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Economic Theory

Green national accounting with a changing population*

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Summary. Following Arrow et al. (2003), this paper considers green national accounting when population is changing and instantaneous well-being depends both on per capita consumption and population size. Welfare improvement is shown to be indicated by an expanded "genuine savings indicator", taking into account the value of population growth, or by an expanded measure of real NNP growth. Under CRS, the measures can be related to the value of per capita stock changes and per capita NNP growth, using a result due to Arrow et al. (2003). The results are compared to those arising when instantaneous well-being depends only on per capita consumption.

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JEL Classification Numbers: D60, D90, O47, Q01.

1 Introduction

How can welfare improvement be measured by national accounting aggregates when population is changing? The answer depends on whether a bigger future population for a given flow of per capita consumption leads to a higher welfare weights for people living at that time, or, alternatively, only per capita consumption mattes.

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When applying discounted utilitarianism to a situation where population changes exogenously through time, it seems reasonable to represent the instantaneous well-being of each generation by the product of population size and the utility derived from per capita consumption. This is the position of 'total utilitarianism', which has been endorsed to by, e.g., Meade (1955) and Mirrlees (1967), and which is the basic assumption in Arrow et al.' (2003) study of savings criteria with a changing population. Within a utilitarian framework, the alternative position of 'average utilitarianism', where the instantaneous well-being of each generation depends only on per capita consumption, have been shown to yield implications that are not ethically defensible.[1]

However, if maximin is applied as a dynamic welfare criterion, then the instantaneous well-being of each generation will be represented by the utility derived from per capita consumption. Moreover, if the welfare criterion cares about sustainability (in the sense that current per capita utility should not exceed what is potentially sustainable), then it becomes important to compare the level of individual utility for different generations, irrespectively of how population size develops. Therefore, utility derived from per capita consumption seems more relevant in a discussion of sustainability.

Following a suggestion by Samuelson (1961, p. 52), the welfare analysis in the present paper does not presuppose a utilitarian framework, and allows for the possibility that a requirement of sustainability is imposed. At this level of generality, one cannot give a definite answer to the question of whether only per capita consumption matters.

In Sections 3–6 of this paper, I follow the basic assumption of Arrow et al. (2003) by letting – in the tradition of 'total utilitarianism' – instantaneous well-being depend not only on per capita consumption, but also population size. Within a model with multiple consumption and capital goods, I derive four ways for indicating welfare improvement, which are generalized or novel results. In Section 7, I compare the results to those arising when instantaneous well-being depends only on per capita consumption, and not on population size. In line with Arrow et al. (2003), I treat population as a form of capital. Section 2 introduces the model, while Section 8 concludes.

2 Model

Following Arrow et al. (2003), I assume that population N develops exogenously over time. The population trajectory $\{N(t)\}_{t=0}^{\infty}$ is determined by the growth function

$$\dot{N} = \phi(N)$$

[1] See Dasgupta (2001b, Section 6.4) for a discussion of the deficiency of 'average utilitarianism'. Dasgupta (2001b, p. 100) suggests 'dynamic average utilitarianism' as an alternative, where discounting seems to arise due to an exogenous and constant per-period probability of extinction. Since the present paper abstracts from any kind of uncertainty, this alternative criterion will not be discussed here.

and the initial condition $N(0) = N^0$. Two special cases are exponential growth,

$$\phi(N) = \nu N\,,$$

where ν denotes the constant growth rate, and logistic growth,

$$\phi(N) = \bar{\nu} N \big(1 - \tfrac{N}{N^*}\big)\,,$$

where $\bar{\nu}$ denotes the maximum growth rate, and N^* denotes the population size that is asymptotically approached. As mentioned by Arrow et al. (2003), the latter seems like the more acceptable formulation in a finite world. In general, denote by $\nu(N)$ the rate of growth of population as a function of N, where

$$\nu(N) = \phi(N)/N\,.$$

Denote by $\mathbf{C} = (C_1, \ldots, C_m)$ the non-negative vector of commodities that are consumed. To concentrate on the issue of intertemporal distribution, I assume that goods and services consumed at any time are distributed equally among the population at that time. Thereby the instantaneous well-being for each individual may be associated with the *utility* $u(\mathbf{c})$ that is derived from the per capita vector of consumption flows, $\mathbf{c} := \mathbf{C}/N$. Assume that u is a time-invariant, increasing, concave, and differentiable function. That u is time-invariant means that all *variable* determinants of current well-being are included in the vector of consumption flows. At any time, labor supply is assumed to be exogenously given and equal to the population size at that time.

