). \tag{2.15}$$

The q-coordinate may then be written down, setting $c_2 = e^{\alpha}$:

$$q = e^t = \phi e^{g(\phi)} = c_2(\phi + 14\phi^2 + 385\phi^3 + 13216\phi^4 + 516852\phi^5 + \cdots). \tag{2.16}$$

2.4 Determining the Yukawa Coupling

We then calculate the Yukawa coupling. This is a symmetric 3-tensor on the parameter space, $\mathbf{P}^1$, which will be globally defined but with poles. The Yukawa coupling is given by

$$\begin{aligned}
\kappa_{ttt} &= \left(\frac{d\log\phi}{dt}\right)^3 \kappa_{\log\phi\log\phi\log\phi} \\
&= \left(\frac{d\log\phi}{dt}\right)^3 \int_{M_\phi} \tilde{\Omega} \wedge \nabla^3_{\phi\frac{d}{d\phi}} \tilde{\Omega} \\
&= \left(\frac{d\log\phi}{dt}\right)^3 \frac{1}{f_0(\phi)^2} \int_{M_\phi} \Omega \wedge \nabla^3_{\phi\frac{d}{d\phi}} \Omega.
\end{aligned} \tag{2.17}$$

To move f_0 to outside the differential, we use Griffiths' transversality property which implies that $\Omega \wedge \nabla^i \Omega = 0$ for $i < 3$.

The term $\int_{M_\phi} \Omega \wedge \nabla^3_{\phi\frac{d}{d\phi}} \Omega$ satisfies a differential equation:

$$\phi\frac{d}{d\phi}\log\left(\int_{M_\phi} \Omega \wedge \nabla^3_{\phi\frac{d}{d\phi}}\Omega\right) = -\frac{A_{\log\phi,3}}{2A_{\log\phi,4}}. \tag{2.18}$$

This gives us

$$\int_{M_\phi} \Omega \wedge \nabla^3_{\phi\frac{d}{d\phi}} \Omega = \frac{c_1(\phi - 3)}{1 - 57\phi - 289\phi^2 + \phi^3}. \tag{2.19}$$

The denominator has zeros at three points in the parameter space. These are the points where the manifold has singularities: where our particular special line in the bigger parameter space $\mathbf{P}^2$ intersects other special lines, and, hence, has 7 double points.

2.5 The Rational Curve Count on the Mirror

The final step is to express κ_{ttt} in terms of q. Using the power series expansion $q = q(\phi)$ and its inverse series giving $\phi = \phi(q)$, and $\frac{d \log \phi}{dt} = \frac{q}{\phi}\frac{d\phi}{dq}$, we may express κ_{ttt} as

$$\begin{aligned} \kappa_{ttt} &= \left(\frac{q}{\phi(q)}\frac{d}{dq}\phi(q)\right)^3 \frac{1}{f_0(\phi(q))^2} \cdot \frac{c_1(\phi-3)}{1-57\phi-289\phi^2+\phi^3} \\ &= -c_1\left(3+14\tfrac{q}{c_2}+714(\tfrac{q}{c_2})^2+24584(\tfrac{q}{c_2})^3+906122(\tfrac{q}{c_2})^4+\cdots\right) \\ &= -c_1\left(3+14\frac{(\frac{q}{c_2})}{1-(\frac{q}{c_2})}+\frac{714-14c_2}{8}\frac{(\frac{q}{c_2})^2}{1-(\frac{q}{c_2})^2}+\frac{24584-14c_2^2}{27}\frac{(\frac{q}{c_2})^3}{1-(\frac{q}{c_2})^3}+\cdots\right). \end{aligned} \tag{2.20}$$

In order that there be only non-negative integers in the last line,[10] we must set $c_2 = 1$.[11] Putting $c_1 = -2m$, we get

$$\kappa_{ttt} = m \cdot \left(6 + 28\frac{q}{1-q} + 175\frac{q^2}{1-q^2} + 1820\frac{q^3}{1-q^3} + 28294\frac{q^4}{1-q^4} + 530992\frac{q^5}{1-q^5} + \cdots\right). \tag{2.21}$$

The actual value of m cannot be seen from this series alone, however, m is supposed to have a fixed value as determined by the value of the Yukawa coupling.

As the general M_A that was initially assumed to be the mirror, has degree 14, and the first term of the q-series of κ_{ttt} which gives the degree of the mirror, gives the degree as a multiple of 6, this assumption must be discarded.

There is another striking observation[12]: the Picard–Fuchs equation is exactly the same as that found on $G(2,7)$ intersected by seven general hyperplanes in $\mathbf{P}^{20}$ ([2]).

2.6 The Picard–Fuchs equation at $\phi = \infty$

In order to study what is going on around $\phi = \infty$, we may introduce the coordinate $\tilde{\phi} = 1/\phi$. Transforming the Picard–Fuchs operator directly (multiplying with $\tilde{\phi}^5$ and using $\tilde{\phi}\frac{d}{d\tilde{\phi}} = -\phi\frac{d}{d\phi}$), we find the Picard–Fuchs operator to be regular at $\tilde{\phi} = 0$. This does not, however, give the correct picture.

Recall that the Picard–Fuchs operator is acting on the section Ω of $F^3 \subset R^3\pi_*\mathbf{C}$, where F^3 is $H^{3,0}$ over each parameter-point. We expressed Ω as the residue of Pf $N_{[\nu_4\nu_5\nu_6\nu_7]}/p_{\nu_1}p_{\nu_2}p_{\nu_3}$ for small y. As y increases toward infinity, this changes as $y^{-7} = \phi^{-1}$. In other words, $F^3 \cong \mathcal{O}_{\mathbf{P}^1}(1)$, and our section Ω has a zero at $\phi = \infty$. To get a non-zero section at

[10]By flipping the sign of c_2, we will still have integers, but many of them will be negative.

[11]This is an observation made from looking at the terms of the power series, but has not been proved in classical mirror symmetry. In the method by Givental ([7]), however, gives $c_2 = 1$.

[12]This observation was made by Duco van Straten.

