Surface mesh generation techniques with guaranteed properties

Trial lecture

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Given a surface $S = \{ x \in \mathbb{R}^3 : f(x) = 0 \}$, generate a mesh $\mathcal{M}$ that represents $S$.

- **Topological properties:**
  - *Manifold surface*: Is the resulting surface $\mathcal{M}$ a valid surface?
  - *Topological type*: Are $S$ and $\mathcal{M}$ the same type of shape?

- **Geometric properties:**
  - *Approximation error*. How good an approximation of $S$ is $\mathcal{M}$?
  - *Triangle density*. We want a minimal amount of triangles, but “enough” triangles in difficult areas.
  - *Shape of triangles*. Avoid long thin triangles.

- **Algorithmic properties:**
  - *Termination*. For what kind of input is the algorithm guaranteed to terminate?
The notion that two objects are of “equal type”

\(\mathcal{M}\) and \(\mathcal{S}\) are homeomorphic

A **homeomorphism** between \(\mathcal{S}\) and \(\mathcal{M}\) is a continuous bijection with a continuous inverse:

Points close on \(\mathcal{S}\) correspond to points close on \(\mathcal{M}\).

A homeomorphism doesn’t imply a continuous deformation between \(\mathcal{S}\) and \(\mathcal{M}\), but an isotopy does:

\(\mathcal{M}\) and \(\mathcal{S}\) are isotopic

An **isotopy** is a continuous map \(\gamma(\cdot, t)\),

\[
\gamma : \mathcal{S} \times [0, 1] \to \mathbb{R}^3, \quad \gamma(S, 0) = \mathcal{S}, \quad \gamma(S, 1) = \mathcal{M},
\]

that is a homeomorphism for any fixed \(t \in [0, 1]\).

If \(\gamma : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3\), \(\gamma\) is an **ambient** isotopy.
Marching cubes assumes $f$ is sampled on a regular 3D grid:

Grid cells have 8 neighbouring samples.
Label samples at corners:
$f < 0$ : inside, $f \geq 0$ : outside.

256 different cell label configurations, can be reduced to 15 using symmetry.

Let configuration of labels determine tessellation of surface inside cell.

Tessellation stored in a fixed table, edge intersections found using linear interpolation.
Labels alone cannot determine full topology in all cases:

E.g. case 6 is *ambiguous:* Two choices of connecting face.

*Inconsistent choices* produce holes in the resulting surface!

And should diagonally opposing corners be connected?

At least $M$ should be a consistent 2-manifold.

Even better if $M$ and $S$ are homeomorphic, or even isotopic...
Montani, Scateni, & Scopigno,  
*A modified look-up table for implicit disambiguation of MC*

- Sacrifice symmetry and extend table of cell configurations.

Nielson & Harmann,  
*The asymptotic decider: Resolving the ambiguity in MC*

- Intersects asymptotic lines of iso-curves of the bilinear interpolant.

Nielson,  
*On marching cubes*

- DeVella’s necklace $DeV(T)$ is intersection of asymptotic planes.
- If $DeV(T)$ exists and is inside cube $\rightarrow$ tunnel.
- 68 basic cases (uses only rotation), some need internal vertices, taken from $DeV(T)$. 
Regular subdivision schemes:

- Modified table guarantees to a consistent 2-manifold.
- Asymptotic decider resolves ambiguities consistently and corners on faces are connected as the bilinear interpolant.
- On marching cubes consistently connect diagonally opposing corners as the trilinear interpolant.

Problems with regular subdivision:

- No guarantee that $M$ and $S$ are of the same type.
- Grid is too coarse: Miss small features.
- Grid is too fine: An excessive amount of triangles.

A strategy is then to _adaptively_ subdivide the grid:

- Subdivide until $S$ is “simple enough” inside a cell.
- Better control on approximation error (bounded by cell size).
- Better control on triangle distribution (governed by cell size).
Snyder’s adaptive refinement algorithm

Subdivide boxes until $S$ is *globally parameterizable* inside each box:

$S$ is globally parameterizable over $X$ if the planar projection of

$S|_X = \{(x_0, x_1, x_2) \in X : f(x_0, x_1, x_2) = 0\}$

along the axis $x_i$ has no fold-overs, that is, $\frac{\partial f}{\partial x_i} \neq 0$ in $X$.

This condition is checked using *interval arithmetics*:

Instead of using a *single scalar* $x$, we calculate with an *interval* $[a, b]$. Thus, $f([a, b])$ is the interval $f$ takes on over the range $[a, b]$.