Denote by $\mathbf{K} = (K_1, \ldots, K_n)$ the non-negative vector of *capital* goods. This vector includes not only the usual kinds of man-made capital stocks, but also stocks of natural resources, environmental assets, human capital, and other durable productive assets. Corresponding to the stock of capital of type j, K_j, there is a net *investment* flow: $I_j := \dot{K}_j$. Hence, $\mathbf{I} = (I_1, \ldots, I_n) = \dot{\mathbf{K}}$ denotes the vector of net investments.

The quadruple $(\mathbf{C}, \mathbf{I}, \mathbf{K}, N)$ is *attainable* if $(\mathbf{C}, \mathbf{I}, \mathbf{K}, N) \in \mathcal{C}$, where $\mathcal{C}$ is a convex and smooth set, with free disposal of consumption and investment flows. The set of attainable quadruples does not depend directly on time. I thus make an assumption of "green" or *comprehensive* accounting, meaning that current productive capacity depends solely on the vector of capital stocks and the population size. If $\mathcal{C}$ is a cone, then the technology exhibits constant returns to scale. An assumption of constant returns to scale will be imposed *only* in Sections 6 and 7.

Society makes decisions according to a *resource allocation mechanism* that assigns to any vector of capital stocks $\mathbf{K}$ and any population size N a consumption-investment pair $(\mathbf{C}(\mathbf{K}, N), \mathbf{I}(\mathbf{K}, N))$ satisfying that $(\mathbf{C}\,(\mathbf{K}, N), \mathbf{I}(\mathbf{K}, N), \mathbf{K}, N)$ is attainable.[2] I assume that there exists a unique solution $\{\mathbf{K}^*(t)\}_{t=0}^{\infty}$ to the differential equations $\dot{\mathbf{K}}^*(t) = \mathbf{I}(\mathbf{K}^*(t), N(t))$ that satisfies the initial condition $\mathbf{K}^*(0) = \mathbf{K}^0$, where $\mathbf{K}^0$ is given. Hence, $\{\mathbf{K}^*(t)\}$ is the capital path that the resource allocation mechanism implements. Write $\mathbf{C}^*(t) := \mathbf{C}(\mathbf{K}^*(t), N(t))$ and $\mathbf{I}^*(t) := \mathbf{I}(\mathbf{K}^*(t), N(t))$.

Say that the program $\{\mathbf{C}^*(t), \mathbf{I}^*(t), \mathbf{K}^*(t)\}_{t=0}^{\infty}$ is *competitive* if, at each t,

[2] This is inspired by Dasgupta (2001a, p. C20) and Dasgupta and Mäler (2000).

1. $(\mathbf{C}^*(t), \mathbf{I}^*(t), \mathbf{K}^*(t), N(t))$ is attainable,
2. there exist present value prices of the flows of utility, consumption, labor input, and investment, $(\mu(t), \mathbf{p}(t), w(t), \mathbf{q}(t))$, with $\mu(t) > 0$ and $\mathbf{q}(t) \geq 0$, such that
   C1 $\mathbf{C}^*(t)$ maximizes $\mu(t)u(\mathbf{C}/N(t)) - \mathbf{p}(t)\mathbf{C}/N(t)$ over all $\mathbf{C}$,
   C2 $(\mathbf{C}^*(t), \mathbf{I}^*(t), \mathbf{K}^*(t), N(t))$ maximizes $\mathbf{p}(t)\mathbf{C} - w(t)N + \mathbf{q}(t)\mathbf{I} + \dot{\mathbf{q}}(t)\mathbf{K}$ over all $(\mathbf{C}, \mathbf{I}, \mathbf{K}, N) \in \mathcal{C}$.