$\phi = \infty$, we may use $\tilde{\Omega} = \phi \cdot \Omega$ instead. We must then modify the Picard–Fuchs operator accordingly: $\phi \frac{d}{d\phi} \mapsto -\tilde{\phi}\frac{d}{d\tilde{\phi}} - 1$. The new Picard–Fuchs operator then becomes

$$\begin{aligned}\sum_{i=0}^{4} \tilde{A}_{\log \tilde{\phi}, i} \left(\tilde{\phi}\frac{d}{d\tilde{\phi}}\right)^i \;=\;& (1 - 289\tilde{\phi} - 57\tilde{\phi}^2 + \tilde{\phi}^3)(1 - 3\tilde{\phi})^2 \left(\tilde{\phi}\frac{d}{d\tilde{\phi}}\right)^4 \\ &+4\tilde{\phi}(3\tilde{\phi} - 1)(143 + 57\tilde{\phi} - 87\tilde{\phi}^2 + 3\tilde{\phi}^3)\left(\tilde{\phi}\frac{d}{d\tilde{\phi}}\right)^3 \\ &+2\tilde{\phi}(-212 - 473\tilde{\phi} + 725\tilde{\phi}^2 - 435\tilde{\phi}^3 + 27\tilde{\phi}^4)\left(\tilde{\phi}\frac{d}{d\tilde{\phi}}\right)^2 \\ &+2\tilde{\phi}(-69 - 481\tilde{\phi} + 159\tilde{\phi}^2 - 171\tilde{\phi}^3 + 18\tilde{\phi}^4)\left(\tilde{\phi}\frac{d}{d\tilde{\phi}}\right) \\ &+\tilde{\phi}(-17 - 202\tilde{\phi} - 8\tilde{\phi}^2 - 54\tilde{\phi}^3 + 9\tilde{\phi}^4).\end{aligned} \tag{2.22}$$

We may proceed as for the previous case, but calculating $\tilde{f}_0$ from the Picard–Fuchs equation rather than the opposite.[13] This gives a Yukawa-coupling[14] in terms of $\tilde{q}$:

$$\kappa_{\tilde{t}\tilde{t}\tilde{t}} = \tilde{c}_1(1 + 42\tilde{c}\tilde{q} + 6958\tilde{c}^2\tilde{q}^2 + \cdots). \tag{2.23}$$

Putting $\tilde{c} = 1$ and $\tilde{c}_1 = 2\tilde{m}$, we get

$$\kappa_{\tilde{t}\tilde{t}\tilde{t}} = \tilde{m}\left(2 + 84\frac{\tilde{q}}{1-\tilde{q}} + 1729\frac{\tilde{q}^2}{1-\tilde{q}^2} + 83412\frac{\tilde{q}^3}{1-\tilde{q}^3} + 5908448\frac{\tilde{q}^4}{1-\tilde{q}^4} + \cdots\right), \tag{2.24}$$

which for $k = 7$ gives degree 14 for the mirror, 588 lines, 12103 conics, etc. This seems to match[15] the general pfaffian M_A.

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[13] At first, the $\tilde{f}_0$ was found explicitly as a series by the same methods as for pfaffian near $\phi = 0$.

[14] Note that the Yukawa-coupling does not change as we go from Ω to $\tilde{\Omega}$ as it is formulated in terms of the normalized version of Ω. We are not, however, guaranteed that the constant c_1 will come out the same when solving the differential equation. However, we may also determine $\tilde{\kappa}$ from κ merely by changing from the coordinate ϕ to $\tilde{\phi}$. This gives $\tilde{c}_1 = -c_1$.

[15] The lines on the general pfaffian has been counted by Ellingsrud and Strømme, and is 588. (Private communication.)

Chapter 3

The Grassmannian Quotient and Curve Count on its Mirror

As the Picard–Fuchs equation on the pfaffian corresponded to that of Plücker-embedding of $G(2,7)$ in projective space intersected by 7 general hyperplanes found in [2], it was natural to look for a closer correspondence: eg. a correspondence between the moduli-spaces of complex structures and therefrom a mirror construction analogous to that of M_y from the general family X_A.

Previously, we have regarded the pfaffian varieties as intersections $X \cap A$ or $X \cap A_y$ where A and A_y are families of 6-planes in $P(E \wedge E)$ and $X = \{N \in E \wedge E \mid \wedge^3 N = 0\}$. The group $G = (\sigma, \tau)$ acting on X was derived from the maps $\sigma(e_i) = e_{i+1}$ and $\tau(e_i) = w^i e_i$.

As was commented before, the B-model Picard–Fuchs equation on the pfaffian quotient M_y proved to be exactly the same as the A-model Picard–Fuchs equation for $G(2,7) \cap \mathbf{P}^{13} \subset \mathbf{P}^{20}$ with $\mathbf{P}^{13}$ a general 13-plane in $\mathbf{P}^{20}$. Hence, it appears that the two may be mirrors.

We may perform a construction on $G(2,7)$ 'dual' to that performed on X in order to get M_y. To do this, select the family $A_y^\vee \subset \mathbf{P}(E^\vee \wedge E^\vee)$ dual to $A_y \subset \mathbf{P}(E \wedge E)$ and make $Y_y = G(2,7) \cap A_y^\vee$. We may then construct N_ϕ from Y_y similarly to the way M_ϕ was constructed from X_y. As in the pfaffian case, dividing out with the group action will have no effect on the Picard–Fuchs equations, except that we should take care to ensure that the objects we define (forms and cycles) are invariant or may be made invariant under the group. Thus, we may effectively work on Y_y.

3.1 Determining the Canonical Form

As $c_1(G(2,7)) = 7H$, the intersection with $\mathbf{P}^{13}$ will make $c_1(Y_y) = 0$, hence, the canonical bundle is trivial.

We will need only a description of the canonical form as a residue defined in a neighborhood of $y = 0$ in the parameter space. Rather than view $G(2,7)$ as embedded in $\mathbf{P}^{20}$, we may look at an affine subset of $G(2,7)$: that given by $u_1 \wedge u_2$ where $u_i = [u_{i,j}]$, $i = 1,2$, $j = 0,\ldots,6$, and where $u_{1,0} = u_{2,2} = 1$, $u_{1,2} = u_{2,0} = 0$. The defining equations then become

$$u_{1,i}u_{2,i+1} - u_{1,i+1}u_{2,i} = y \cdot (u_{1,i-2}u_{2,i+3} - u_{1,i+3}u_{2,i-2}), \quad i \in \mathbf{Z}_7. \tag{3.1}$$

Now, we may define the canonical form Ω as the residue of

$$\Psi = \frac{\bigwedge_{i=1,3,4,5,6} du_{1,i} \wedge du_{2,i}}{\prod_{i\in\mathbf{Z}_7}(u_{1,i}u_{2,i+1} - u_{1,i+1}u_{2,i} - y \cdot (u_{1,i-2}u_{2,i+3} - u_{1,i+3}u_{2,i-2}))}. \tag{3.2}$$

3.2 Determining f_0

For $y = 0$, the variety decomposes. One of the components may be given affinely by $u_1 \wedge u_2$ where $u_1 = [1,0,0,0,0,0,0]$, $u_2 = [0,0,1,u_{2,3},u_{2,4},u_{2,5},0]$. We may define the 3-cycle $\gamma_0(0)$ by $|u_{i,j}| = \epsilon$ on Y_0, and extend this to a neighborhood of $y = 0$: say, $|y| < \delta$. As for the pfaffian, we will rather use the 10-cycle Γ in $\mathbf{P}^{20}$ defined by $|u_{i,j}| = \epsilon_{i,j}$, where again $u_i = [u_{i,j}]$ with $u_{1,0} = u_{2,2} = 1$, $u_{1,2} = u_{2,0} = 0$. The actual choices of δ the $\epsilon_{i,j}$ will be made so as to make the quotients $v_{i,j}$ defined below sufficiently small, but will otherwise be of no importance.

