Dispersion

- So far, we have studied particles with reference momentum $p = p_0$.
- A dipole field disperses particles according to their energy.
- This introduces an **x-E correlation in the beam**.

- Off-momentum: $p = p_0 + \Delta p = p_0 (1 + \Delta p/p_0)$. Hill’s equation:
  
  $$x'' + K(s)x = \frac{1}{\rho} \frac{\Delta p}{p_0}$$

- Solution gives an extra **dispersion term** to the homogenous solution $x_{\Delta p=0}(s)$
  
  $$x(s) = x_{\Delta p=0}(s) + D_x(s) \frac{\Delta p}{p_0}$$

  $D(s)$: “dispersion function”
Dispersion suppression

Rms beam size increases with dispersion, calculated as:

\[ \sigma_{\text{rms}}(s) = \sqrt{\varepsilon_{\text{rms}} \beta(s) + (D(s) \sigma_p/p_0)^2} \]

D = 0

D > 0

The dispersion can be **locally suppressed** by lattice design. An energy spread does then not contribute to the beam-size.

Important for example when you want a **very small beam size**, for example at a collider interaction region.
Dispersion in linear lattices

- Lattice elements may add or subtract to the dispersion in the beam (x-E correlation). Assuming we know the dispersion at one location, we can calculate how the dispersion propagates through the lattice:

\[
\frac{\Delta p}{p_0} = 0 \quad \frac{\Delta p}{p_0} < 0 \\
D(s) = 0
\]

- Dispersion is affected by quadrupoles as well, and propagates along the lattice in the same manner as a particle. We may use the matrix framework for particle tracking to calculate dispersion. To add dispersion generated from dipoles and other elements to the framework, we add a third row:

\[
\begin{bmatrix}
D \\
D' \\
1 
\end{bmatrix}_1 = 
\begin{bmatrix}
m_{11} & m_{12} & \Delta D \\
m_{21} & m_{22} & \Delta D' \\
0 & 0 & 1 
\end{bmatrix}_0 
\begin{bmatrix}
D \\
D' \\
1 
\end{bmatrix}_0
\]

where \(m_{ij}\) are the elements of the regular transport matrix and \(\Delta D, \Delta D'\) is the generated dispersion.
Dispersion in rings

- In rings, the design dispersion is uniquely defined by the lattice.
- Without quadrupole focusing, assuming constant bending field, the circumference would increase by $\frac{\Delta p}{p_0}$ yielding a constant dispersion, $D(s) = \frac{\Delta x(s)}{(\Delta p/p_0)} = \frac{\Delta \rho}{(\Delta \rho/\rho)} = \rho$.

- Quadrupole focusing modifies the dispersion function. We define the momentum compaction factor (cf. longitudinal dynamics) as:

$$\alpha = \frac{\Delta R / R}{\Delta p / p} = \langle D_x \rangle / \rho$$

- The momentum compaction factor can be calculated by computer codes. The value is usually $> 0$ and $<< 1$. 
Chromaticity

- Particles with \( \Delta p \neq 0 \) focuses differently in quadrupoles
  - Optics analogy, “chromatic aberration”

- Focal length: \( f = \frac{1}{kl}, \ k \alpha \frac{1}{p}, \ f \alpha p=p_0(1+ \Delta p/p_0). \)

- Detrimental effects on beam:
  - Focusing (beta function) depends on energy: “projected” emittance growth in lines
  - The accelerator tunes, \( Q \), depends on energy; energy-spread -> tune-spread. Unstable resonance values might be hit.
Chromaticity in lines

- The focusing properties of a lattice depend on the beam energy.
- Chromaticity in lines can be quantified by $\sim \frac{d\beta}{\beta_0}/ \frac{\Delta p}{p_0}$ – the “W-function”.
- The projected emittance, calculated as the rms emittance for particles of all energies, increases:

If particles with the nominal energy are at a waist (focused), off-momentum particles are not at waist and the projected emittance increases.

\[ \Delta p/p_0 = 0 \]
\[ \Delta p/p_0 = -5\% \]
\[ \Delta p/p_0 = +5\% \]
Chromaticy versus dispersion