Here C1 corresponds to utility maximization, while C2 corresponds to intertemporal profit maximization. Assume that the implemented program $\{\mathbf{C}^*(t), \mathbf{I}^*(t), \mathbf{K}^*(t)\}_{t=0}^{\infty}$ is competitive with finite utility and consumption values,

$$\int_0^\infty \mu(t)N(t)u(\mathbf{C}^*(t)/N(t))dt \text{ and } \int_0^\infty \mathbf{p}(t)\mathbf{C}^*(t)dt \text{ exist},$$

and that it satisfies a capital value transversality condition,

$$\lim_{t\to\infty} \mathbf{q}(t)\mathbf{K}^*(t) = 0\,. \tag{1}$$

It follows that the implemented program $\{\mathbf{C}^*(t), \mathbf{I}^*(t), \mathbf{K}^*(t)\}_{t=0}^{\infty}$ maximizes

$$\int_0^\infty \mu(t)N(t)u(\mathbf{C}/N(t))dt$$

over all programs that are attainable at all times and satisfies the initial condition. Moreover, writing $\mathbf{c}^*(t) := \mathbf{C}^*(t)/N(t)$, it follows from C1 and C2 that

$$\mathbf{p}(t) = \mu(t)\nabla_{\mathbf{c}} u(\mathbf{c}^*(t))\,, \tag{2}$$

$$w(t) = \mathbf{p}(t)\frac{\partial \mathbf{C}(\mathbf{K}^*(t), N(t))}{\partial N} + \mathbf{q}(t)\frac{\partial \mathbf{I}(\mathbf{K}^*(t), N(t))}{\partial N}\,, \tag{3}$$

$$-\dot{\mathbf{q}}(t) = \mathbf{p}(t)\nabla_{\mathbf{K}}\mathbf{C}(\mathbf{K}^*(t), N(t)) + \mathbf{q}(t)\nabla_{\mathbf{K}}\mathbf{I}(\mathbf{K}^*(t), N(t))\,. \tag{4}$$

3 Welfare analysis

Write $U(K, N) := Nu(\mathbf{C}(K, N)/N)$ and $U^*(t) := U(K^*(t), N(t))$ for the flow of *total utility*. In line with the basic analysis of Arrow et al. (2003), I assume for the next four sections that $U^*(t)$ measures the social level of instantaneous well-being at time t.

Assume that, at time t, society's *dynamic welfare* is given by a Samuelson-Bergson welfare function defined over paths of total utility from time t to infinity, and that this welfare function does not depend on t. Moreover, assume that, for a given initial condition, the optimal path is time-consistent, and that society's resource allocation mechanism implements the optimal path. If the welfare indifference surfaces in infinite-dimensional utility space are smooth, then, at time t, $\{\mu(s\}_{s=t}^{\infty}$ are local welfare weights on total utility flows at different times.[3] Following a standard argument in welfare economics, as suggested by Samuelson (1961,

[3] By identifying the social level of instantaneous well-being at time t with $U^*(t)$, I assume that there are stable welfare indifference surfaces in infinite-dimensional space when the well-being of each generation is measured by total utility, irrespectively of how consumption flows and population size develop. Discounted total utilitarianism leads to linear indifference surfaces in this space.

p. 52) in the current setting, one can conclude that dynamic welfare is increasing at time t if and only if

$$\int_t^\infty \mu(s)\dot{U}^*(s)ds > 0\,. \tag{5}$$

To show that this welfare analysis includes discounted total utilitarianism, assume for the rest of this paragraph only that society through its implemented program maximizes the sum of total utilities discounted at a constant rate ρ. Hence, the dynamic welfare of the implemented program at time t is

$$\int_t^\infty e^{-\rho(s-t)}U^*(s)ds\,.$$

Then the change in dynamic welfare is given by

$$\begin{aligned}\frac{d}{dt}\Big(\int_t^\infty e^{-\rho(s-t)}U^*(s)ds\Big) &= -U^*(t) + \rho\int_t^\infty e^{-\rho(s-t)}U^*(s)ds \\ &= e^{\rho t}\int_t^\infty e^{-\rho s}\dot{U}^*(s)ds\,,\end{aligned}$$

where the second equality follows by integrating by parts. Hence, (5) follows by setting $\{\mu(t)\}_{t=0}^\infty = \{e^{-\rho t}\}_{t=0}^\infty$.