We may now rewrite the residual form so as to suite our purpose of evaluating it as a power series in y:

$$\Psi = \frac{1}{\prod_i(1 - \sum_j v_{i,j})} \cdot \bigwedge_{i=1,3,4,5,6} \frac{du_{1,i} \wedge du_{2,i}}{u_{1,i}u_{2,i}} \tag{3.3}$$

where

$$\begin{gathered}
v_{1,1} = -y \cdot \tfrac{u_{1,5}u_{2,3}}{u_{2,1}}, \quad v_{1,2} = y \cdot \tfrac{u_{1,3}u_{2,5}}{u_{2,1}}, \\
v_{2,1} = -y \cdot \tfrac{u_{1,6}u_{2,4}}{u_{1,1}}, \quad v_{2,2} = y \cdot \tfrac{u_{1,4}u_{2,6}}{u_{1,1}}, \\
v_{3,1} = y \cdot \tfrac{u_{2,5}}{u_{1,3}}, \\
v_{4,1} = \tfrac{u_{1,3}u_{2,4}}{u_{1,4}u_{2,3}}, \quad v_{4,2} = y \cdot \tfrac{u_{1,1}u_{2,6}}{u_{1,4}u_{2,3}}, \quad v_{4,3} = -y \cdot \tfrac{u_{1,6}u_{2,1}}{u_{1,4}u_{2,3}}, \\
v_{5,1} = \tfrac{u_{1,4}u_{2,5}}{u_{1,5}u_{2,4}}, \quad v_{5,2} = -y \cdot \tfrac{1}{u_{1,5}u_{2,4}}, \\
v_{6,1} = \tfrac{u_{1,5}u_{2,6}}{u_{1,6}u_{2,5}}, \quad v_{6,2} = y \cdot \tfrac{u_{1,3}u_{2,1}}{u_{1,6}u_{2,5}}, \quad v_{6,3} = -y \cdot \tfrac{u_{1,1}u_{2,3}}{u_{1,6}u_{2,5}}, \\
v_{7,1} = y \cdot \tfrac{u_{1,4}}{u_{2,6}}.
\end{gathered} \tag{3.4}$$

In order that the power series expansion be possible, we must have $\sum_j |v_{i,j}| < 1$. For this, we need $\epsilon_{1,i}/\epsilon_{2,i} < \epsilon_{1,j}/\epsilon_{2,j}$ for $3 \le i < j \le 6$, and δ sufficiently small.

If we look at the ring generated by the $v_{i,j}$, the subring of elements that do not contain terms $u_{i,j}$ is generated by (see A.3 for method)

$$\begin{aligned} r_1 &= v_{1,1}v_{2,1}v_{3,1}v_{4,2}v_{5,2}v_{6,2}v_{7,1} = y^7 = \phi, \\ r_2 &= v_{1,2}v_{2,1}v_{3,1}v_{4,3}v_{5,2}v_{6,1}v_{6,3}v_{7,1} = -y^7 = -\phi, \\ r_3 &= v_{1,1}v_{2,1}v_{3,1}v_{4,1}v_{4,3}v_{5,1}v_{5,2}v_{6,1}v_{6,3}v_{7,1} = y^7 = \phi, \\ r_4 &= v_{1,1}v_{2,2}v_{3,1}v_{4,1}v_{4,3}v_{5,2}v_{6,3}v_{7,1} = -y^7 = -\phi. \end{aligned} \tag{3.5}$$

There are no relations between these generators, as may be seen from that fact that the terms $v_{4,2}$, $v_{1,2}$, $v_{5,1}$, and $v_{2,2}$ are unique to their respective generators.

For any $r^m = \prod_i r_i^{m_i}$, the corresponding $u^n = \prod_{i,j} u_{i,j}^{n_{i,j}}$ occurs $\prod_i \binom{n_i}{n_{i,1},\ldots}$ number of times, $n_i = \sum_j n_{i,j}$. The power series expansion for $f_0 = \int_{\gamma_0} \Omega$ will then be given by

$$\begin{aligned} \frac{1}{(2\pi i)^7}\int_\Gamma \Psi &= \sum_{\substack{(m_i)\in\mathbf{N}_0^4 \\ m=\sum_i m_i}} (-1)^{m_2+m_4}\phi^m \cdot \prod_{i=1}^{7} \binom{n_i}{n_{i,1},\ldots} \\ &= \sum_{\substack{(m_i)\in\mathbf{N}_0^4 \\ m=\sum_i m_i}} (-1)^{m_2+m_4}\phi^m \cdot \binom{m}{m_2}\binom{m}{m4}\binom{m+m_3}{m} \\ &\qquad \cdot \binom{m+m_2+m_3}{m_1, m_2+m_3, m_2+m_3+m_4} \\ &\qquad \cdot \binom{m+m_3+m_4}{m_1, m_3+m_4, m_2+m_3+m_4} \\ &= 1 + 5\phi + 109\phi^2 + 3317\phi^3 + 121501\phi^4 + 4954505\phi^5 + \cdots \end{aligned} \tag{3.6}$$

which may be recognized as exactly the same series as for the pfaffian quotient.

3.3 The Rational Curve Count on the Mirror

As the power series for f_0 is exactly as in the case of the pfaffian quotient, the Picard–Fuchs equation, the natural coordinate, the Yukawa-coupling, and the curve count on the mirror, will all be exactly the same, except that the constants may differ. Hence, the rational curve count of the mirror is given by the coefficients of

$$\kappa_{ttt} = m \cdot \left(6 + 28\tfrac{q}{1-q} + 175\tfrac{q^2}{1-q^2} + 1820\tfrac{q^3}{1-q^3} + 28294\tfrac{q^4}{1-q^4} + 530992\tfrac{q^5}{1-q^5} + \cdots\right). \tag{3.7}$$

3.4 The Grassmannian Quotient near $\phi = \infty$

The case for the Grassmannian quotient becomes exactly the same as for the pfaffian quotient. Again the residue acts like $y^{-7} = \phi^{-1}$ as ϕ increases, so we again shift to $\phi \cdot \Omega$, with the transformation on the Picard–Fuchs operator being the same as for the pfaffian quotient.

Again, this can be checked explicitly by finding $\tilde{f}_0$ in terms of a binomial sum series, etc. This gives the same answer though.

Chapter 4

Conclusions and Final Remarks

4.1 Mirror Conjecture

Apparently, there is a strong relation between the varieties defined by the pfaffians and the grassmannian $G(2,7)$. The B-models of the M_y and W_y are isomorphic, and according to mirror symmetry and assuming that we actually have the mirrors, the A-models of the general pfaffian and general $G(2,7)$ sections should also be isomorphic, and vice versa. It may of course be possible that we have found models with the same B-model but different A-models, in which case they would not be mirrors.

