Gauge Theory of Electro-Weak Interactions

• Read Appendix D of Book
  – Gauge Theories

• Following slides based among others on
  – The ideas of particle physics: an introduction to scientists, by Coughlan, Dodd, Gripaios
    http://www.amazon.ca/gp/reader/0521677750/ref=sib_rdr_zmin?p=S001&j=1#reader-page
  – Djouadi
    • Anatomy of EWSB
      (1) SM Higgs (2) SUSY Higgs
      (1) Talk (2) Talk (3) Talk
19. Motivation for the Theory

• Current-current theory of WIs
  – Good account of low energy experimental observations
  – Heavy W particles as intermediate bosons

• Why not a QED-like theory? A predictive theory
  – dealing with higher order calculations
  – revealing new phenomena
  – allowing possibility for electro-weak unification and more

• Problems with W bosons
  – Spin 1 \textit{and} non-zero mass
  – Spin transforms as vector under Lorentz transformations
    • 3 components to define orientation / polarisation at any point
      – 2 transversal and 1 longitudinal
      – OK for massive spin-1 particle
    • Massless spin-1 particle (photon): longitudinal degree of freedom has no meaning
      – Photon can always be parameterised in a plane perpendicular to its direction of motion
• Difference becomes important when propagators carry very high momentum
  – Transverse propagator for massless photon behaves like $1/p^2 \rightarrow 0$ at high momentum
  – Massive vector particle
    • Presence of extra longitudinal component in its propagator spoils this behaviour
    • At very high momentum, massive propagator approaches value of $1/M^2$

• In perturbation theory (PT)
  – Probability of occurrence of an event given by sum of contributions from a series of increasingly more complex Feynman diagrams
    • Each contribution should be a dimensionless number
  – Furthermore, it is necessary to sum over all the possible values for the unobserved momenta of all the internal virtual particles in any diagram
  – However, at very high momenta a massive W boson propagator contributes a factor of $1/M^2$
    • To compensate this, multiply by factors leading to dimensionless contribution of the form: $p^2/M^2c^2$
    • This leads to diverging diagrams when summed over all possible internal momenta

• Mass factor in W-boson propagator leads to ever-increasing number of infinite contributions to perturbation series
  – Not possible to be reabsorb into redefinitions of the masses and couplings
  – Bad behaviour of theory at high energies exhibited by some processes involving W-bosons …
Lagrangian formalism

Classically, \( L(q, \dot{q}, t) = T(\dot{q}) - U(q) \)

\[ = \frac{1}{2} m \dot{q}^2 - U(q) \]

\( T \) : kinetic energy; \( U \) : potential energy \((U = mgq\) for gravity\)

Euler – Lagrange equations lead to equations of motion:

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 \quad \Rightarrow \quad F = \frac{dU}{dq} = ma \]

In QM or QFT, dealing with wavefunctions or fields, define Lagrange density \( \mathcal{L} \) as a functional of the field \( \Phi(x^\mu) \).

The integral over the 3-dimensional space leads to: \( L = \int \mathcal{L}(\Phi, \partial_\mu \Phi) \)

The Euler - Lagrange equations become:

\[ \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \right) = 0 \]
From Lagrangian to equations of motion

- Lagrangian density of a free electron in terms of its wavefunction or field
  - And it is trivial to go from the Lagrangian to the equations of motions

- The situation is more complex in case of elementary particles in interaction
  - Equations in general not known
  - When known, difficult to solve

For scalar fields (neutral pion, Higgs),
\[ \mathcal{L} = \frac{1}{2} \left( \partial_\mu \Phi \partial^\mu \Phi \right) - \frac{1}{2} M^2 \Phi^2 \]
leads to the Klein-Gordon equation
\[ (\partial_\mu \partial^\mu + M^2) \Phi = 0; \text{ where } \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \]
Making correspondence rules in E-p relations leads to KG-equation

For a spin \( \frac{1}{2} \) Dirac field (electron)
\[ \mathcal{L} = \bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi \]
leads to the Dirac equation:
\[ (i \gamma^\mu \partial_\mu - m) \Psi(x) = 0 \]

To describe elementary particle interactions, it is necessary to propose an expression for the Lagrangian of the interacting quantum fields!
20. Gauge theory

- Principle of gauge invariance is at the origin of the fundamental forces
  - Applies to all of the 4 known forces
  - May provide with the basis for a comprehensive unified theory
- Basic method of gauge theory
  - To ensure that the Lagrangian describing the interaction between particle WFs remain invariant under certain symmetry transformations
    - Which reflect conservation laws observed in nature
- 20.2 The formulation of QED
  - QED seeks to explain the interaction of charged particles, such as $e^-$ and $e^+$, in such a way that the total electric charge $Q$ is conserved.
  - Global phase transformations: identical operations at all points in space-time
    - The laws of physics are independent of the choice of phase convention
    - Weak constraint on the form of the lagrangian

\[
\text{Global phase transformation: } G\mathcal{L}(\Psi_e) \to \mathcal{L}(\Psi_e^*)
\]

(1) $\psi(x)$ describes wavefunction
(2) The Lagrangian describes interactions $L(\psi)$
(3) A group of transformations $G$ shifts the phase of the wavefunction
  $G\psi(x) = \psi^*(x)$
  $G L(\psi) = L(\psi^*)$
(4) Invariance requires $L(\psi) = L(\psi^*)$. This limits the possible functional form of $L$
20.2 The formulation of QED (II)

- Local gauge transformation: choose a convention for defining the phase of the electron WF, which is different at different space-time points
  - The theory is not invariant under the more demanding symmetry!
  - However, by introducing another field, which compensates for the local change in the electron WF, it is possible to obtain a lagrangian which exhibits such symmetry!

  
  
  \[
  \text{Local phase transformation: } G(x)\ell(\Psi_e) \rightarrow \ell^*(\Psi_e^*)
  \]

  where \( x=(\vec{x},t) \Rightarrow \ell \neq \ell^* \)

- Required field must have infinite range
  - Massless field \( \rightarrow \) EM field whose quantum is the photon

- Introduction of photon leads to local gauge invariance
  - Photon communicates the different space-time-dependent conventions, which define the phase of the electron WF, between different points in space-time

  
  
  \[
  \text{Local phase transformation: } G(x)\ell(\Psi_e, A) \rightarrow \ell^*(\Psi_e^*, A^*)
  \]

  \[
  \Rightarrow \ell = \ell^*
  \]

- The presence in the lagrangian of a mass term would destroy the gauge invariance
  - Massive spin-1 particles generally give rise to non-renormalisable theories.
  - Local gauge invariance \( \rightarrow \) good guide for the renormalisability of theories
→ Simple 2-dimensional harmonic oscillator (SHO):
  x and y coordinates oscillate sinusoidally with the same frequency
  Particle in general describes an elliptic trajectory in xy-plane
  2 motions combined into motion of a single complex variable
  \[ z = x + iy \]
  Position of particle in xy-plane labelled by \( z \) (Argand diagram or Gauss plane)
→ SHO described by equation
  \[ \frac{d^2z}{dt^2} + \omega^2 z = 0 \]
  \( \omega = 2\pi v = 2\pi/T \) is the angular frequency.