We *assume* that

If $[a, b] \to 0$ then $f([a, b]) \to 0$,

which is not always the case.
Initialize $A$ with bounding box and while there exist a box $X$ in $A$:

- if $0 \notin f(X)$, then $X$ is void of surface and is discarded.
- if $0 \notin \frac{\partial f}{\partial x_i}(S|_X)$ then $S|_X$ is globally parameterizable, put in $B$.
- Otherwise, subdivide $X$ and put subdivided boxes in $A$.

Then, mesh each box $x$ in $B$, sorted from small to large:

- $S|_X$ is globally parameterizable along $x_i$, can be projected onto the $\{x_0, x_1, x_2\}\{x_i\}$-plane w/o folding
- Intersect with side walls s.t. curves look as side was top-side.
- Triangulate projected regions.
- Propogate face-surface configuration to neighbouring boxes.

If no generated cube face are tangent to $S$:

Snyder’s algorithm terminates.

**Small normal variation** is a stronger condition, but relaxes requirement on projection direction:

The small normal variation condition requires that

\[ \langle \nabla f(a), \nabla f(b) \rangle \geq 0, \quad \forall a, b \in S|_X, \]

i.e., all normal vectors are inside a 90° cone.

Property implies global parameterizability along cone axis:

\( S|_X \) can be projected along cone axis onto a plane w/o foldover
Small normal variation and mean value theorem bound the curve:

The curvature is limited, and cells are equally-sided cubes, the surface cannot escape “too far” through a neighbouring cell.

The surface can be “isotopically pushed” so that $\mathcal{M}$ intersects an edge only once.

An equally-sided bounding box is adaptively refined as an oct-tree:

The tree is balanced such that the size of adjacent cells maximally differ by a factor of two.

This reduces the number of configurations.

On faces, edge intersections are joined, and for all cases but one, the cell contains a single loop.
Start with bounding box, subdivide oct-tree until every leaf $X$ is

- either void of $S$
  (check if $f(X) \neq 0$ using interval arithmetics)
- or $S|_X$ satisfies the small normal variation criterion
  (use interval arithmetics)
- and boxes are balanced
  (two adjacent boxes differ maximally in size by a factor of 2).

Then, build $\mathcal{M}$ by meshing the cells in the oct-tree:

- All edges with sign-changes get a vertex inserted.
- For each face in the oct-tree, connect vertices.
- For each leaf cell, connect loops of edges.

$S$ continuous & non-degenerate & interval arithmetics converges:

Algorithm terminates and $\mathcal{M}$ and $S$ are isotopic.

... but what about triangle shapes?
Delaunay refinement in the plane

Farthest point Delaunay refinement \textit{improves triangle shapes}:

While a triangle $T$ of low quality exists:
- Insert a vertex at the circumcircle of $T$.
- Circumcircle of $T$ is no longer empty, $T$ will be retriangulated.

Let $r$ be the circumradius and $l$ be the shortest edge of $T$.

Bounds on $r/l$ implies bounds on the smallest angle $\theta$ of $T$:

\[
\frac{r}{l} = \frac{1}{2 \sin(\theta)}, \quad \text{and thus,} \quad \frac{r}{l} > B \quad \iff \quad \theta < \theta_{\min}.
\]

When $T$ is refined:
- Three new edges have length $r$, the rest are longer.
- New edges are at least $Bl$ long.
- $B > 1$, i.e. $\theta_{\min} < 30^\circ$: no new edges shorter than $l$. 
How to define a Delaunay triangulation on a surface?

Chew, 
*Guaranteed-quality mesh generation for curved surface.*

Generalize circle criterium using the surface Delaunay ball:

The *surface Delaunay ball* for a triangle $T$ in a mesh $\mathcal{M}$ of set of points $P \subset S$ is the sphere through the corners of $T$ and with its center on $S$.

Generalization is consistent for “reasonable surfaces”:

If $\nabla S$ over two adjacent triangles is inside a $\frac{\pi}{2}$-cone, apices of the two triangles are consistently outside their opposing circumcircle.

And calculating the criterium amounts to intersecting $S$ with a line:

Centers of all spheres circumscribing $T$ lies on a line.
Surface Delaunay refinement procedure:

- Initialize with a coarse mesh $\mathcal{M}$ of points $P \subset S$.
- Build constrained Delaunay triangulation.
- While not finished:
  - Find triangles that either
    - has a minimal angle $< 30^\circ$, or
    - violates a user-specified size criterion.
  - Insert circumcentre of triangle with the largest circumcircle.
  - Update triangulation.