Assuming that we actually have mirror symmetry, it would appear that varying the complex structure on M_y or W_y leads to a transition from the Kähler structure on the pfaffian section X_A to that of the grassmannian section Y_A.

Conjecture 4.1 The pairs $M_y + W_y$ is the mirror family of $X_A + Y_A$ where M_y and W_y (resp. X_A and Y_A) form different parts of the A-model (Kähler) moduli space. ■

It seems as if the pfaffians correspond the grassmanian section with 'negative Kähler form', and vice versa. Previous realizations of field theories corresponding to 'negative Kähler structures' have been made using Landau–Ginzburg models ([8], [21]).

A birational map between the varieties X_Y and Y_Y has been found ([14]), which may shed some extra light on the relation between these two families.

It is worth noting that the smooth varieties X_A and Y_A cannot be birationally equivalent. As $h^{1,1} = 1$, this has a unique positive integral generator (the dual of a line); if birational, these two must correspond up to a rational factor. Integrating the third power of this over the variety gives the degree; the ratio of the degrees would then be the third power of a rational number, which is not possible for $42/14 = 3$.

4.2 The Web of Calabi–Yau Manifolds

The family $\widetilde{X_y/H}$ comes from a $\mathbf{P}^1$-subfamily of the general 50-parameter family X_A; this is conjectured to be the mirror of X_A. However, there are other subfamilies: in particular the $\mathbf{P}^2$-family X_Y.

Reasoning about the link between the complex deformations and the deformations of the Kähler structure, we may draw some conclusions. Let $\pi : \widetilde{X} \to X$ be a resolution of double-points on X and let ω be a Kähler form on X. Then we get a degenerate Kähler form on $\widetilde{X}$: it is zero on the exeptional lines. Conversely, if we have a degenerate Kähler form, we may blow down lines on which it is zero to produce a Calabi–Yau variety with double-points. In other words, resolving the double-points may give a larger space of Kähler forms.

Alternatively, rather then resolving the double-points, we may deform them. In this way, the space of Kähler-forms remains the same, however, the deformation space allow deformations away from the family of singular varieties.

The above is what happens at X_Y: we may deform it to give any X_A, or we may resolve the double-points to give $\widetilde{X_Y}$. Starting with the mirror symmetry $X_A \leftrightarrow \widetilde{X_y/H}$, we may extend the picture as in the following table. From right to left, the left-hand varieties are deformed from singular to less singular varieties, and the right-hand varieties are resolved bit-by-bit. From top to bottom, it is the other way around.

$\boldsymbol{X_A \leftrightarrow \widetilde{X_y/H}}$	$X_{A_H} \leftrightarrow \widetilde{X_y}/H$	$X_Y \leftrightarrow \widehat{X_y}/H$	$X_y \leftrightarrow X_y/H$	
		$\boldsymbol{\widetilde{X_Y} \leftrightarrow \widetilde{X_Y}/H}$	$\widehat{X_y} \leftrightarrow X_Y/H$	(4.1)
			$\boldsymbol{\widetilde{X_y} \leftrightarrow X_{A_H}/H}$	

Smooth Calabi–Yau manifolds are displayed in bold face. The variety $\widehat{X_y}$ is X_y with only the 49 double-points inherited from the X_Y-family resolved. With X_{A_H}, we denote the varieties X_A where A is a 7-plane to which the group H restricts, making it a 8-parameter subfamily of the 50-parameter family X_A; these are generally smooth.

It may be worth noting the existence of the special varieties X_y for the special values of y such that X_y contains the fixed-points under, not only the subgroup H of G, but also of another subgroup H' of G. These additional fixed-points will also be double-points and resolving these will make the Calabi–Yau manifold rigid: there are no deformations of the complex structure, hence, $h^{2,1} = 0$. This would extend the diagram. It is, however, not clear what the mirrors would be; however, it would be natural to assume that it should have some geometric link to X_A.

In fact, it has been conjectured that the entire moduli space of Calabi–Yau manifolds is linked together through such transitions ([8]).

Appendix A

Mathematical Tools

A number of standard tools are used. The following sections gives a quick review of the tools and results used.

A.1 Resolution of Double-Points

A.1.1 On a Simple Class of Double-Points

A double-point is a singularity such that there is a local definition by a complete intersection $g_1(x) = g_2(x) = \cdots = g_r(x) = 0$ in, say, $\mathbf{C}^n$ with jacobian of rank $r-1$ at this point, and there is a vector v such that $\partial q_i/\partial v = v \cdot \nabla q_i = 0$ at this point for all i but $\partial^2 q_j/\partial v^2 \neq 0$ for some j. By a coordinate change moving the double-point to origo and a linear transformation of the polynomials, this can be transformed to $g'_i(x') = x'_i$ for $i = 1, \ldots, r-1$ where $g'_i = \nabla g'_i = 0$ in origo, and $\partial^2 g'_r/\partial x'^2_r \neq 0$. By eliminating $x'_1, \ldots, x'_{r-1}$ from g'_r, the equation $g'_r = 0$ becomes $\sum_{i,j \geq r} a_{ij} x'_i x'_j = h(x')$ where h is a polynomial in $x'_r, \ldots, x'_n$ which has no constant, linear, or quadratic terms. The matrix given by a_{ij} may be taken to be symmetric, and a linear transformation can turn it into a diagonal matrix with only 1's and 0's on the diagonal.

As the coordinates $x'_1, \ldots, x'_{r-1}$ were eliminated from the equation, we can leave them out entirely. Hence, assume $r = 1$. The equation may then be written $x'^2_1 + \cdots + x'^2_p = h(x)$ with h as above. By an analytic transformation, the variables $x'_1, \ldots, x'_p$ may be eliminated from $h(x')$. In our case, p will be maximal, so the equation simply is $x'^2_1 + \cdots + x'^2_p = 0$.

In the case of $p = 4$, the equation may be rewritten $(x'_1 + ix'_2)(x'_1 - ix'_2) = (ix'_3 + x'_4)(ix'_3 - x'_4) = 0$ which gives the convenient form, after a change of coordinates, $x''_1 x''_2 = x''_3 x''_4$.

A.1.2 Standard Model of a Double-Point

The standard model of a double-point on a 3-fold is the variety V in $\mathbf{P}^4$ defined by $x_1x_2 = x_3x_4$. This has a double-point in $[x_0 : x_1 : x_2 : x_3 : x_4] = [1 : 0 : 0 : 0 : 0]$ (the point $(0,0,0,0) \in \mathbf{C}^4 \subset \mathbf{P}^4$ if we set $x_0 = 1$), hereafter called origo. There are two families of hyperplanes on V passing through origo.

If we look at the subvariety $x_1 = 0$, we find that this has two components, $x_3 = 0$ and $x_4 = 0$, both passing through origo. Let us choose $x_1 = x_3 = 0$ as our plane, H. In order to resolve the double points, we blow the variety up along H.

