Effective field theory for photons in media

Finn Ravndal, Institute of Physics, University of Oslo, Norway.

- Physical optics
- Maxwell equations and symmetries
- Minkowski (1908) and Abraham (1909) theories
- Covariant theory and photon mass
- Effective field theory
- Conclusion

ArXiv: 0804.4013, 0805.2606
Refraction:

\[ \sin \theta_0 = n \sin \theta \]

• In vacuum:
  
  \[ \nu_0 \lambda_0 = c_0 \equiv 1 \]

• In medium:

  \[ \nu \lambda = c = \nu_0 \cdot \lambda_0 / n = 1 / n \]
Light wave going through a material medium:

Scattered waves from each atom interfere in the forward direction to give a new wave with shorter wavelength.
Maxwell equations:

\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0, \quad \nabla \cdot \mathbf{D} = 0 \]

Constitutive relations:

\[ \mathbf{D} = \varepsilon \mathbf{E} \quad \mathbf{B} = \mu \mathbf{H} \]

Gives wave equation

\[ \left( n^2 \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{E}(\mathbf{r}, t) = 0 \]

with \[ n = \sqrt{\varepsilon \mu} \]
Covariant formulation

Maxwell equations in rest frame:

\[ F^{\mu\nu} = \begin{pmatrix} 0 & -E \\ E & -B_{ij} \end{pmatrix} \]
\[ \partial_\lambda F^{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0 \]

\[ H^{\mu\nu} = \begin{pmatrix} 0 & -D \\ D & -H_{ij} \end{pmatrix} \]
\[ \partial_\nu H^{\mu\nu} = 0 \]

Boost velocity:

\[ V^\mu = \frac{(1, v)}{\sqrt{1 - v^2}} \]

Covariant form of constitutive equations:

\[ \mu H_{\mu\nu} = F_{\mu\nu} + (n^2 - 1)(F_\mu V_\nu - F_\nu V_\mu) \]
with \[ F^\mu = F^{\mu \nu} V_\nu \]

Can then write \[ \mu H_{\mu \nu} = X^\rho_\mu X_\nu^\sigma F_{\rho \sigma} \]

with \[ X_{\mu \nu} = \eta_{\mu \nu} + (n^2 - 1)V_\mu V_\nu \]

Lagrangian: \[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu} H^{\mu \nu} \]

or \[ \mu \mathcal{L} = -\frac{1}{4} F_{\rho \sigma} F^{\mu \nu} X^\rho_\mu X^\sigma_\nu \]

But Lorentz boosts are not symmetry transformations!
Symmetries

• Invariance under Lorentz transformations with \textit{reduced} speed of light $c=1/n$

• Gauge invariance: $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \Phi$ invariant under

\[
\begin{align*}
\mathbf{A} & \rightarrow \mathbf{A} + \nabla \chi \\
\Phi & \rightarrow \Phi - \partial \chi/\partial t
\end{align*}
\]

Massless photons?
Energy density

\[ E = \frac{1}{2}(E \cdot D + B \cdot H) \]

and conservation:

\[ \frac{\partial E}{\partial t} + \nabla \cdot N = 0 \]

where \( N = E \times H \) is the Poynting vector

Momentum density

\[ G = D \times B \]

and conservation

\[ \frac{\partial G}{\partial t} + \nabla \cdot T = 0 \]

where

\[ T_{ij} = -(E_i D_j + B_i H_j) + \frac{1}{2} \delta_{ij}(E \cdot D + B \cdot H) \]

is the Maxwell stress tensor
With $x^\mu = (t, x)$ can combine to 4-dimensional conservation law $\partial_\nu T^\mu_\nu = 0$ where

$$T^\mu_\nu = \begin{pmatrix} \mathcal{E} & N \\ G & T_{ij} \end{pmatrix}$$

is the Minkowski energy-momentum tensor

Tensor only symmetric in vacuum where $G = N$
Abraham (1909):

\[ T_{M}^{\mu\nu} = T_{A}^{\mu\nu} + (n^2 - 1) \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} \]

where symmetric energy-momentum tensor

\[ T_{A}^{\mu\nu} = \begin{pmatrix} \mathcal{E} & N \\ N & T_{ij} \end{pmatrix} \]