It is assumed that normals over triangles are inside a $\frac{\pi}{2}$-cone.

*If* algorithm halts:

- No interior triangle with minimum angle $< 30^\circ$.
- No triangle is larger than user-specified size criterion.

But *no* guarantees on topological relationships of $\mathcal{M}$ and $S$:

*Surface-based schemes have trouble with topology changes!*
Using the 3D Delaunay triangulation

Boissonnat and Oudot,
*Provably good sampling and meshing of surfaces.*

The **restricted Delaunay triangulation (RDT)** is a subset of the 3D Delaunay triangulation:

The RDT $\mathcal{M}$ for a set of points $P \subset S$ is the set of faces from the 3D Delaunay triangulation whose dual Voronoi-edges intersects $S$.

**Surface Delaunay balls** are empty in a RDT:

A triangle $T$ in a RDT is characterized by that the sphere through the corners of $T$ with center on $S$ is empty.

We know how to get a surface from $P$...

*Can we guarantee that $\mathcal{M}$ and $S$ are isotopic?*

Yes! If the samples of $P$ are dense enough.
The sample density of a $\psi$-sample is bound by $\psi : S \rightarrow \mathbb{R}^+$:

For any point $x$ on $S$, there is a point $p \in P$ maximally $\psi(x)$ away.

The density of $P$ is compared to the *local feature size*:

The $\text{lfs}(x)$ is the Euclidean distance from $x$ to the medial axis, and if density of $P$ is bound by $\psi(x) = \epsilon \text{lfs}(x)$, then $P$ is an $\epsilon$-sample.

*Weak $\epsilon$-samples* only require condition on Delaunay balls centers.

The following theorem guarantees an isotopic mesh:

**Theorem** (Amenta & Bern, Boissonnat & Oudot)

If $P$ is a weak $\epsilon$-sample with $\epsilon < 0.1$ and $M = \text{RDT}(P)$ with at least a triangle on every component of $S$, then $M$ is homeomorphic and ambient isotopic to $S$. 
Algorithm for building an $\epsilon$-sample $P$:

- Initialize with at least one triangle on each component of $S$.
- While not finished:
  - Intersect every Voronoi-edge with $S$.
  - If intersection $x$ exists,
    it is the center of a surface Delaunay ball with radius $r$.
  - If $r \geq \psi(x)$, insert $x$ into $P$ and update $\mathcal{M}$.

For 1-Lipschitz $\psi$ where $0 < \psi(x) < \epsilon \text{lfs}(x)$:

- The result is a weak $\psi$-sample.
- If $\epsilon < 0.1$ then $\mathcal{M}$ is ambient isotopic to $S$
- Number of points bounded so the algorithm terminates:
  \[ |P| < O(H(\psi, S)), \quad H(\psi, S) := \int_{x \in S} \frac{1}{\psi(x)^2} \, dx \]
- Combines with $\theta_{\text{min}}$-predicate and terminates if $\theta_{\text{min}} < \frac{\pi}{6}$.

need an explicit apriori lower bound on $\text{lfs}(x)$!

... can it be avoided?
Asserting the topological ball property

Cheng, Dey, Ramos, & Ray, *Sampling & meshing a surf. w/ guarant. topology and geometry*.

An alternative requirement is the *topological ball property*:

A point set $P$ on $S$ has the *topological ball property* if any $k$-dim face of $\text{Vor}(P)$ intersects $S$ in a closed $(k-1)$-dim ball or is $\emptyset$.

And with this property, the topological space can be triangulated:

**Theorem** Edelsbrunner & Shah

If $P \subset S$ satisfies the topological ball property, and $\mathcal{M}$ is the restricted Delaunay triangulation of $P$, then $\mathcal{M}$ is homeomorphic to $S$. 
Insert points on $S = \{ f(x) = 0 \}$ where top. ball property fails:

1. For Voronoi edge $e$, $p = e \cap S$ must be a single point:
   If $e$ intersects $S$ more than once, insert farthest intersection.

2. $\mathcal{M}$ is 2-manifold:
   If triangle fan of $p$ contains *multiple cycles* or edge $[p, q]$ is not shared by *exactly two triangles*,
   insert farthest intersection of $S$ with edges of Vor-cell of $p/q$.

3. For Voronoi face $F$, $s = S \cap F$ must be a single segment:
   *If $s$ has a closed loop, $s$ is at $x$ tangent to a dir $d$ in $F$.*
   Insert intersection of $s \cap \ell(x, d)$ farthest from $x$.
   Tests 1 & 2 excludes that $s$ is of multiple segments.