Dispersion effect: linear

Chromatic effect: non-linear

From D. Gamba
Chromaticity in rings

- Chromaticity in rings, $\xi$, is defined by the tune shift per momentum change:

$$\Delta Q = \xi \frac{\Delta p}{p_0}$$

- The accelerator tune, $Q$, depends on energy; energy-spread -> tune-spread -> low order resonance values might be crossed -> beam loss.
- $\Delta p > 0$ in a FODO lattice leads to weaker focusing and thus $\Delta Q < 0$. Therefore, a linear lattice naturally generates a negative chromaticity, $\xi < 0$.
- The negative natural chromaticity may be adjusted using sextupoles magnets.
Magnet multipole expansion

- We discussed earlier the normalized magnet strengths:
  
  **Dipole**
  \[
  \frac{1}{\rho} = \frac{eB}{p} \Leftrightarrow \frac{1}{\rho} [m^{-1}] = 0.3 \frac{B[T]}{p[GeV/c]}
  \]

  **Quadrupole**
  \[
  k = \frac{eg}{p} \Leftrightarrow k [m^{-2}] \approx 0.3 \frac{g[T/m]}{p[GeV/c]}
  \]

- We can generalize this concept to magnetic multipole components \( k_n \) for a \( 2(n+1) \)-pole:

  \[
  k_n = \frac{e}{p} \frac{\partial^n B_y}{\partial x^n}, \text{ with unit } [m^{-(n+1)}]
  \]

- Furthermore, the **kicks on a particle** from a magnetic \( 2(n+1) \)-pole can be expressed as a combination of multipole components:

  \[
  \Delta x' + i\Delta y' = \frac{k_n l}{n!} (x + iy)^n
  \]

  - \( n=1 \): the quadrupole linear terms
  - \( n=2 \): sextupole terms:

    \[
    \Delta x' = \frac{1}{2} k_2 l(x^2 - y^2)
    \]
    \[
    \Delta y' = \frac{1}{2} k_2 lxy
    \]

  - Can be derived by Laplace eq. for the B-field.
  - See **Wille Ch. 3.**
Sextupoles
Correction with sextupoles

Sextupoles as chromaticity correctors

- Sextupoles can be used to correct chromaticity.
- Sextupole fields\(^*\):  
  \[ B_x = \frac{\partial^2 B_y}{\partial x^2} xy \quad B_y = \frac{\partial^2 B_y}{\partial x^2} \frac{1}{2}(x^2 - y^2) \]
- Add dispersion in \(x\):  
  \[ x \rightarrow x + \delta D_x \]
- Fields transform to:

\[ \delta = \frac{\Delta p}{p_0} \]

Sextupole can be set in order to cancel chromaticity induced by the quadrupoles.

*assuming perfect thin sextupoles
Geometric terms of sextupoles may be cancelled by imposing $M = -1$ between two equal strength sextupoles.

Result: lattice with the first order chromaticity corrected, $(d\beta / \beta_0)/(\Delta p/p_0) = 0$, and geometric terms cancelled for the nominal energy. Application example: final focusing for a linear collider.
Can you also make apochromatic particle beam focusing analogous to how achromatic camera lenses are made?

To ensure energy-independence of the beam focusing (Twiss parameters), the answer is yes.

To ensure energy-independent phase-advance of the individual particles (cf. tune), the answer is no.

This means that sextupoles are required in rings to mitigate energy dependence of the tune.

Non-linear terms

- Sextupoles introduce higher order non-linear terms, $f(x^2, y^2, \ldots)$. If not cancelled by $-I$ transforms, these terms add **non-linear terms** to the particle dynamics in the accelerator.

- Real magnets contain small amount of higher order multipole fields, which also adds to the non-linear terms.

- For circular accelerators, the orbit stability now becomes a non-linear problem.

- The part of the transverse phase-space which is stable can be studied by particle tracking for many, many turns. The resulting stable phase-space is called the **dynamics aperture**.

- The non-linear dynamics can also be studied analytically, using Hamiltonian dynamics.