→ Elliptical trajectory of a 2-dimensional harmonic oscillator in the complex plane representation.

  \[ z = x + iy = r e^{i\theta}; \]
  \[ r = \sqrt{x^2 + y^2}; \tan \theta = y/x \]
  Absolute origin of \( \theta \) is irrelevant, freedom of choice (re-gauge \( \theta \)):
  \[ \theta \not\equiv \theta - \alpha \]
  → equivalent to multiply equation by \( e^{-i\alpha} \)
  → phase is absorbed by redefining \( z' = z e^{-i\alpha} \)
  \[ \frac{d^2z}{dt^2} + \omega^2 z = 0 \rightarrow \frac{d^2z'}{dt^2} + \omega^2 z' = 0 \]
  possible because \( \alpha \) is constant → global gauge freedom and invariance
Let us now require local gauge invariance: time-dependent shift $\alpha(t)$

$\to e^{-i\alpha(t)}$ can no longer be absorbed by redefining $z$

$\to$ Additional terms $A(t)$ involving time-derivatives of $\alpha(t)$

$\to$ Compensation of additional terms is possible by:

- replacing $\frac{d}{dt} \to \frac{d}{dt} - iA(t)$

- re-gauging $A(t) \to A(t) - \frac{d\alpha}{dt}$

  when re-gauging $\theta \otimes \theta - \alpha(t)$

$\to A(t)$ is the compensatory or gauge field

**Anything gained?**

Shift $\alpha(t)$ as rotation of reference frame with angular velocity $d\alpha/dt$

$\to$ "fictitious forces": centrifugal (radially outward) and Coriolis (sideways)

$\to$ Gauge field $A(t)$ generates precisely these forces automatically

$\to$ Homework: do the maths and compare to results you can find in a Mechanics text book.

$\to$ Requirement of local gauge invariance created the right kind of forces acting on the particle in accord with experience!
The Lagrangian for a spin-$\frac{1}{2}$ Dirac field of mass $m$ is

$$\mathcal{L}_D = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi.$$ 

The local $U(1)$ gauge transformations are defined as

$$\psi \rightarrow \psi' = U(x) \psi(x) = e^{-ieQ\alpha(x)} \psi,$$

where $\alpha(x)$ is an arbitrary scalar function and $Q$ the charge operator (i.e. $Q\psi = -\psi$ for an electron). For infinitesimal symmetries

$$\delta U(x) = [1 - ieQ\alpha(x)] \psi(x).$$

The charge operator $Q$ is the generator of the unitary group $U(1)$. In order for the Lagrangian $\mathcal{L}_D$ to remain gauge invariant a gauge field $A_\mu$ is required. The electric field $\vec{E}$ and the magnetic field $\vec{B}$ can be deduced from a scalar potential $\phi$ and a vector potential $\vec{A}$

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t},$$

$$\vec{B} = \nabla \times \vec{A},$$

suggested by the equations of motion of the electromagnetic field (Maxwell equations):

$$\nabla \vec{E} = 0,$$

$$\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t},$$

$$\nabla \vec{B} = 0,$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$ 

To have a relativistic notation, $\phi$ and $\vec{A}$ are put in 4-vector:

$$A_\mu \equiv \left( \phi, \vec{A} \right),$$
The covariant derivative is defined by

\[ D_\mu \equiv \partial_\mu + i e \, Q \, A_\mu \]

and transforms as follows

\[ D_\mu \rightarrow D'_\mu = \partial_\mu + i e \, Q \, A'_\mu = e^{-i e Q \alpha(x)} D_\mu \]

The gauge field, the photon, transforms as

\[ A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x). \]

The Lagrangian of interaction between the electron and photon is written

\[ \mathcal{L}_I = -e J_{em}^\mu A_\mu \]

\[ \mathcal{L}_I = -\bar{\psi} \gamma^\mu Q \psi A_\mu , \]

where \( J_{em}^\mu = \bar{\psi} \gamma^\mu Q \psi \) is the electromagnetic current. For free photons one has an additional term

\[ \mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \]

where the electromagnetic antisymmetric field tensor is

\[ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu , \]
The QED-Lagrangian is now made of three main members:

\[
\mathcal{L}_{QED} = \mathcal{L}_D + \mathcal{L}_I + \mathcal{L}_\gamma
\]

\[
= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu Q\psi A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]

\[
= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]

The Euler-Lagrange equations

\[
\partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi_i)} \right] = \frac{\partial \mathcal{L}}{\partial \Phi_i}
\]

lead to the equations of motion of electromagnetism

- for \( \Phi_i = \psi \),

\[
(i\gamma^\mu D_\mu - m)\psi = 0,
\]

corresponding to the Dirac equation (without field \( A_\mu \ D_\mu \leftarrow \partial_\mu \)). The minimal coupling of the photon to spinors is introduced through the covariant derivative \( D_\mu \psi \) and is determined purely by the transformation properties of \( \psi \) under the gauge group.
• for $\Phi_i = A_\mu$:

\[ \partial_\nu F^{\mu\nu} = J^\mu_{em} \equiv (\rho, \vec{j}_{em}) \]

\[ \Box A^\mu - \partial^\mu (\partial_\nu A^\nu) = J^\mu_{em}, \]

where $\Box \equiv \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$. If we choose the Lorentz gauge

\[ \partial_\mu A^\mu = 0, \]

we arrive to the Maxwell equations

\[ \Box A^\mu = J^\mu_{em}. \]

The electromagnetic current is conserved

\[ \partial_\mu J^\mu_{em} = \partial_\mu \partial_\nu F^{\mu\nu} = 0 \]

implying conservation of electric charge $q$

\[ q = \int J^0_{em} d^3x = \int \rho d^3x \]
For a massive vector field of mass $M$ we obtain the field Lagrangian by adding a mass term

$$\mathcal{L}_\gamma \rightarrow -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu.$$ 

The corresponding Euler-Lagrange equations are

$$\left(\Box + M^2\right) A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = J_{em}^\mu,$$

For a free field ($J_{em}^\mu = 0$) and by applying $\partial_\mu$ from the left side:

$$\left(\Box + M^2\right) \partial_\mu A^\mu - \partial_\nu \partial_\mu \partial^\mu A^\nu = 0,$$

$$M^2 \partial_\mu A^\mu = 0 \iff \partial_\mu A^\mu = 0.$$ 

In QED, the photon is massless because U(1) is an exact symmetry:

Any mass term for the photon would spoil gauge invariance

$$\frac{1}{2} M_A^2 A_\mu A^\mu \rightarrow \frac{1}{2} M_A^2 \left(A_\mu + \partial_\mu \alpha\right) \left(A_\mu + \partial_\mu \alpha\right) \neq \frac{1}{2} M_A^2 A_\mu A^\mu$$
\[ \mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu - m \right) \psi \]

\[ = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi - e Q A_\mu \left( \bar{\psi} \gamma^\mu \psi \right) \]