The blow-up, $\widetilde{V} \subset \mathbf{P}^4 \times \mathbf{P}^1$, is defined by the equations for V, $x_1x_2 = x_3x_4$, and the relation[1] $[y_0 : y_1] = [x_1 : x_3]$, where $[y_0 : y_1]$ are coordinates on $\mathbf{P}^1$, as $x_1 = x_3 = 0$ are the defining equations for H. This will however contain two components: one corresponding to V and the tube $H \times \mathbf{P}^1$. In order to get the proper $\widetilde{V}$, remove the $H \times \mathbf{P}^1$ and take the closure. This amounts to using the fact that $[x_1 : x_3] = [x_4 : x_2]$, adding the restriction $[x_4 : x_2] = [y_0 : y_1]$. This new equation only affects $H \times \mathbf{P}^1$ as it is automaticly satisfied elsewhere. Over the point $x_1 = x_2 = x_3 = x_4 = 0$ in V, the fibre of $\widetilde{V} \longrightarrow V$ is still $\mathbf{P}^1$ though. Elsewhere, it is only a single point.

If we choose coordinates $z_{ij} = x_iy_j$ for $\mathbf{P}^4 \times \mathbf{P}^1 \subset \mathbf{P}^9$, we get equations

$$\begin{aligned} z_{ij}z_{kl} &= z_{il}z_{kj} \quad \text{for all } i,j,k,l, \\ z_{1j}z_{2j} &= z_{3j}z_{4j} \quad \text{for all } j, \\ z_{i0}z_{3j} &= z_{i1}z_{1j} \quad \text{for all } i,j, \\ z_{i0}z_{2j} &= z_{i1}z_{4j} \quad \text{for all } i,j. \end{aligned} \tag{A.1}$$

The three sets of equations correspond to restricting $\mathbf{P}^9$ to $\mathbf{P}^4 \times \mathbf{P}^1 \subset \mathbf{P}^9$, to $V \times \mathbf{P}^1$, to $\widetilde{V} \cup H \times \mathbf{P}^1$, and finally to the compactification $\widetilde{V}$ of the subset corresponding to $V \setminus H$ in this space. This is easily seen to be smooth (it suffices to check at the fibre over origo).

A.1.3 General Blow-Up along a Subvariety

If $V \subset \mathbf{P}^n$ is defined by $p_1 = \cdots = p_r = 0$ and the subvariety W is defined by restricting to $q_0 = \cdots = q_s = 0$, we define the blow-up of V along W as follows.

Let U be the points in $(V \setminus W) \times \mathbf{P}^s \subset \mathbf{P}^{ns+n+s}$ such that $[y_0 : \ldots : y_s] = [q_0(x) : \ldots : q_s(x)]$; if the q_i have different degrees, you will have to use a weighted projective space instead of just $\mathbf{P}^r$. Then, take $\widetilde{V}$ to be the closure of U in $\mathbf{P}^n \times \mathbf{P}^s$.

We notice that the blowing up along a subvariety is a local process: it can be performed on open subsets of the variety, and glued together. More precisely, if $q_0 = \cdots = q_s$ and

[1] This equation is more properly written $y_0x_3 = y_1x_1$. Over H, $x_1 = x_3 = 0$, so this gives no restriction on H. Hence, further restrictions must be put on the $H \times \mathbf{P}^1$ component.

$q'_0 = \cdots = q'_{s'} = 0$ both define the same subvariety on some open set, $q' = Aq$ and $q = A'q'$ for some matrices A and A'. This gives transformations $y' = Ay$ and $y = A'y'$. By this gluing together, the variety will not be contained in $\mathbf{P}^n \times \mathbf{P}^s$ as in the previous case.

A.1.4 Resolution of Double-Points

Let $V \subset \mathbf{P}^n$ be a variety with a double-point in P, and let W be a smooth codimension 1 subvariety passing through P. Take $\widetilde{V}$ to be the blow-up of V along W. In local coordinates around P, V will look like the standard double-point (analytically). Hence, the blow-up along W will resolve the double point replacing it with a $\mathbf{P}^1$.

At smooth points of V, as W is smooth, W is locally a complete intersection, ie., defined by only one equation. The blow-up will thus have no effect as the blow-up will take place in $V \times \mathbf{P}^0$.

A.2 Cohomology Change by Creation and Resolution of a Double Point

We wish to know what happens to the cohomology when the double points are created and later resolved.

A.2.1 A Family of Varietes with a Double Point in a Special Parameter-Point

The standard double point on a 3-fold may be represented by the double point of $x_1x_2 = x_3x_4$ in $\mathbf{P}^4$. If we have a parameter family of varieties such that this double point appeares only on a subfamily, this may be represented by $x_1x_2 - x_3x_4 = \lambda x_0^2$ where λ is the parameter. For $\lambda = 0$ we have a double point, whereas for $\lambda \neq 0$, the variety is smooth.

In the same way all double points may be identified with this special case, families of varieties where a special point in the parameter space gives a variety with a double point may locally be identified with the above family.

As has already been demonstrated, a double point is resolved by blowing it up to a $\mathbf{P}^1$: topologically, a point is removed and a $\mathbf{P}^1$ inserted at its place. So, what happens when the double point is created?

If we write $x_j = a_j + ib_j$ with $a_j, b_j \in \mathbf{R}$, the equation $x \cdot x = \sum_{j=1}^4 x_j^2 = \lambda$, using this form for the quadric this time, may be written $a \cdot a - b \cdot b = \lambda$, $a \cdot b = 0$ assuming $\lambda \in \mathbf{R}$. If we let $\lambda \to 0$, $\lambda > 0$, the variety changes continuously. In fact, for $\lambda > 0$, we can map $V_{\lambda_0} - V_\lambda$

by $(a,b) \mapsto (\frac{b^2+\lambda}{a^2}a, b)$. We see that when $\lambda \to 0$, the subset with $b = 0$ (ie. $x \in \mathbf{R}^4$) maps to the origin. Thus, the set $\{x \in \mathbf{R}^4 | x^2 = \lambda > 0\} \cong S^3$ collapses to a point.

A.2.2 The Cohomology of This Family

If we look at a general variety in the family, $\lambda \neq 0$, we get a smooth hypersurface in $\mathbf{P}^4$. The cohomology of this is $h^{p,q} = 1$ for $0 \leq p = q \leq 3$, zero otherwise.

The special case of $\lambda = 0$ is resolved blowing up along a hyperplane: eg., $x_1 = x_3 = 0$. Using $[y_0 : y_1] = [x_1 : x_3] = [x_4 : x_2]$ we get a projection $\tilde{V} \to \mathbf{P}^1$ with fibre $\mathbf{P}^2$. For $\omega \in H^{1,1}(\mathbf{P}^2)$, the cohomology of $\mathbf{P}^2$ is generated by ω_2^i for $i = 0,1,2$ as inherited from $\mathbf{P}^4$. This makes $R^*\pi_*\mathbf{C}$ free on $\mathbf{P}^1$, and, hence, $H^*(\tilde{V},\mathbf{C}) = H^*(\mathbf{P}^1,\mathbf{C}) \otimes H^*(\mathbf{P}^2,\mathbf{C})$. This makes (for $\tilde{V}$) $h^{0,0} = h^{3,3} = 1$, $h^{1,1} = h^{2,2} = 2$, and $h^{p,q} = 0$ otherwise.