Turn now to the question of how to determine (5) by means of current prices and quantities. Since u is concave and differentiable, and $\mathcal{C}$ is a convex and smooth set, with free disposal of consumption flows, it follows that, at each t,

$$\mathcal{U}(t) := \{(U, \mathbf{I}, \mathbf{K})|\ U = N(t)u(\mathbf{C}/N(t)) \text{ and } (\mathbf{C}, \mathbf{I}, \mathbf{K}, N(t)) \in \mathcal{C}\}$$

is a convex and smooth set. Furthermore, it follows from C1 and C2 that, at each t, $(U^*(t), \mathbf{I}^*(t), \mathbf{K}^*(t))$ maximizes

$$\mu(t)U + \mathbf{q}(t)\mathbf{I} + \dot{\mathbf{q}}(t)\mathbf{K}$$

over all $(U, \mathbf{I}, \mathbf{K}) \in \mathcal{U}(t)$. In particular,

$$-\dot{\mathbf{q}}(t) = \mu(t)\nabla_{\mathbf{K}}U(\mathbf{K}^*(t), N(t)) + \mathbf{q}(t)\nabla_{\mathbf{K}}\mathbf{I}(\mathbf{K}^*(t), N(t))\,. \tag{6}$$

Denote by $\psi(t)$ the marginal value of population growth, measured in present value terms. Since $\psi(t)$ is measured in present value terms, the decrease of the value of population growth, $-\dot{\psi}(t)$, equals the marginal productivity of the population stock:

$$\begin{aligned}-\dot{\psi}(t) = \mu(t)&\frac{\partial U(\mathbf{K}^*(t), N(t))}{\partial N} \\ &+\mathbf{q}(t)\frac{\partial \mathbf{I}(\mathbf{K}^*(t), N(t))}{\partial N} + \psi(t)\phi'(N(t))\,.\end{aligned} \tag{7}$$

It follows from (2) and the definition of $U(K, N)$ that

$$\mu(t)\frac{\partial U(\mathbf{K}^*(t), N(t))}{\partial N} = v(t) + \mathbf{p}(t)\frac{\partial \mathbf{C}(\mathbf{K}^*(t), N(t))}{\partial N}\,,$$

where $v(t) := \mu(t)\big(u(\mathbf{c}^*(t)) - \nabla_{\mathbf{c}} u(\mathbf{c}^*(t))\mathbf{c}^*(t)\big)$ denotes the marginal value of consumption spread, measured in present value terms. Hence, using (3), equation (7) can be rewritten as

$$-\dot{\psi}(t) = v(t) + w(t) + \psi(t)\phi'(N(t)) , \tag{8}$$

and increasing N leads to three different kinds of marginal contributions:

1. *Consumption is spread on more people*: $v(t)$,
2. *Output increases*: $w(t)$,
3. *Population growth increases*: $\psi(t)\phi'(N(t))$.

By combining (6) and (7), one obtains

$$\begin{aligned} \mu\dot{U}^* &= \mu\big(\nabla_{\mathbf{K}} U \cdot \mathbf{I}^* + \tfrac{\partial U}{\partial N} \cdot \phi(N)\big) \\ &= -\big(\dot{\mathbf{q}}\mathbf{I}^* + \mathbf{q}\dot{\mathbf{I}}^* + \dot{\psi}\phi(N) + \psi \tfrac{d}{dt}\big(\phi(N)\big)\big) \\ &= -\tfrac{d}{dt}\big(\mathbf{q}\mathbf{I}^* + \psi\phi(N)\big) . \end{aligned} \tag{9}$$

Assuming that

$$\lim_{t\to\infty} \big(\mathbf{q}(t)\mathbf{I}^*(t) + \psi(t)\phi(N(t))\big) = 0$$

holds as an investment value/population growth value transversality condition, one arrives at the following result by integrating (9) and using (5) as an indicator of welfare improvement.

Proposition 1. *Dynamic welfare is increasing at time t if and only if*

$$\mathbf{q}(t)\mathbf{I}^*(t) + \psi(t)\phi(N(t)) > 0 .$$

This formally generalizes the "genuine savings indicator" to a case with population change by indicating welfare improvement by means of a positive value of net investments and population growth. However, while $\mathbf{q}$ can in principle be observed as market prices in a perfect market economy or calculated as efficiency prices provided the resource allocation mechanism implements an efficient program, one needs to consider how to calculate ψ. This question is posed in the next section.

4 Value of population growth

How can the marginal value of population growth, $\psi(t)$, be calculated? Solving (8) and imposing

$$\lim_{t\to\infty} \psi(t) = 0$$

as a terminal condition yields

$$\psi(t) = \int_t^\infty \tfrac{\phi(N(s))}{\phi(N(t))}\big(v(s) + w(s)\big)ds . \tag{10}$$

Hence, the value of population growth is the integral of an expression that consists of two factors, $v(s) + w(s)$ and $\phi(N(s))/\phi(N(t))$. Let us investigate (10) by discussing these factors.