**Kinetic term:**

\[ \mathcal{L}_K = - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \]

\[ \partial_\mu F^{\mu \nu} = e Q \left( \bar{\psi} \gamma^\nu \psi \right) \]

**Mass term:**

\[ \mathcal{L}_M = \frac{1}{2} m_\gamma^2 A_\mu A^\mu \]

Not Gauge Invariant \[ \Rightarrow m_\gamma = 0 \]

\[ \text{Gauge Symmetry} \quad \Rightarrow \quad \text{QED Dynamics} \]

[exp: \( m_\gamma < 6 \times 10^{-17} \text{ eV} \)]

The Standard Model

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20.3. Generalised gauge invariance

- Attempt to describe strong interactions
  - Strong force does not distinguish between $p$ and $n$
    - Isopin symmetry $\rightarrow$ invariance under rotations in isospin space
    - SU(2) symmetry rotates $p$ into $n$ and vice-versa
      - EM force breaks isospin symmetry
    - Lagrangian invariant under the group of global isospin rotations SU(2)

  $\text{Global phase transformation: } G^{SU(2)}(N) \rightarrow N^*$

- Require that theory be symmetric under local gauge transformations
  - Necessary to introduce massless gauge particle $\rho$ (whose source is isospin) to ensure invariance of Lagrangian
  - To explain nucleons with or without electric charge, $\rho=(\rho^+, \rho^0, \rho^-)$
  - $\rho$ carries its own isospin $\rightarrow$ self-interaction, in contrast to photons in QED
    - QED Abelian: In a simple shift in phase, a series of transformations can be performed in any order to produce the same effect as one big transformation
    - SU(2) gauge theory non-Abelian: a series of transformations does depend on the order of operation (just like rotations in 3 dimensions).

- Gauge invariance first generalised to isospin in 1954
  - $\rho$, unfortunately, massive “bound states” of 2 pions and cannot be considered as candidates for the fundamental role of gauge bosons.

Nucleon: $N \equiv \begin{pmatrix} p \\ n \end{pmatrix}$; $I = \frac{1}{2} = \begin{pmatrix} I_3 = +1/2 \\ S = -1/2 \end{pmatrix}$

$Q = I_3 + \frac{Y}{2}$

$Q$: electric charge; $I_3$: 3rd component of isospin

B: baryon number; S: strangeness; $Y = B + S$: hypercharge

$\rho$ - meson: $I = 1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\rho$ carries its own isospin $\rightarrow$ self-interaction, in contrast to photons in QED
20.4. Gauge invariance and WIs

- Incorporation of basic laws of leptonic physics
  - Lepton number conservation
  - WIs independent of electric charge
    - WIs sees only a lepton and cannot distinguish between neutrino and electron
  - Define weak isospin
    - Require WIs to be invariant under rotations in this weak isospin space
    - Require Lagrangian to be invariant under SU(2)_W of weak isospin

- Enforcing local symmetry introduces massless gauge particles, W,
  - To guarantee the invariance of the Lagrangian
  - W boson communicates between interacting leptons the locally defined convention governing the mixture of ‘electron’ and ‘neutrino’ constituting the lepton.
  - W: charge triplet (W^+, W^0, W^-)
    - W^+, W^-: charged currents; W^0: neutral currents

- How to reconcile gauge invariance and heavy gauge bosons?

\[ l_e \equiv \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}; \quad l_\mu \equiv \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}; \quad I = \frac{1}{2} = \begin{pmatrix} 3+1/2 \\ 1/2 \end{pmatrix} \]

\[ Q = I_3 + \frac{Y}{2} \]

Global phase transformation:
\[ G^{SU(2)}_W \mathbf{L}(l_e, l_\mu) \rightarrow \mathbf{L}(l_e^*, l_\mu^*) \]

Local phase transformation:
\[ G^{SU(2)}_W(x) \mathbf{L}(l_e, l_\mu, W) \rightarrow \mathbf{L}(l_e^*, l_\mu^*, W^*) \]
QUANTUM CHROMODYNAMICS

FREE QUARKS: \( N_c = 3 \)

\[ \mathcal{L} = \bar{q} \left[ i \gamma^\mu \partial_\mu - m \right] q \]

SU(3) Colour Symmetry: \( q \to U q = \exp \left\{ i \frac{\lambda^a}{2} \theta_a \right\} q \)

Gauge Principle: Local Symmetry \( \theta_a = \theta_a(x) \)

\[ D^\mu q \equiv (I_3 \partial^\mu + i g_s G^\mu) q \to U D^\mu q \]

\[ D^\mu \to U D^\mu U^\dagger \]

\[ G^\mu \to U G^\mu U^\dagger + \frac{i}{g_s} (\partial^\mu U) U^\dagger \]

\[ [G^\mu]_{\alpha \beta} = \frac{1}{2} (\lambda^a)_{\alpha \beta} G_a^\mu(x) \]

8 Gluon Fields
\[ \mathcal{L}_{\text{QCD}} = -\frac{1}{2} \text{Tr} \left( G^{\mu \nu} G_{\mu \nu} \right) + \overline{q} \left[ i \gamma^{\mu} D_{\mu} - m_q \right] q \]

\[ = -\frac{1}{4} \left( \partial^{\mu} G_{a}^{\nu} - \partial^\nu G_{a}^{\mu} \right) \left( \partial_\mu G_{v}^{a} - \partial_\nu G_{a}^{\mu} \right) + \sum_q \overline{q}_\alpha \left[ i \gamma^{\mu} \partial_\mu - m_q \right] q_\alpha \]

\[ - \frac{1}{2} g_s G_{\mu}^{a} \sum_q \left[ \overline{q}_\alpha \left( \lambda^{a}_{\alpha \beta} \gamma^{\mu} q_\beta \right) \right] \]

\[ + \frac{1}{2} g_s f_{abc} \left( \partial_\mu G_{v}^{a} - \partial_\nu G_{a}^{\mu} \right) G_{b}^{\mu} G_{c}^{\nu} - \frac{1}{4} g_s^2 f_{abc} f_{ade} G_{b}^{\mu} G_{c}^{\nu} G_{d}^{\gamma} G_{v}^{e} \]

- **Gluon Self – interactions** \[ G^3, \ G^4 \]
- **Universal Coupling** \[ g_s \] (No Colour Charges)

The Standard Model  
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\( \text{SU}(2)_L \otimes \text{U}(1)_Y \) **Gauge Theory**

<table>
<thead>
<tr>
<th>Fields</th>
<th>( \psi_1(x) )</th>
<th>( \psi_2(x) )</th>
<th>( \psi_3(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quarks</strong></td>
<td>( \begin{pmatrix} q_u \ q_d \end{pmatrix}_L )</td>
<td>( (q_u)_R )</td>
<td>( (q_d)_R )</td>
</tr>
<tr>
<td><strong>Leptons</strong></td>
<td>( \begin{pmatrix} \nu_i \ l^- \end{pmatrix}_L )</td>
<td>( (\nu_i)_R )</td>
<td>( (l^-)_R )</td>
</tr>
</tbody>
</table>