*Example of LHC dynamic aperture simulation study*

The red area represents initial conditions that are stable up to 100,000 turns around the LHC. The blue circles represent unstable initial conditions: their radius is proportional to the stability time.
We have studied the transverse optics of a circular accelerator and we have had a look at the optics elements,

- the dipole for bending
- the quadrupole for focusing
- the sextupole for chromaticity correction
- In LHC: also octupoles for controlling non-linear dynamics

Example: LHC lattice

The periodic structure in the LHC arc section
Synchrotron radiation

- Charged particles undergoing acceleration emit electromagnetic radiation
  
- Main limitation for circular electron machines
  - RF power consumption becomes too high

- The main limitation factor for LEP...
  - ...the main reason for building LHC

- However, synchrotron radiations is also useful
Characteristic of SR: power

Lorentz invariant formula for power radiated by accelerated charged particles:

\[ P_S = \frac{e^2 c}{6\pi \varepsilon_0} \frac{1}{(m_0 c^2)^2} \left( \frac{dp}{d\tau} \right)^2 \]

(This is Larmor’s non-relativistic formula with the substitutions \( dt \to d\tau \) and \( p \to p^\mu \)). Two cases:

1) Linear acceleration \( \frac{dv}{dt} \parallel v \):

\[ P_S = \frac{e^2 c}{6\pi \varepsilon_0} \frac{1}{(m_0 c^2)^2} \left( \frac{dp}{dt} \right)^2 \]

\( \frac{dp}{dt} = \frac{dE}{dx} \) is in the order 10-100 MV/m in today’s accelerators. \( P_S \) compared to power provided by the accelerator to increase the energy: \( \eta = \frac{P_S}{dE/dt} \sim 10^{-14} \Rightarrow \) linear acceleration gives negligible radiation.

2) Circular acceleration \( \frac{dv}{dt} \perp v \):

\[ P_S = \frac{e^2 c}{6\pi \varepsilon_0} \frac{\gamma^2}{(m_0 c^2)^2} \left( \frac{dp}{dt} \right)^2 = \frac{e^2 c}{6\pi \varepsilon_0} \frac{1}{(m_0 c^2)^4} \frac{E^4}{R^2} \]

Radiated power increase with \( E^4 \) (!).
Characteristics of SR: distribution

- Electron rest-frame: radiation distributed as a "Hertz-dipole"

\[
\frac{dP_s}{d\Omega} \propto \sin^2 \psi
\]

- Relativist electron: Hertz-dipole distribution in the electron rest-frame, but transformed into the laboratory frame the radiation form a very sharply peaked light-cone

We assume a photon emitted in the rest frame y-direction, while the particle is moving in the z-direction (acceleration in the x-direction), \( p^\mu = [E/c, 0, \gamma p_y, \gamma p_z] \)

\[
p'^\mu = \begin{bmatrix} \gamma & 0 & 0 & \beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta \gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} E/c \\ 0 \\ p_y \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma E/c \\ 0 \\ p_y \\ \gamma \beta E/c \end{bmatrix}
\]

\[ \Rightarrow \tan \theta = \frac{p_y'}{p_z'} = \frac{p_y}{\gamma \beta E/c} \approx \frac{E/c}{\gamma E/c} = \frac{1}{\gamma} \Rightarrow \text{The light-cone has an extremely small angle!} \]
Characteristics of SR: spectrum

Synchrotron radiation has a broad spectrum, due to short pulses as seen by an observer in the lab frame.
50% of radiation power is contained within a well defined "critical frequency" :

$$\omega_c = \frac{3c\gamma^3}{2R}$$

Advantages of synchrotron radiation as a light source :
1. High intensity
2. Spectrum that cannot be covered easy with other sources
3. Critical frequency is tunable

Radiation spectrum for different electron beam energies

See Wille (2000) for a derivation of the full spectrum.
Synchrotron radiation centers

Synchrotrons, or storage rings with boosters, which circulate low-emittance electron beams of with beam energy of ~1 GeV to ~8 GeV.