The sign of $v(s) + w(s)$

If one assumes that the value of consumption, $\mathbf{pC}^*$, exceeds the total functional share of labor, wN, then it follows from (2) and the definition of v that $u(\mathbf{c}^*) \leq 0$ is a sufficient condition for $v + w$ to be negative. That $u(\mathbf{c}^*)$ is negative, means that instantaneous well-being is reduced if an additional person is brought into society and offered the existing per capita consumption flows.[4] Note that, since u is increasing, $u(\mathbf{c})$ is negative for vectors with small consumption flows. E.g., if c is one-dimensional and $u(c) = \ln c$, then $u(c) \leq 0$ if and only if $c \leq 1$.

Since per capita consumption and, thus, utility derived from per capita consumption increases with development, one obtains the conclusion that $v + w < 0$ is more likely to hold for less developed societies.

Since $w > 0$, it is sufficient for $v + w$ to be positive that v and, thus,

$$u(\mathbf{c}^*) - \nabla_{\mathbf{c}} u(\mathbf{c}^*)\mathbf{c}^*$$

are non-negative. That $u(\mathbf{c}^*) - \nabla_{\mathbf{c}} u(\mathbf{c}^*)\mathbf{c}^*$ is positive, means that instantaneous well-being is increased if an additional person is brought into society even when the total consumption flows are kept fixed and must be spread on an additional person.[5] Note that it follows from the concavity of u that $u(\mathbf{c}) - \nabla_{\mathbf{c}} u(\mathbf{c})\mathbf{c}$ is non-decreasing as $\mathbf{c}$ increases along a ray where different commodities are consumed in fixed proportions. E.g., if c is one-dimensional and $u(c) = \ln c$, then $d\big(u(c) - u'(c)c\big)/dc = 1/c$, and $u(c) - u'(c)c \geq 0$ if and only if $c \geq e$.

Since it is reasonable to assume that per capita consumption as well as the marginal productivity of labor increases with development, one obtains the conclusion that $v + w > 0$ is more likely to hold for more developed societies.

Note that v and, thus, ψ are *not* invariant under an additive shift in the utility function (cf. Arrow et al. 2003, p. 224).

The development of $\phi(N(s))/\phi(N(t))$

If there is constant absolute population growth at all future times, then $\phi(N(s))/\phi(N(t))$ equals 1 throughout and (10) simplifies to

$$\psi(t) = \int_t^{\infty} \big(v(s) + w(s)\big) ds \,.$$

If future absolute population growth is lower than the present – which occurs on the decreasing part of a logistic growth function – then $\phi(N(s))/\ \phi(N(t))$ is smaller than 1 throughout and it holds that

$$\psi(t) < \int_t^{\infty} \big(v(s) + w(s)\big) ds \,,$$

[4] Observe that u has not been normalized to satisfy $u(\mathbf{0}) = 0$. The analysis allows for the possibility that there are per capita consumption flows, $\mathbf{c} \geq 0$, such that $u(\mathbf{c}) < 0$. Cf. the concepts of 'well-being subsistence', as discussed by Dasgupta (2001b, Ch. 14) in the tradition of Meade (1955) and Dasgupta (1969), and 'critical-level utilitarianism', as proposed by Blackorby and Donaldson (1984).

[5] As discussed by Dasgupta (2001b, Ch. 14), the condition $u(\mathbf{c}) - \nabla_{\mathbf{c}} u(\mathbf{c})\mathbf{c} = 0$ is important in classical utilitarian theories of optimal population; see also Meade (1955) and Dasgupta (1969).

provided that $v(s) + w(s) > 0$.

If future absolute population growth is higher than the present – which occurs with exponential growth, entailing that the rate of growth of population, $\nu(N) = \phi(N)/N$, is constant – then $\phi(N(s))/\phi(N(t))$ is greater than 1 throughout and it holds that

$$\psi(t) > \int_t^\infty \big(v(s) + w(s)\big) ds \,,$$

provided that $v(s) + w(s) > 0$.

Measuring the value of population growth by the present value of future wages

If it holds that the total value of the current population $N(t)$ valued by $\psi(t)$ is approximated the present value of future wages, then the total value of population growth, $\psi(t)\phi(t)$, can be approximated through multiplying the present value of future wages by the population growth rate, $\nu(N) = \phi(N)/N$.