**Free Lagrangian for Massless Fermions:**

\[
\mathcal{L}_0 = \sum_j i \bar{\psi}_j \gamma^\mu \partial_\mu \psi_j
\]

**SU(2)_L \otimes U(1)_Y Flavour Symmetry:**

\[
\begin{align*}
   \psi_1 & \rightarrow e^{iy_1\beta} U_L \psi_1 \\
   \bar{\psi}_1 & \rightarrow \bar{\psi}_1 U_L^\dagger e^{-iy_1\beta} \\
   \psi_2 & \rightarrow e^{iy_2\beta} \psi_2 \\
   \bar{\psi}_2 & \rightarrow \bar{\psi}_2 e^{-iy_2\beta} \\
   \psi_3 & \rightarrow e^{iy_3\beta} \psi_3 \\
   \bar{\psi}_3 & \rightarrow \bar{\psi}_3 e^{-iy_3\beta}
\end{align*}
\]

The Standard Model

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\[
\mathcal{L}_{cc} = - \frac{g}{2\sqrt{2}} W^\mu_\mu \left[ \bar{q}_u \gamma^\mu (1 - \gamma_5) q_d + \bar{v}_l \gamma^\mu (1 - \gamma_5) l \right] + \text{h.c.}
\]

\[
\mathcal{L}_{NC} = - g \ W^3_\mu \bar{\psi}_1 \gamma^\mu \frac{\sigma_3}{2} \psi_1 - g' B_\mu \sum_j \gamma_j \bar{\psi}_j \gamma^\mu \psi_j
\]

\[
\mathcal{L}_0 = \sum_j i \bar{\psi}_j \gamma^\mu \partial_\mu \psi_j
\]

\[
\mathcal{L}_k = - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = - \frac{1}{2} \text{Tr}(W_{\mu\nu} W^{\mu\nu}) = \mathcal{L}_{\sin} + \mathcal{L}_3 + \mathcal{L}_4
\]

\[
\mathcal{L}_3 = i e \cot \theta_w \left\{ (\partial^\mu W^\nu - \partial^\nu W^\mu) W^\mu_\mu Z_\nu - (\partial^\mu W^\nu - \partial^\nu W^\mu) W^\mu_\nu Z_\nu + W^\mu_\mu W^\nu_\nu \left( \partial^\mu Z^\nu - \partial^\nu Z^\mu \right) \right\}
+ i e \left\{ (\partial^\mu W^\nu - \partial^\nu W^\mu) W^\mu_\nu A_\nu - (\partial^\mu W^\nu - \partial^\nu W^\mu) W^\mu_\nu A_\nu + W^\mu_\mu W^\nu_\nu \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) \right\}
\]

\[
\mathcal{L}_4 = - \frac{e^2}{2 \sin^2 \theta_w} \left\{ \left( W^\mu_\mu W^\nu_\nu \right)^2 - W^\mu_\mu W^\nu_\nu W^\nu_\nu \right\} - e^2 \cot^2 \theta_w \left\{ W^\mu_\mu W^\nu_\nu Z_\nu Z^\nu - W^\mu_\mu Z_\nu Z^\nu W^\nu_\nu \right\}
- e^2 \cot \theta_w \left\{ 2 W^\mu_\mu Z_\nu W^\nu A^\nu - W^\mu_\mu Z_\nu W^\nu A^\nu - W^\mu_\nu A^\mu W^\nu Z^\nu \right\} - e^2 \left\{ W^\mu_\mu A_\mu A^\nu W^\nu A^\nu \right\}
\]
21. Spontaneous symmetry breaking

- Existence of asymmetric solutions to a symmetric theory
  - Ordinary magnet
    - Magnetic field defines preferred direction in space $\rightarrow$ rotational symmetry is broken
    - Whereas equations governing the motions of the individual atoms in the magnet are entirely rotationally symmetric
    - Why?
      - The symmetric state is not the state of minimum energy, i.e. the ground state (GS)
      - In the process of evolving towards the GS, the intrinsic symmetry of the system has been broken
  - Law of gravitation attraction spherically symmetric but trajectory of Earth around Sun is elliptic (but closes up!)
- A simple mechanical example of spontaneous symmetry breaking is the behaviour of a marble inside the bottom of a wine bottle.
  - The initial position of the marble is symmetric but not minimum energy.
  - A small perturbation will cause the rotational symmetry to be broken and the system to assume the state of minimum energy.

- Spontaneous symmetry breaking: When the symmetry of a physical system is broken by an asymmetric ground state
  - Why spontaneous? Disturbance needed to break symmetry can be made arbitrarily small (even tiny thermal jiggling of molecules in atom)
21.2 Spontaneous breaking of global symmetry

- Has spontaneously broken symmetry anything to do with gauge boson mass?
  - Asymmetric ground state with symmetric lagrangian
- Hypothetical spinless particle / quantum field
  - consisting of 2 components \( \Phi = (\phi_1, \phi_2) \) (think of analogy N=(p,n) for example

**Lagrangian specifying interaction between \((\phi_1, \phi_2)\)**

- Choose interaction energy such that state of minimum energy corresponds to a non-zero value of the field
- Minimum around the circle \( \phi_1^2 + \phi_2^2 = R^2 \)
  - Vacuum states of theory characterised by non-zero average values for \( \phi_1 \) and \( \phi_2 \)
  - Lagrangian still symmetric under transformations between \( \phi_1 \) and \( \phi_2 \), i.e. under rotations in the plane \( \phi_1 - \phi_2 \)

- Global phase transformation:
  \[
  G \mathcal{L}(\phi_1, \phi_2) \rightarrow \mathcal{L}(\phi_1^*, \phi_2^*)
  \]
21.2 Spontaneous breaking of global symmetry

- Consider particular vacuum state given by: \( \phi_1 = 0, \phi_2 = R \)
  - Can be obtained by simple redefinition of the fields (new axes through point R)
    \( \phi'_1 = \phi_1; \phi'_2 = \phi_2 - R \)
  - Both Lagrangians describe the same physics

  \[ \mathcal{L}(\phi_1, \phi_2) = \mathcal{L}'(\phi'_1, \phi'_2) \]

- Interesting features arise in this redefined system:
  1. Vacuum is not invariant under original group \( \mathbb{G} \) circle \( \phi_1 + \phi_2 = R^2 \)
  2. Lagrangian now describes \( \phi'_2 \) as massive particle (mass proportional to R) and \( \phi'_1 \) as massless particle
- The global symmetry of the original Lagrangian necessary broken
  - One of the particles has been given a mass, whilst the other remains massless

- Do particle masses originate this way?
  - But what about the presence of the massless spin-0 particle?
  - Goldstone theorem: “Whenever a global symmetry is spontaneously broken, a massless, spin-0 “Goldstone boson” appears
    - No physical meaning ...
  - What about local gauge symmetry?
21.3 Spontaneous breaking of local symmetry – the Higgs mechanism