Conservation \[ \partial_\nu T_{M}^{\mu\nu} = 0 \] gives

\[ \partial_\nu T_{A}^{\mu\nu} + K^\mu = 0 \]

where Abraham force

\[ K = (n^2 - 1) \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{H}) \]
Quantization

Photon energy: \( E = \hbar \omega \)

Photon momentum: \( p = \hbar k = nE \)

from Minkowski momentum density \( G = D \times B \)

and \( p = \hbar k / n^2 \) from Abraham momentum density

\[
N = E \times H = G / n^2
\]

Photon mass:

\[
m^2_\gamma = E^2 - p^2 = E^2 \cdot \begin{cases} 1 - n^2 < 0 & \text{Minkowski} \\ 1 - 1/n^2 > 0 & \text{Abraham} \end{cases}
\]
Effective Theory

Light cone in medium rest frame: \( x = \pm t/n \)

Metric: \( ds^2 = \frac{dt^2}{n^2} - dx^2 = \eta_{\mu\nu}dx^\mu dx^\nu \)

with \( x^\mu = (t/n, x) \) or \( \partial_\mu = (n\partial_t, \nabla) \)

so that covariant 4-momentum \( p^\mu = (nE, p) \)

Massless excitation \( p^\mu p_\mu = 0 \) or \( E = p/n \)

as for Minkowski photon.
Covariant electromagnetism

4-vector potential

$A^\mu = (n\Phi, A)$. 

and field tensor

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

or

$F^{\mu\nu} = \begin{pmatrix} 0 & -nE \\ nE & -B_{ij} \end{pmatrix}$

Lagrangian:

$\mu \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$= \frac{1}{2} (n^2 E^2 - B^2)$
Standard energy-momentum tensor

\[ \mu T^{\mu \nu} = F^\mu_\alpha F^{\alpha \nu} + \frac{1}{4} \eta^{\mu \nu} F^\alpha_\beta F^{\alpha \beta} \]

with components

\[ T^{\mu \nu} = \begin{pmatrix} \mathcal{E} & n \mathbf{N} \\ n \mathbf{N} & T_{ij} \end{pmatrix} \]

is now symmetric and conserved on both indices

\[ \partial_\mu T^{\mu \nu} = \partial_\nu T^{\mu \nu} = 0 \]

giving

\[ \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{N} = 0 \]

and

\[ \frac{\partial \mathbf{G}}{\partial t} + \nabla \cdot \mathbf{T} = 0 \]

i.e. Minkowski momentum density!
Physical consequences:

1) Casimir effect:  
\[ E_0 = \sum_k \hbar \omega_k = \frac{1}{n} \sum_k \hbar |k| \]  
i.e.  
\[ E_0 = \frac{1}{n} E_{vac} \]  
I. Brevik and K. Milton, arXiv 0802.2542

2) Thermal radiation:  
\[ U = \int \frac{d^3 k}{(2\pi)^3} \frac{\hbar \omega_k}{e^{\beta \hbar \omega_k} - 1} \]  
i.e.  
\[ U = n^3 U_{vac} \propto n^3 T^4 \]  
Landau & Lifschitz, 1960


**Interactions**

Dielectric has \( \mu = 1 \):

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{d_1}{M^2} \mathbf{E} \cdot \nabla^2 \mathbf{E} + \frac{d_2}{M^4} (\nabla^2 \mathbf{E})^2 + \frac{a}{M^4} (\mathbf{E} \cdot \mathbf{E})^2
\]

describes dispersion

\[
n(\lambda) = A + \frac{B_1}{\lambda^2} + \frac{B_2}{\lambda^4}
\]

\( B_{1,2} \propto d_{1,2} \)

and the Kerr effect

\[
n(E) = n + K E^2
\]

\( K \propto a \)

**Characteristic mass:** \( M = (5 - 10) \text{eV} \)
Conclusion

• Electromagnetic field in medium is like any other excitation in condensed matter physics, theory defined in rest frame only

• Theory similar to Minkowski formulations in most physical predictions

• Effective theory describes dispersion in a natural way

• Quantum corrections can be calculated