As we see, the effect of creating and resolving the double point was to increase $h^{1,1}$ and $h^{2,2}$ by one.

One specific comment can be made about the new elements of $H^{1,1}$ when the point $[1 : 0 : 0 : 0 : 0]$ is resolved in this manner. The new $(1,1)$-form inherited from the $\mathbf{P}^1$ may be described as

$$\omega_0 = \partial\bar{\partial}\log(y_0\bar{y}_0 + y_1\bar{y}_1). \tag{A.2}$$

This is equal to $\partial\bar{\partial}\log(x_1\bar{x}_1 + x_3\bar{x}_3)$ and $\partial\bar{\partial}\log(x_2\bar{x}_2 + x_4\bar{x}_4)$ whereever these are defined.

A.2.3 The Change of Cohomology due to the Double Points

If a variety V has r double points, each of these may be resolved locally. Each such replacing of a point with a $\mathbf{P}^1$ increases the Euler-number by one: either by increasing h^2 by one or by reducing h^3 by one. If the new $\mathbf{P}^1$ gives a new 2-cycle in H_2, h^2 is increased by one; if the new $\mathbf{P}^1$ does not give a new homology, h^3 must drop by one.

We are also interested in what happens to the cohomology as the double points are created: ie. with V_λ as $\lambda \to 0$, $V = V_0$. Again, we may use the above model for double points and identify this locally with the creation of a double point in V_λ as $\lambda \to 0$. In the model, h^4 was increased by one as an S^3 collapsed to a point. This implies that either h^4 is increased by one or h^3 is reduced by one.

To prove the above claims, let V_λ be a smooth variety for λ in a punctured neighborhood of zero with r double points for V_0. Let $p \in V_0$ be such a double point. As $\lambda \to 0$, an S^3 is collapsed to a point, which in turn is blown up to a $\mathbf{P}^1$. Hence, topologically, we remove an S^3 from a compact manifold X and glue in a $\mathbf{P}^1$ to produce $\tilde{X}$. We may then use that ([6], theorem 27.3) states that

$$\cdots \to H^{p-1}(A) \to H_c^p(X \setminus A) \to H^p(X) \to H^p(A) \to H_c^{p+1}(X \setminus A) \to \cdots \tag{A.3}$$

for a suitably nice closed set (eg. compact real manifold) A.

We start with the compact manifold X and remove an S^3 leaving $U = x\backslash S^3$. As $H^0_c(U) = 0$, $H^0(X) \cong H^0(S^3)$, we now get

$$\begin{gathered} 0 \to H^1_c(U) \to H^1(X) \to H^1(S^3) = 0, \\ 0 = H^1(S^3) \to H^2_c(U) \to H^2(X) \to H^2(S^3) = 0, \\ 0 = H^2(S^3) \to H^3_c(U) \to H^3(X) \to H^3(S^3) = \mathbf{C} \to H^4_c(U) \to H^4(X) \to H^4(S^3) = 0, \\ 0 = H^{p-1}(S^3) \to H^p_c(U) \to H^p(X) \to H^p(S^3) = 0 \quad \text{for } p \geq 5. \end{gathered} \tag{A.4}$$

Gluing in $\mathbf{P}^1$ to make $\tilde{X}$ then gives similarly

$$\begin{gathered} 0 \to H^1_c(U) \to H^1(\tilde{X}) \to H^1(\mathbf{P}^1) = 0, \\ 0 = H^1(\mathbf{P}^1) \to H^2_c(U) \to H^2(\tilde{X}) \to H^2(\mathbf{P}^2) = \mathbf{C} \to H^3_c(U) \to H^3(\tilde{X}) \to H^3(\mathbf{P}^2) = 0, \\ 0 = H^{p-1}(\mathbf{P}^1) \to H^p_c(U) \to H^p(\tilde{X}) \to H^p(\mathbf{P}^2) = 0 \quad \text{for } p \geq 4. \end{gathered} \tag{A.5}$$

The manifolds X and $\tilde{X}$ both being compact complex manifolds, $h^{1,2} = h^{2,1}$ for both. Hence, as h^3 is even, so either h^2 and h^4 are increased by one each or h^3 is decreased by two.

If $W = \{l_0 = \cdots = l_k = 0\}$ is a smooth codimension 1 subvariety in a variety $V \subset \mathbf{P}^n$, blowing up along this variety gives a variety $\tilde{V} \subset \mathbf{P}^n \times \mathbf{P}^k$ ($\mathbf{P}^k$ weighted by degrees of l_j). Assume that this blows up double points $p_1, \ldots, p_r$. This gives the $(1,1)$-form inherited from $\mathbf{P}^k$:

$$\omega = \partial\bar{\partial} \log \Big(\sum_{j=0}^{k} (y_j \bar{y}_j)^{1/\deg l_j} \Big) = \partial\bar{\partial} \log \Big(\sum_{j=0}^{k} (l_j \bar{l}_j)^{1/\deg l_j} \Big). \tag{A.6}$$

As this blows at least one double point to a $\mathbf{P}^1$ and the integral of ω over such a $\mathbf{P}^1$ is non-zero, it extends the H^2-cohomology by one dimension.

If the blowing up of V along W resolved s double points, r 2-cycles appared. In passing from V_λ to V, r 3-cycles, each one an S^3 located where the double-points appared, vanish. The surface W induces a relation between these 3-cycles, however, there may be more relations than just this one. In order to determine the Hodge numbers, one may determine the number, d, of linearly independent relations on the vanishing 3-cycles; then, $h^{1,1}$ and $h^{2,2}$ increase by d whereas $h^{1,2}$ and $h2,1$ decrease by $s - d$ ([4]).

A.3 Finding Generators of Subring

Assuming we have a list of variables, $x_1, \ldots, x_n$, and Laurent-monomials $v_j = \alpha_j \prod_i x_i^{a_{j,i}}$, with $a_{j,i} \in \mathbf{Z}$, we wish to find $r_k = \prod_j v_j^{b_{k,j}}$ so that $r_k(x)$ is independent of the x_i and generates the ring of polynomials in v_j which become constant in the x_i: ie., that

$$0 \longrightarrow \mathbf{C}[r] \longrightarrow \mathbf{C}[v] \longrightarrow \mathbf{C}[x, 1/x] \tag{A.7}$$

where $\mathbf{C}[x, 1/x] = \mathbf{C}[x_1, 1/x_1, \ldots, x_n, 1/x_n]$ etc., is exact (there is no need for the rightmost map to be surjective). This may not be possible, so alternatively we may weaken our demand to r_k Laurent-monomials in v_j: eg., that

$$0 \longrightarrow \mathbf{C}[r, 1/r] \longrightarrow \mathbf{C}[v, 1/v] \longrightarrow \mathbf{C}[x, 1/x] \tag{A.8}$$

is exact. Nor this need be the case: that the kernel of $\mathbf{C}[v, 1/v] \longrightarrow \mathbf{C}[x, 1/x]$ is a freely generated ring. One may, indeed, in general find the need for $r_k = \prod_j v_j^{b_{k,j}}$ where $b_{k,j}$ are rational numbers.