• Consider the original Lagrangian and demand local gauge invariance
  – Must introduce a gauge particle, $A$, in order to maintain invariance
    \[ G(x) \mathcal{L}(\phi_1, \phi_2, A) \rightarrow \mathcal{L}(\phi_1^*, \phi_2^*, A) \]
  – Redefine the fields (such that axes through point of minimum energy
    $\phi'_1 = \phi_1$; $\phi'_2 = \phi_2 - R$
  – Both Lagrangians describe the same physics
    \[ \mathcal{L}(\phi_1, \phi_2, A) \equiv \mathcal{L}'(\phi'_1, \phi'_2, A) \]
  – Something remarkable happens in this last step:
    1. Redefined $\phi'_2$ particle acquires a mass proportional to $R$
    2. Massless Goldstone boson disappears
    3. Formerly massless gauge particle $A$ now acquires a mass, again proportional to $R$

• What happened?
  – The original Lagrangian describes a two component particle $\Phi = (\phi_1, \phi_2)$ and a massless gauge particle $A$, consisting of 2 spin polarisation states
  – The redefined Lagrangian describes one massive spinless particle $\phi'_2$ and one massive vector particle $A'$, containing now 3 polarisation states
  – Total number of degrees of freedom remains the same, 4

• No Goldstone boson … but price to pay is now to find “Higgs boson”!

Table 21.1. The Higgs mechanism

<table>
<thead>
<tr>
<th>Before</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>massive $A'$</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td></td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>massive $\phi'_2$</td>
</tr>
</tbody>
</table>
Take a simple scalar real field $\phi$ with the usual Lagrangian:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \quad \text{with} \quad V(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \lambda \phi^4$$

If $\mu^2 > 0$ : $V(\phi) \geq 0$ if $\lambda \geq 0$ $\Rightarrow$ minimum of $V$ at $\phi_0 = 0$.

$\mathcal{L}$ is the Lagrangian of a spin–zero particle of mass $\mu$.

$\mathcal{L}$ is invariant under the reflexion: $\phi \rightarrow -\phi$ (no cubic terms).

If $\mu^2 < 0$ : the potential $V(\phi)$ has a minimum when:

$$\partial V / \partial \phi = \mu^2 \phi + \lambda \phi^3 = 0 \Rightarrow \phi_0^2 = -\mu^2 / \lambda = v^2$$

and not at $\phi_0^2 = 0$. $v \equiv \langle 0 | \phi | 0 \rangle$ is the vacuum expectation value.
To interpret the theory, expand around one of the minima \( \nu \) by defining a field \( \sigma \phi = \nu + \sigma \). In terms of the new field, \( \mathcal{L} \) is

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - (\mu^2) \sigma^2 - \sqrt{-\mu^2 \lambda} \sigma^3 - \frac{\lambda}{4} \sigma^4 + \text{const}.
\]

Theory of scalar field of mass \( m^2 = -2\mu^2 \) and \( \sigma^3, \sigma^4 \) interactions. Reflexion symmetry is broken (not anymore apparent in \( \mathcal{L} \)).

\[ \Rightarrow \text{simplest example of a spontaneously broken symmetry.} \]
Make things more complicated: 4 scalar fields $\phi_i$ with $i = 0, 1, 2, 3$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2} \mu^2 \phi_i^2 - \frac{1}{4} \lambda (\phi_i \phi_i)^2$$

which is invariant under the rotation group in four dimensions $O(4)$

$$\phi_i(x) = R_{ij} \phi_j(x) \quad \text{for any orthogonal matrix } R$$

Again, for $\mu^2 < 0$, the potential has a minimum at: $\phi_i^2 = -\frac{\mu^2}{\lambda} \equiv v^2$.

Expand around minima: $\phi_0 = v + \sigma$ and rewrite the fields $\phi_i = \pi_i$

$$\Rightarrow \mathcal{L} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} (-2\mu^2)\sigma^2 - \lambda v \sigma^3 - \frac{\lambda}{4} \sigma^4$$

$$+ \frac{1}{2} \partial_\mu \pi_i \partial^\mu \pi_i - \frac{\lambda}{4} (\pi_i \pi_i)^2 - \lambda v \pi_i \pi_i \sigma - \frac{\lambda}{2} \pi_i \pi_i \sigma^2$$

As expected, we have a massive $\sigma$ boson with $m^2 = -2\mu^2$ but also massless pions! There is still an $O(3)$ symmetry among the $\pi_i$ fields.

The Goldstone Theorem:

For every spontaneously broken continuous symmetry, the theory will contain massless scalar particles (that is spin–0 bosons).
Let us consider first the rather simple abelian U(1) case:

A complex scalar field $\phi$ coupled to itself and an EM field $A_\mu$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^* D^\mu \phi - V(\phi)$$

with $D_\mu = \partial_\mu - ieA_\mu$ and potential $V(\phi) = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$

The Lagrangian $\mathcal{L}$ is invariant under the local U(1) transformations

$$\phi(x) \rightarrow e^{i\alpha(x)} \phi(x) \quad A_\mu(x) \rightarrow A_\mu(x) - (1/e) \partial_\mu \alpha(x)$$

$\mu^2 > 0$: simply $\mathcal{L}_{\text{QED}}$ for particle of mass $\mu$ and $\phi^4$ int.

$\mu^2 < 0$: $\phi(x)$ will acquire a vev and the minimum of $V$ will be at

$$\langle \phi \rangle = \left( -\mu^2 / 2\lambda \right)^{1/2} \equiv (v/\sqrt{2})$$
As usual, we expand the Lagrangian around the vacuum state $\langle \phi \rangle$

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \phi_1(x) + i\phi_2(x)] \Rightarrow$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial^\mu + i e A^\mu)\phi^*(\partial_\mu - i e A_\mu)\phi - \mu^2 \phi^*\phi - \lambda(\phi^*\phi)^2 + \cdots$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - v^2 \lambda \phi_1^2 + \frac{1}{2}e^2 v^2 A_\mu A^\mu - ev A_\mu \partial^\mu \phi_2$$

- There is a photon mass term: $\frac{1}{2}M_A^2 A_\mu A^\mu$ with $M_A = ev = -e\mu^2/\lambda$!
- We still have a scalar particle $\phi_1$ with a mass $M_{\phi_1}^2 = -2\mu^2$.
- Apparently, we have a massless particle $\phi_2$, a Goldstone boson.
But, problem: before, we had 4 dof (2 for $\phi$, 2 for $A_\mu$) and now 5 dof (1 for $\phi_1$, 1 for $\phi_2$ and 3 for massive $A_\mu$). There must be a non–physical field involved and indeed, there is a term $evA_\mu\partial\phi_2$!

To get rid of it, notice that at 1st order, we have for the original $\phi$

$$\phi = \frac{1}{\sqrt{2}}(v + \phi_1 + i\phi_2) \equiv \frac{1}{\sqrt{2}}(v + \phi_1)e^{i\phi_2(x)/v}$$

By using freedom of gauge transf. and making $A_\mu \rightarrow A_\mu - (1/ev)\partial_\mu\phi_2(x)$ this term (and all $\phi_2$ terms) disappear: this is the unitary gauge.

The moral of the story is:

- The photon (2 dof) has absorbed the Goldstone boson (1 dof) and became massive (i.e. with 3 dof): the longitudinal polarisation of the photon is the would be Goldstone boson.
- $U(1)$ symmetry is no more apparent: it is spontaneously broken.