Some applications of photon science with Synchrotron Radiation:
- material/molecule analysis (UV, X-ray)
- crystallography; photo voltaic
- Life sciences; protein compositon

Example: the synchrotron SOLEIL, France

About ~40 synchrotron light sources exists throughout the world, and more are under constructions.
Radiation damping in storage rings

Synchrotron radiation: particle loses momentum on very close to direction of motion (within angle $1/\gamma$):

Rf cavities replenishes momentum in longitudinal direction:

The net effect is damping of the transverse phase-space of an electron beam in a storage ring – radiation damping. The emittance can be reduced by a several orders of magnitude in ~10 ms. A topic for the Linear Collider lectures.
Case study: the LHC
CERN accelerator complex

- LHC is responsible for accelerating protons from 450 GeV up to 7000 GeV
- 450 GeV protons injected into LHC from the SPS
- PS injects into the SPS
- LINACs injects into the PS
- The protons are generated by a proton source where a H₂ gas is heated up to provide protons
- The limitations in the earlier part of the acceleration chain originates from space charge -> collective effects lecture
- Circumference = 26658.9 m

- Four interactions points, where the beams collide, and massive particle physics experiments record the results of the collisions (ATLAS, CMS, ALICE, LHCb)

- Eight straight sections, containing the IPs, around 530 m long

- Eight arcs with a regular lattice structure, containing 23 arc cells

- Each arc cell has a periodic FODO-lattice, 106.9 m long
LHC bending magnets

8.3 T maximum field (allows for 7 TeV per proton beam). Generated by a current of 12 kA in the superconducting Rutherford coils.

Developments for higher energy hadron colliders (HE-LHC, FCC) : Nb_3Sn, HTS
LHC cavities

- Superconducting RF cavities. Standing wave, $f = 400$ MHz
- Each beam: one cryostat at 4.5 K, 4+4 cavities in each cryostat
- 5 MV/m accelerating gradient, 16 MeV energy gain per turn
LHC: the collision point

\[ \sigma(s) = \sigma^* \sqrt{1 + (s/\beta^*)^2} \]

\[ \sigma_{arc} = \sqrt{\varepsilon \beta_{typ}} \approx 0.3 \text{mm} \]

\[ \sigma_{IP} = \sqrt{\varepsilon \beta^*} \approx 17 \mu\text{m} \]

Collision region:
- * very strong quadrupoles,
- * close to the interaction point

\( \beta_{typ} \approx 180m, \beta^* = 0.55m, \varepsilon \approx 0.5nm \times rad \)

Upgraded magnets, more beta squeeze, more luminosity.
LHC

- **proton-proton collisions**
  \[ \Rightarrow \text{two vacuum chambers, with opposite bending field} \]

- Proton chosen as particle type due to **low synchrotron radiation**

- Magnetic **field-strength limiting factor** for particle energy

- **RF cavities**
  \[ \Rightarrow \text{bunched beams} \]

- **Superconducting lattice magnets** and **superconducting RF cavities**

- **Synchrotron** with **alternating-gradient focusing**; regular **FODO arc-sections** with **sextupoles** for chromaticity correction and octupoles for controlling the non-linear dynamics
### LHC nominal parameters  
(at collision energy)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particle type</td>
<td>p, Pb</td>
</tr>
<tr>
<td>Proton energy $E_p$ at collision</td>
<td>7000 GeV</td>
</tr>
<tr>
<td>Peak luminosity (ATLAS, CMS)</td>
<td>$1 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$</td>
</tr>
<tr>
<td>Circumference C</td>
<td>26 658.9 m</td>
</tr>
<tr>
<td>Bending radius $\rho$</td>
<td>2804.0 m</td>
</tr>
<tr>
<td>RF frequency $f_{RF}$</td>
<td>400.8 MHz</td>
</tr>
<tr>
<td># particles per bunch $n_p$</td>
<td>$1.15 \times 10^{11}$</td>
</tr>
<tr>
<td># bunches $n_b$</td>
<td>2808, 25 ns spacing</td>
</tr>
</tbody>
</table>