Notice, that if we make a matrix $A = [a_{j,i}]$ of the powers of x_i in v_j, and similarly let $B = [b_{k,j}]$ be the powers of v_j in r_k, then the r_k are independent of x_i if $B \cdot A = 0$. In other words, for the above sequences to be exact, the rows of B must be linearly independent and give the kernel of $- \cdot A$ (over $\mathbf{N}_0$, $\mathbf{Z}$ or $\mathbf{Q}$ respectively).

Appendix B

Computer Calculations

Computer algebra programs, Maple[1] and Macaulay 2[2] (and, at first, the older version Macaulay), have been used extensively throughout my studies. Partly to test hypotheses and get some overview, but in some cases also to prove results. The following will present the calculations. I do not give all the programs as that would be of little interest from a mathematical point of view, but have taken care to ensure that sufficient information is given to enable a reasonably computer literate mathematician to confirm my findings.

B.1 The Polynomials

First, the matrix N_Y, $Y = [y_1, y_2, y_3]$, is given by

$$N_Y = \begin{bmatrix} 0 & x_1y_1 & x_2y_2 & x_3y_3 & -x_4y_3 & -x_5y_2 & -x_6y_1 \\ -x_1y_1 & 0 & x_3y_1 & x_4y_2 & x_5y_3 & -x_6y_3 & -x_0y_2 \\ -x_2y_2 & -x_3y_1 & 0 & x_5y_1 & x_6y_2 & x_0y_3 & -x_1y_3 \\ -x_3y_3 & -x_4y_2 & -x_5y_1 & 0 & x_0y_1 & x_1y_2 & x_2y_3 \\ x_4y_3 & -x_5y_3 & -x_6y_2 & -x_0y_1 & 0 & x_2y_1 & x_3y_2 \\ x_5y_2 & x_6y_3 & -x_0y_3 & -x_1y_2 & -x_2y_1 & 0 & x_4y_1 \\ x_6y_1 & x_0y_2 & x_1y_3 & -x_2y_3 & -x_3y_2 & -x_4y_1 & 0 \end{bmatrix}, \tag{B.1}$$

[1]In addition to using Maple as a general tool for doing formal arithmetic and algebra, I have used the grobner package for calculating Gröbner bases. This has the advantage of working over characteristic 0 and with free parameters (ie., general point in parameter space).

[2]Macaulay is very efficient for calculating Gröbner bases, unlike Maple's package for Gröbner bases. There are, however, some disadvantages: it works over characteristic $p > 0$, allows no free parameters, and is generally less flexible and programable than Maple. By default, I have used $p = 31991$ which has the advantage that as $p \equiv 1 \bmod 7$, $w^7 - 1$ has seven roots over $\mathbf{Z}_p$.

though the case of interest will be restricted to $y_3 = 0$, denoting $y_2/y_1 = y$, giving the matrix

$$N_y = \begin{bmatrix} 0 & x_1 & x_2y & 0 & 0 & -x_5y & -x_6 \\ -x_1 & 0 & x_3 & x_4y & 0 & 0 & -x_0y \\ -x_2y & -x_3 & 0 & x_5 & x_6y & 0 & 0 \\ 0 & -x_4y & -x_5 & 0 & x_0 & x_1y & 0 \\ 0 & 0 & -x_6y & -x_0 & 0 & x_2 & x_3y \\ x_5y & 0 & 0 & -x_1y & -x_2 & 0 & x_4 \\ x_6 & x_0y & 0 & 0 & -x_3y & -x_4 & 0 \end{bmatrix}. \tag{B.2}$$

The seven pfaffians, one for each principal 6×6-minor, are

$$\begin{aligned} p_0 &= -y^3x_0x_1x_6 + y^2x_1{x_3}^2 + y^2{x_4}^2x_6 + yx_0x_2x_5 - x_0x_3x_4, \\ p_1 &= -y^3x_1x_2x_3 + y^2x_1{x_6}^2 + y^2x_3{x_5}^2 + yx_0x_2x_4 - x_2x_5x_6, \\ p_2 &= -y^3x_3x_4x_5 + y^2{x_0}^2x_5 + y^2{x_1}^2x_3 + yx_2x_4x_6 - x_0x_1x_4, \\ p_3 &= -y^3x_0x_5x_6 + y^2x_0{x_2}^2 + y^2{x_3}^2x_5 + yx_1x_4x_6 - x_2x_3x_6, \\ p_4 &= -y^3x_0x_1x_2 + y^2x_0{x_5}^2 + y^2x_2{x_4}^2 + yx_1x_3x_6 - x_1x_4x_5, \\ p_5 &= -y^3x_2x_3x_4 + y^2{x_0}^2x_2 + y^2x_4{x_6}^2 + yx_1x_3x_5 - x_0x_3x_6, \\ p_6 &= -y^3x_4x_5x_6 + y^2{x_1}^2x_6 + y^2{x_2}^2x_4 + yx_0x_3x_5 - x_1x_2x_5, \end{aligned} \tag{B.3}$$

where p_i refers to the pfaffian of N_y with the i'th row and column[3] removed.

B.2 Double-Points on Varieties

B.2.1 General 49 Double-Points

The general variety X_Y has 49 double-points: at $[0 : y_1 : y_2 : y_3 : -y_3 : -y_2 : -y_1]$ and the images of this point under the group action G. For simplicity, I restrict myself to X_y which is sufficient in my case.

Let us look at the defining polynomials at this point. The upper leftmost 4×4-minor has pfaffian equal to y at this point, hence, the polynomials p_4, p_5, p_6 defines the variety close to this point: ie., the three polynomials give a complete intersection at this singularity.

If we look at local coordinates $[dx_0 : dx_1 + 1 : dx_2 + y : dx_3 : dx_4 : dx_5 - y : -1]$ we may look at the lowest (ie. first and second) degree terms. This gives first degree terms of the three polynomials:

$$\begin{aligned} p_4 &= -y\,dx_3 + y\,dx_4 + O(dx^2), \\ p_5 &= -y^2\,dx_3 + y^2\,dx_4 + O(dx^2), \\ p_6 &= -y^2\,dx_1 + y\,dx_2 - y\,dx_5 + O(dx^2). \end{aligned} \tag{B.4}$$

[3]Note, that the enumeration is over Z_7: ie., $i = 0, 1, \ldots, 6$.