This is the Higgs mechanism.
(1) To understand the Higgs mechanism, imagine that a room full of physicists quietly chattering is like space filled only with the Higgs field.

(2) ... a well-known scientist walks in, creating a disturbance as he moves across the room, and attracting a cluster of admirers with each step ...

(3) ... this increases his resistance to movement, in other words, he acquires mass, just like a particle moving through the Higgs field ...
(4) ... if a rumour crosses the room ...

(5) ... it creates the same kind of clustering, but this time among the scientists themselves. In this analogy, these clusters are the Higgs particles.
22. The Glashow-Salam-Weinberg model

• Formulation of unified theory of weak and EM interactions
  – 1967-68 S. Weinberg, A. Salam
  – Based in part on work developed previously by S. Glashow

• Theory
  – describes interactions of leptons by the exchange of W bosons and photons
  – Incorporates Higgs mechanism to generate masses of W bosons, while keeping the photon massless
  – G. t'Hooft (1971) demonstrated that the EW theory is renormalisable, i.e. higher order perturbation theory leads to finite
  – The extension of the theory to the hadronic sector is accomplished
  – through the quark model and the GIM mechanism
  – (1973)
Glashow (1961): idea that EM and weak interactions could be unified within the frame of a gauge theory based on

\[ U(1)_Y \times SU(2)_L \]

- abelian group \( U(1)_Y \) related to hypercharge \( Y \),
- non-abelian group \( SU(2)_L \) related to weak isospin \( T \).

Hypercharge related to electric charge \( Q \) and weak charge \( T_3 \) (3\(^{rd}\) component of \( T \)):

\[ Q = T_3 + \frac{Y}{2} \]

Two experimental facts suggested the classification of fermions in weak isodoublets of negative helicity and in weak isosinglets of positive helicity:

- The existence of transitions of type \( \nu_e \rightarrow e \) and \( \nu_\mu \rightarrow \mu \) in weak interactions.
- The occurrence in weak charged currents of fermions of only negative helicity states.

<table>
<thead>
<tr>
<th>Table 22.1. The weak quantum numbers of the leptons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antileptons have opposite values of ( I^w ), ( I^y ), ( Y^w ) and ( Q ).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( I^w )</th>
<th>( I^y )</th>
<th>( Y^w )</th>
<th>( Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_e )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>(-1)</td>
<td>(0)</td>
</tr>
<tr>
<td>( e_L )</td>
<td>( \frac{1}{2} )</td>
<td>(-\frac{1}{2} )</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>( e_R )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>(-2)</td>
<td>(-1)</td>
</tr>
</tbody>
</table>
The three fermion families or generations of SM, separated in right $R$ and left $L$ handed components:

<table>
<thead>
<tr>
<th>Helicity</th>
<th>Generation</th>
<th>$Q$</th>
<th>$T_3$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1.</td>
<td>2.</td>
<td>3.</td>
</tr>
<tr>
<td></td>
<td>($\nu_e$, $e^-$)</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>-1</td>
</tr>
<tr>
<td>$L$</td>
<td>($\nu_\mu$, $\mu^-$)</td>
<td>-1</td>
<td>$-\frac{1}{2}$</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>($\nu_\tau$, $\tau^-$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>($u$, $d'$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>($c$, $s'$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>($t$, $b'$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td>$e_R$</td>
<td>-1</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>$\mu_R$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tau_R$</td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td>$u_R$</td>
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<tr>
<td></td>
<td>$c_R$</td>
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<td></td>
<td>$t_R$</td>
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<td></td>
<td>$d_R$</td>
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<td></td>
<td>$s_R$</td>
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</tr>
<tr>
<td></td>
<td>$b_R$</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>
Local gauge invariance through transformations of the group $SU(2)_L \times U(1)$ imposed to the Lagrangian introduces 4 gauge fields:

- 3 components of an isotriplet $\bar{W}_\mu$ corresponding to the generators of $SU(2)_L$
- an isosinglet $B_\mu$ corresponding to the generators of $U(1)_Y$

In order not to violate the local gauge invariance of the theory, the Lagrangian must not contain quadratic mass terms, such as $M_W^2 W^\mu W_\mu$, where $M_W$ is the mass of the boson $W$. For the same reason, fermions must remain massless.

- **Interaction between leptons conserves**
  - Weak isospin
  - Weak hypercharge

Local (space - time dependent) transformations:

$$G^{SU(2)_L^W}(x) \ell(l_L, W) \rightarrow \ell(l_L^*, W^*)$$

$$G^{U(1)^W}(x) \ell(l_L, e_R, B) \rightarrow \ell(l_L^*, e_R^*, B^*)$$

Total gauge invariance:

$$G^{SU(2)_L^W \times U(1)^W}(x) \ell(l_L, e_R, W, B) \rightarrow \ell(l_L^*, e_R^*, W^*, B^*)$$
EW-symmetry, $SU(2)_L \times U(1)_Y$, spontaneously broken (Salam, Weinberg, 1967) through the Higgs mechanism (1964):

- The starting point is a theory with a scalar field doublet (neutral-charged) $\Phi \equiv \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}$ and its anti-doublet, making 4 real components.

- Three of the Higgs boson scalar fields are absorbed by the longitudinal components of the vector bosons, $W^\pm$ an $Z^0$ which acquire a mass, whereas the photon remains massless.

- The photon remains massless and $U(1)_{EM}$ symmetry remains exact.

- A massive neutral spin-0 boson, the Higgs particle, remains and constitutes currently the weakest point of the GSW theory. The search for it has been/is/will be one of the highest priorities at various experiments at LEP/Tevatron/LHC/LC.

- SSB generates fermion masses through the same doublet of Higgs as for the gauge boson case, by adding a Yukawa interaction of lepton and $\Phi$ fields, which is invariant under $SU(2) \times U(1)$ gauge interactions.
22.2.2 SSB

- Gauge invariance of total Lagrangian under SU(2)xU(1) broken by the neutral Higgs component taking a non-zero vacuum value, $\Phi^0 = R$
  - Corresponding to the state of minimum energy – vacuum expectation value
  - Must redefine the Higgs field so that it is zero at the state of minimum energy
- Weak isospin and weak hypercharge are no longer conserved charges
  - SU(2)xU(1) symmetry now broken
  - However, combination corresponding to electric charge $Q = I_3 + Y/2$ is still conserved
  - U(1) gauge symmetry of WED remains unbroken, and the photon remains massless
- Gauge boson masses are generated by the mixing of R with W and B in $\mathcal{L}_2$
  - Just as Q is mixture of $I_3$ and Y, so is the EM gauge particle a mixture of the neutral gauge particles of weak isospin $W^0$ and weak hypercharge B
  - The remaining parts of $W^0$ and B wavefunctions also mix together to produce another gauge particle $Z^0$

$$A = W^0 \cos \theta_W + B \sin \theta_W \quad ; \quad Z^0 = W^0 \cos \theta_W - B \sin \theta_W$$

Lagrangian terms involving the Higgs field (interaction with gauge bosons and leptons): $\mathcal{L}_2(\Phi, B, W)$ and $\mathcal{L}_3(l_L, e_R, \Phi)$

Total Lagrangian: $\mathcal{L}_1(l_L, e_R, W, B) + \mathcal{L}_2(\Phi, B, W) + \mathcal{L}_3(l_L, e_R, \Phi)$

After redefinition: $\phi^0' = \phi^0 - R$; $\phi^- = \phi^-$

Lagrangian must still describe same physics

$$\mathcal{L}_2(\Phi, B, W) \equiv \mathcal{L}_2'(\Phi', B, W)$$

$$\mathcal{L}_3(l_L, e_R, \Phi) \equiv \mathcal{L}_3'(l_L, e_R, \Phi')$$

Table 22.2. The electroweak Higgs mechanism

Spontaneous symmetry breaking in the electroweak model leads to three massive vector bosons and one massive Higgs boson. Both before and after symmetry breaking, the total number of physical degrees of freedom is 12.