4. For Voronoi volume $V$, $m = S \cap V$ must be a single disc:
   *If the silhouette $\langle \nabla m, \nabla f(p) \rangle = 0$ is outside $V$, $m$ is a disc.*
   Break silhouette loops: insert points tangent to a $d' \perp \nabla S(p)$.
   Insert points where silhouette intersects $\partial V$. 
When algorithm terminates:

- $P$ satisfies topological ball property:
  \[ M \text{ homeomorphic to } S. \]
  (No guarantee for isotopy.)

- For smooth $S$:
  New point $q$ inserted is $\|p - q\| \leq 0.06 \text{lfs}(p)$ for $p \in P$.

- Algorithm terminates.

- Not necessary to know lfs!

- Needs up to second order derivatives of $f$.

No consideration for triangle quality, but can be extended:

Repeat until stable:

- Run a pass of e.g. Chew’s algorithm (may destroy topology).
- Run a pass of steps 1–4 (may destroy triangle shape).

Extension terminates for smooth surfaces.

...what about surfaces with degenerate points?
Mourrain & Tecourt, *Isotopic meshing of a real algebraic surface*. The algorithm is based on the concept of the *polar variety*

The *polar variety* $C$ is the set of points satisfying

$$ f(x, y, z) = 0, \quad \frac{\partial f}{\partial z}(x, y, z) = 0. $$

This is the silhouette along the $z$-axis and is usually a set of curves.

The polar variety is segmented using *slab points*:

*Slab points* are the $x$-coordinates where $C$ is singular or has tangent perpendicular to $x$-axis.
The polar variety slices the space into a set of vertical slabs, and the cross sections are meshed using a planar algorithm:

- Find critical points $X$ of $C$:
  - $C$ is tangent to $y$,
  - $C$ intersects itself, or
  - $C$ has another singularity.

  Between critical points $C$ is $x$-monotonous.

- Insert intermediate $x$-values into $X$ between critical points.

- Find intersections of $C$ and vertical lines through $x \in X$:
  - Intersections of critical points are multiple roots.
  - Intersections of intermediate points are simple roots.

- Connect intersections, based on multiplicity of roots.

The resulting curve is isotropic to $C$. 
To build a mesh $\mathcal{M}$ from a surface $\mathcal{S}$:

1. Find all slab points $X$ and insert intermediate $x$-coordinate.
2. For each $x_i \in X$, intersect $\mathcal{S}$ with the $yz$-plane through $x_i$. Build cross section using planar meshing.
3. Project cross sections onto $xy$-plane.
4. Connect critical points of polar variety. Regions are stacks of $xy$-monotone pieces.
5. Triangulate the regions using points from cross sections.
6. Multiplicate and raise the planar triangles to fill the 3D shape.

For distinct slab-points $\mathcal{M}$ is ambient isotropic to $\mathcal{S}$.

# vertices is bound by $O(d^7)$ for an algebraic surface of degree $d$. 
Summary

We have looked at some approaches, each approach has different strength and weaknesses.

Approaches based on subdivision of space:

- Marching cubes approaches
  - produce consistent surface $\mathcal{M}$, but
  - cannot guarantee that $\mathcal{M}$ is of correct topological type.

- Snyder’s adaptive algorithm
  - cannot handle singularities,
  - interval arithmetic must converge,
  - requires that generated faces are not tangent to $S$.

- Pantinga & Vegter’s small normal variation approach
  - creates an mesh isotopic to $S$, but
  - cannot handle singularities,
  - interval arithmetic must converge.
Delaunay-based approaches:

- Chew’s farthest point strategy
  - guarantees triangle size and shape, but
  - no guarantee that $\mathcal{M}$ is of correct topological type.

- Boissonnat & Oudots $\epsilon$-sample strategy
  - creates a mesh isotopic to *non-singular* $S$, but
  - and requires explicit apriori knowledge of lfs.

- Cheng, Dey, Ramos & Ray’s topological ball approach
  - creates a mesh *homeomorphic* to *non-singular* $S$,
  - without explicit knowledge of lfs, but
  - needs second order derivatives of $f$.

And finally an approach based on sweeping through space:

- Mourrain & Tecourt’s space-sweeping approach
  - creates a mesh *isotopic* to $S$ with degenerate points, but
  - requires distinct slab-points, and
  - no control of triangle size and shape.