As can easily be seen, this is degenerate as the second is y times the first. Hence, we must look at the second degree term of p_5. Substituting $dx_3 = dx_4$ and $dx_2 = dx_5 + y\,dx_1$ gives

$$\begin{aligned} &-y^5 dx_1 dx_0 - y^4 dx_0 \left(dx_5 + y\,dx_1\right) - 2\,y^4 dx_0 dx_5 \\ &\qquad + 2\,y^4 dx_4{}^2 + y^2 dx_1 dx_4 - 2\,y\,dx_4 dx_5 - y^3 dx_0{}^2 - dx_0 dx_4 \\ &= \; dx_0 \left(-y^3 dx_0 - 3\,y^4 dx_5 - 2\,y^5 dx_1 - dx_4\right) + y\,dx_4 \left(-2\,dx_5 + 2\,dx_4 y^3 + y\,dx_1\right). \end{aligned} \tag{B.5}$$

Hence, we have proved that the singularity is a double-point, and we have the defining equations up to the necessary degree available for later use.

B.2.2 Double-Points at Fixed-Points

As for the point $[1:0:0:0:0:0:0]$ (and the other fixed points by maps from G and from F), we have the same kind of calculation. As the only principal 4×4-minor which is non-degenerate is that containing rows 1, 3, 4, and 6 (counting in $\mathbf{Z}_7$ starting with 0), its pfaffian being $-y$, the definig polynomials are p_0, p_2, p_4.

The first order terms of the three in coordinates $[1 : dx_1 : dx_2 : dx_3 : dx_4 : dx_5 : dx_6]$ are

$$\begin{aligned} p_0 &= 0 + O(dx^2), \\ p_2 &= y^2 dx_5 + O(dx^2), \\ p_5 &= y^2 dx_2 + O(dx^2), \end{aligned} \tag{B.6}$$

which forces us to look at the second degree term of p_0. This is, using that $dx_2 = dx_5 = 0$,

$$-y^3 dx_1 dx_6 - dx_4 dx_3. \tag{B.7}$$

Hence, we have proved that this singularity, too, is a double-point.

B.3 Desingularization

In order to desingularize the double-points, I will need a codimension 1 variety passing through the double-point, which is not a complete intersection in X. As before, I will do the calculations on X_y rather than on X_Y as this is all I need.

B.3.1 General 49 Double-Points

Again, let us look at the point $[0 : 1 : y : 0 : 0 : -y : -1]$. If we look at the subvariety defined by $x_0 = 0$, we find that this is reducible: looking at the polynomials of the Gröbner basis, we find that some of them are reducible, and that some of their irreducible factors correspond to codimension 1 components of the variety. Each time one of the polynomials in the basis is reducible, replace it in turn with each of the factors, and check if the result has the correct dimension. The dimension and degree may be found from the Poincaré series (or the Hilbert polynomial).

Intersecting X_y with $x_0 = 0$ and decomposing it into irreducible components, we find that on component is given by

$$x_0 = x_3 = x_6 = y^2 x_4 x_2 - x_1 x_5 = 0, \tag{B.8}$$

which is a smooth codimension 1 surface of degree 2. Locally, in a neighborhood of the singular point, $x_0 = x_3 = 0$ suffices; globally, this is reducible, but the other components lie away from the point.

B.3.2 Double-Points at Fixed-Points

We now look at the fixed-point $[1 : 0 : 0 : 0 : 0 : 0 : 0]$. We now look at the subvariety defined by $x_1 = 0$ and again find that this may be decomposed as in the previous case. Indeed, we find that there is a component given by

$$x_1 = x_2 = x_5 = y^2 x_4 x_6 - x_0 x_3 = 0, \tag{B.9}$$

which is the same as one of the components found for one of the general 49 double-points.

Locally, in a heighborhood of the fixed-point, $x_1 = y^2 x_4 x_6 - x_0 x_3 = 0$ suffices, with the other components lying away from the point.

B.4 General Number of Singularities

Finding all singularities on a given variety generally reduces to finding the subvariety defined by the determinants of the minors of size equal to the codimension in the jacobian of the defining polynomials. Ie., to find the points where the rank of the jacobian is less than the codimension of the variety. Given a specific variety, this can be solved, eg., using Gröbner bases. Given a family of varieties, however, finding all singularities, or all singular varieties where the general variety in the family is smooth, is a harder problem. (It can be reduced to finding a Gröbner basis, but with more equations and more variables than in the above, and in reality it may not be possible to compute.)

In this case, it is not neccessary to find all singular varieties or all singularities. It suffices to find the singularites on a general variety. We have already proved that on X_Y there are 49 double-points and on X_y there are another 7. If we can find a single case of each, we have proved that the general case has no more than these singularities. The reason for this is that the varieties with more singularities will give a variety in the parameter space. If we can find a single point in each parameter space (provided the parameter space is nice, eg., some projective space as in our case) such that the corresponding varieties do not have extra singularities, then the subfamily of varietes with extra singularities must have positive codimension.

I have used Macaulay to find Gröbner bases and calculate the number of singularities. The following program does this:

```
x=?
y=?
z=?
KK=ZZ/31991
R=KK[a,b,c,d,e,f,g]
M=matrix {{0,-b*x,-c*y,-d*z,e*z,f*y,g*x},
 {b*x,0,-d*x,-e*y,-f*z,g*z,a*y},
 {c*y,d*x,0,-f*x,-g*y,-a*z,b*z},
 {d*z,e*y,f*x,0,-a*x,-b*y,-c*z},
 {-e*z,f*z,g*y,a*x,0,-c*x,-d*y},
 {-f*y,-g*z,a*z,b*y,c*x,0,-e*x},
 {-g*x,-a*y,-b*z,c*z,d*y,e*x,0}}
P=pfaffians(6,M)
PS=P+minors(codim(P),jacobian(P))
print("dim X",dim P)
print("dim sing X",dim PS)
print("deg sing X",degree PS)
```

In the first three lines, the ?-signs are replaced by the appropriate numbers. The dimension (reduce by one to get projective dimension) of the variety and of the singular locus, and the degree of the singular locus.

B.4.1 The X_Y Family

If we set $y_1 = y_2 = y_3 = 1$, the above program produces:

```
(dim X, 4)
(dim sing X, 1)
(deg sing X, 49)
```

Hence, the 49 double-points are the only general singularities in this family.

B.4.2 The X_y Family

If we set $y = 1$ (or $y_1 = y_2 = 1, y_3 = 0$), we get:

```
(dim X, 4)
(dim sing X, 1)
(deg sing X, 56)
```

These are exactly the $49 + 7$ singularities we knew. Hence, these are the only general singularities in this family.

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There was a point to this story, but it has temporarily escaped the chronicler's mind.

—From *So long and thanks for all the fish* by Douglas Adams