<table>
<thead>
<tr>
<th>Before</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi^-$; $W^-$</td>
<td>$\rightarrow$</td>
</tr>
<tr>
<td>$\phi^+$; $W^-$</td>
<td>$\rightarrow$</td>
</tr>
<tr>
<td>$\phi^0$, $\tilde{\phi}^0$; $W^0$, B</td>
<td>$\rightarrow$</td>
</tr>
</tbody>
</table>

$$A = W^0 \cos \theta_W + B \sin \theta_W \quad ; \quad Z^0 = W^0 \cos \theta_W - B \sin \theta_W$$
Let us switch to the more complicated (non-abelian) SM case:

- We need to generate masses for three gauge bosons $W^\pm$ and $Z$.
- Photon should stay massless since QED is an exact symmetry.

$\implies$ we need at least 3 degrees of freedom for the scalar fields.

The simplest choice is a complex SU(2) doublet of scalar field $\phi$

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{with } Y_\phi = +1$$
To the SM Lagrangian discussed previously,

\[ \mathcal{L}_{\text{SM}} = -\frac{1}{4} F_{\mu\nu}^a F_{\alpha}^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \overline{L} i D_\mu \gamma^\mu \gamma L + \overline{e}_R i D_\mu \gamma^\mu e_R \cdots \]

we need to add the invariant terms of the scalar field part:

\[ \mathcal{L}_S = (D^\mu \Phi) \dagger (D_\mu \Phi) - \mu^2 \Phi \dagger \Phi - \lambda (\Phi \dagger \Phi)^2 \]

with the covariant derivative \( D_\mu \) defined as usual:

\[ D_\mu = \partial_\mu - i g_2 T_a W_\mu^a - i g_1 \frac{Y}{2} B_\mu \]

For \( \mu^2 < 0 \), the neutral component of the doublet \( \Phi \) will develop a vacuum expectation value [the vev should not be in the charged direction to preserve \( U(1)_{\text{QED}} \)]:

\[ \langle \Phi \rangle_0 \equiv \langle 0 | \Phi | 0 \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \text{ with } v = \left( -\frac{\mu^2}{\lambda} \right)^{1/2} \]
We can make now, the usual exercise:

- write $\Phi$ in terms of four fields $\theta_{1,2,3}(x)$ and $H(x)$ at 1st order:

$$
\Phi(x) = e^{i\theta_a(x)\tau^a(x)/v} \left( \begin{array}{c} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{array} \right) \approx \frac{1}{\sqrt{2}} \left( \begin{array}{c} \theta_2 + i\theta_1 \\ v + H - i\theta_3 \end{array} \right)
$$

- make a gauge transformation on $\Phi$ to go to the unitary gauge:

$$
\Phi(x) \rightarrow e^{-i\theta_a(x)\tau^a(x)} \Phi(x) = \frac{1}{2} \left( \begin{array}{c} 0 \\ (v + H(x)) \end{array} \right)
$$
then fully develop the term $|D_\mu \Phi|^2$ of the Lagrangian $\mathcal{L}_S$:

$$|D_\mu \Phi|^2 = |(\partial_\mu - ig_1 \frac{\tau^a}{2} W^a_\mu - ig_2 \frac{1}{2} B_\mu) \Phi|^2$$

$$= \frac{1}{2} \left| \begin{pmatrix} \partial_\mu - \frac{i}{2} (g_2 W^3_\mu + g_1 B_\mu) & -\frac{ig_2}{2} (W^1_\mu - iW^2_\mu) \\ -\frac{ig_2}{2} (W^1_\mu + iW^2_\mu) & \partial_\mu + \frac{i}{2} (g_2 W^3_\mu - g_1 B_\mu) \end{pmatrix} \begin{pmatrix} 0 \\ v + H \end{pmatrix} \right|^2$$

$$= \frac{1}{2} (\partial_\mu H)^2 + \frac{1}{8} g_2^2 (v + H)^2 |W^1_\mu + iW^2_\mu|^2 + \frac{1}{8} (v + H)^2 |g_2 W^3_\mu - g_1 B_\mu|^2$$

define the new fields $W^\pm_\mu$ and $Z_\mu$ [$A_\mu$ is the orthogonal of $Z_\mu$]:

$$W^\pm = \frac{1}{\sqrt{2}} (W^1_\mu \mp W^2_\mu) \ , \ Z_\mu = \frac{g_2 W^3_\mu - g_1 B_\mu}{\sqrt{g_2^2 + g_1^2}} \ , \ A_\mu = \frac{g_2 W^3_\mu + g_1 B_\mu}{\sqrt{g_2^2 + g_1^2}}$$

$$\sin^2 \theta_W \equiv \frac{g_2}{\sqrt{g_2^2 + g_1^2}} = \frac{e}{g_2} \quad (2)$$

and pick up the terms which are bilinear in the fields $W^\pm, Z, A$:

$$M_W^2 W^\mu_\mu W^-\mu + \frac{1}{2} M_Z^2 Z_\mu Z^\mu + \frac{1}{2} M^2 A_\mu A^\mu$$

with the masses: $M_W = \frac{1}{2} v g$, $M_Z = \frac{1}{2} v \sqrt{g^2 + g_1^2}$ and $M_A = 0$
We have achieved (half of) our goal!

By spontaneously breaking the symmetry $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{QED}}$, 3 Goldstones have been eaten by the $W^\pm, Z$ boson to get the masses. The photon remains massless: $U(1)_{\text{QED}}$ symmetry is still unbroken.

What about the masses of the fermions?
Use again the Higgs field $\Phi$ with $Y_\Phi = 1$ and also the isodoublet

$$\tilde{\Phi} = i\tau_2 \Phi^* \quad \text{with } Y_{\tilde{\Phi}} = -1$$

and introduce the $\text{SU}(2) \times \text{U}(1)$ invariant Yukawa Lagrangian:

$$\mathcal{L}_F = -f_e \bar{L} \Phi e_R - f_d \bar{Q} \Phi d_R - f_u \bar{Q} \tilde{\Phi} u_R + \text{h.c.}$$

and repeat the same exercise as before, i.e. $\Phi^* \rightarrow \frac{1}{\sqrt{2}} (0, v + H)$ etc...

$$\mathcal{L}_F = -\frac{1}{\sqrt{2}} f_e (\bar{\nu}_e, \bar{e}_L) \begin{pmatrix} 0 \\ v + H \end{pmatrix} e_R + \cdots = -\frac{1}{\sqrt{2}} (v + H) \bar{e}_L e_R + \cdots$$

Constant term in front of $\bar{e}_L e_R \ (+ \text{h.c.})$ identified with the mass:

$$m_e = f_e v / \sqrt{2} \ , \ m_u = f_u v / \sqrt{2} \ , \ m_d = f_d v / \sqrt{2}$$

So with the same isodoublet $\Phi$ of scalar fields, we have generated the masses of both the gauge bosons and fermions, while preserving the $\text{SU}(2) \times \text{U}(1)$ gauge symmetry (which is now hidden). Not bad!!

**What about the Higgs field itself?**
The kinetic part of the field $H$, $\frac{1}{2}(\partial_\mu H)^2$, comes from the term $|D_\mu \Phi|^2$. Mass and self-interaction part from $V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$:

$$V = \frac{\mu^2}{2}(0, v + H) \begin{pmatrix} 0 \\ v + H \end{pmatrix} + \frac{\lambda}{2}(0, v + H) \begin{pmatrix} 0 \\ v + H \end{pmatrix}^2$$

$$= -\frac{1}{2}\lambda v^2 (v + H)^2 + \frac{1}{4}\lambda (v + H)^4 \quad \text{[using } v^2 = -\mu^2/\lambda]\]

Doing the exercise you find that the Lagrangian containing $H$ is,

$$\mathcal{L}_H = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - V = \frac{1}{2}(\partial^\mu H)^2 - \lambda v^2 H^2 - \lambda v H^3 - \frac{\lambda}{4} H^4$$

The Higgs boson mass is given by: $M_H^2 = 2\lambda v^2 = -2\mu^2$. The Feynman rules* for the Higgs self–interaction vertices are:

$$g_{H^3} = (3!i)\lambda v = 3i \frac{M_H^2}{v} , \quad g_{H^4} = (4!i)\lambda/4 = 3i \frac{M_H^2}{v^2}$$

What about the Higgs boson couplings to gauge bosons and fermions? They were almost derived previously, when we calculated the masses:

$$\mathcal{L}_{M_V} \sim M_V^2 (1 + H/v)^2 , \quad \mathcal{L}_{m_f} \sim -m_f (1 + H/v)$$

Therefore, the Higgs couplings to gauge bosons and fermions are:

$$g_{Hf} = im_f/v , \quad g_{HVV} = -2iM_V^2/v , \quad g_{HHVV} = -2iM_V^2/v^2$$

By the way, the value of the vev is fixed in terms of the $W$ mass:

$$M_W = \frac{1}{2}gv = \left(\frac{\sqrt{2}g^2}{8G_F}\right)^{1/2} \Rightarrow v = 1/(\sqrt{2}G_F)^{1/2} \sim 246 \text{ GeV}$$
The GSW model – the Standard Model – predicted the existence of neutral currents mediated by $W^0 = Z^0$, as well as the masses of the weak intermediate bosons, discovered at CERN.

The most important confirmation of the model:

- 1973: discovery in $\nu - N$ and $\nu - e^-$ of neutral currents $\implies Z^0$ exists!
- 1983: discovery in $\bar{p}p$ annihilations at the CERN $S'p\bar{p}S$ collider of $W^\pm$ and $Z^0$ bosons, propagators of weak interactions, at a mass of $\sim 80$ and $\sim 90 \text{ GeV}/c^2$.

- GSW predicted:

$$M_W = \frac{38.5}{\sin^2 \theta_W} \text{GeV} \quad M_Z = \frac{m_W}{\cos \theta_W} \text{GeV}$$

$$= 80 \text{ GeV} \quad = 91 \text{ GeV}$$

for $\sin^2 \theta_W = 0.23$!

- Although lepton and quark masses are also generated by the same doublets of Higgs – $l_L$ and $l_R$ mixing with $R$ in $\mathcal{L}_3$
  - The predictive power is not the same as the mass generation for the gauge bosons
  - Appropriate coefficients are freely chosen in $\mathcal{L}_3$
  - so as to guaranty the correct lepton mass
Comparison of CC vs NC lead to first determination of $\sin^2 \theta_W$

$$R_{NC/CC} = \frac{\sigma(\nu_\mu + p \rightarrow \nu_\mu + p + \pi^0)}{\sigma(\nu_\mu + p \rightarrow \mu^- + p + \pi^+)} = 0.51 \pm 0.25$$

Currently:

$$\sin^2 \theta_W = 0.23117 \pm 0.00016$$

This leads to prediction: $M_{W^\pm} \approx 80 \text{ GeV}$ ; $M_{Z^0} \approx 90 \text{ GeV}$

The ratios of the cross sections

$$R_\nu = \frac{\sigma_{NC}^{\nu}}{\sigma_{CC}^{\nu}} \quad \text{and} \quad R_\bar{\nu} = \frac{\sigma_{NC}^{\bar{\nu}}}{\sigma_{CC}^{\bar{\nu}}}$$

are given in the electroweak theory in terms of the Weinberg angle $\theta_W$:

$$R_\nu = \frac{1}{2} - \sin^2 \theta_W + (1 + r)\frac{5}{3}\sin^4 \theta_W$$

(1)

and

$$R_\bar{\nu} = \frac{1}{2} - \sin^2 \theta_W + (1 + 1/r)\frac{5}{9}\sin^4 \theta_W ,$$

(2)

where $r$ is the ratio of antineutrino to neutrino CC total cross sections: $r = \sigma_{CC, \bar{\nu}} / \sigma_{CC, \nu} = 0.48 \pm 0.02$ experimentally. On the basis of these ratios, the experiment yielded

The elastic-scattering reactions of neutrinos on atomic electrons,

$$\nu_\mu + e^- \rightarrow \nu_\mu + e^- \quad \text{and} \quad \bar{\nu}_\mu + e^- \rightarrow \bar{\nu}_\mu + e^- ,$$

proceed via NC's. They are characterized by small cross sections—smaller than their hadronic counterparts by the mass ratio $m_e/m_\rho$ because of the smaller c.m. energies—and, for the same reason, by small electron production angles, $\theta_e \approx \sqrt{m_e/E_\nu}$. Until now, these angles have not been resolved by the experiments, so only total cross sections have been measured. The expectations in the electroweak theory are

$$\sigma_{\nu,e} = \frac{G_F^2 E_\nu}{\pi} (1 - 4\sin^2 \theta_W + \frac{16}{3}\sin^4 \theta_W)$$

and

$$\sigma_{\bar{\nu},e} = \frac{G_F^2 E_\nu}{\pi} \left( \frac{1}{3} - \frac{4}{3}\sin^2 \theta_W + \frac{16}{3}\sin^4 \theta_W \right).$$