To investigate the merits of such an approximation, note that

$$\begin{aligned} -\frac{d(\psi N)}{dt} &= -\big(\dot{\psi} N + \psi \dot{N}\big) \\ &= \big(v + w + \psi\phi'(N)\big)N - \psi\phi(N) \\ &= wN \;+\; \mu\big(u(\mathbf{c}^*) - \nabla_{\mathbf{c}} u(\mathbf{c}^*)\mathbf{c}^*\big)N \;+\; \nu'(N)N\psi N, \end{aligned} \tag{11}$$

where $\phi'(N) = d(\nu(N)N)/dN = \nu'(N)N + \nu(N)$ has been used to establish the last equality. Hence, the total value of the current population $N(t)$ valued by $\psi(t)$ can be expressed as follows:

$$\begin{aligned} \psi(t)N(t) = &\int_t^\infty w(s)N(s)ds \\ &+ \int_t^\infty \mu(s)\big(u(\mathbf{c}^*(s)) - \nabla_{\mathbf{c}} u(\mathbf{c}^*(s))\mathbf{c}^*(s)\big)N(s)ds \\ &+ \int_t^\infty \nu'(N(s))N(s)\psi(s)N(s)ds \,, \end{aligned} \tag{12}$$

provided that the following population value transversality condition holds:

$$\lim_{t\to\infty} \psi(t)N(t) = 0 \,. \tag{13}$$

In the special case where u is homogeneous of degree 1, it follows that $u(\mathbf{c}^*) - \nabla_{\mathbf{c}} u(\mathbf{c}^*)\mathbf{c}^* = 0$ throughout, so that in (12) the second term on the rhs. is equal to zero. In the special case where growth is exponential so that the growth rate $\nu(N)$ is constant, it follows that $\nu'(N) = 0$ throughout, so that in (12) the third term on the rhs. is equal to zero. Hence, a linearly homogeneous u combined with exponential population growth are sufficient for the total value of population growth, $\psi(t)\phi(t)$, to be equal to

$$\nu(N) \cdot \int_t^\infty w(s)N(s)ds \,. \tag{14}$$

In a developed society one would expect that

$$\int_t^\infty \mu(s)\big(u(\mathbf{c}^*(s)) - \nabla_\mathbf{c} u(\mathbf{c}^*(s))\mathbf{c}^*(s)\big)N(s)ds > 0\,,$$

while

$$\int_t^\infty \nu'(N(s))N(s)\psi(s)N(s)ds < 0\,,$$

since exponential growth cannot be maintained indefinitely. Hence, in a developed society, the two additional terms in (12) go in different directions, implying that it cannot easily be determined whether (14) over- or underestimates the total value of population growth, $\psi(t)\phi(t)$.

5 Real NNP growth as a welfare indicator

To investigate to what extent real NNP growth indicates welfare improvement in the presence of a changing population, I follow Asheim and Weitzman (2001) and Sefton and Weale (2000) by using a Divisia consumption price index when expressing comprehensive NNP in real prices. The application of a price index $\{\pi(t)\}$ turns the present value prices $\{\mathbf{p}(t), \mathbf{q}(t)\}$ into *real* prices $\{\mathbf{P}(t), \mathbf{Q}(t)\}$,

$$\mathbf{P}(t) = \mathbf{p}(t)/\pi(t)$$
$$\mathbf{Q}(t) = \mathbf{q}(t)/\pi(t)\,,$$

implying that the *real* interest rate, $R(t)$, at time t is given by

$$R(t) = -\tfrac{\dot{\pi}(t)}{\pi(t)}\,.$$

A *Divisia* consumption price index satisfies

$$\frac{\dot{\pi}(t)}{\pi(t)} = \frac{\dot{\mathbf{p}}(t)\mathbf{C}^*(t)}{\mathbf{p}(t)\mathbf{C}^*(t)}\,,$$

implying that $\dot{\mathbf{P}}\mathbf{C}^* = 0$:

$$\dot{\mathbf{P}}\mathbf{C}^* = \tfrac{d}{dt}\left(\frac{\mathbf{p}}{\pi}\right)\mathbf{C}^* = \frac{\pi\dot{\mathbf{p}}\mathbf{C}^* - \dot{\pi}\mathbf{p}\mathbf{C}^*}{\pi^2} = 0\,.$$

Define comprehensive *NNP in real Divisia prices*, $Y(t)$, as the sum of the real value of consumption and the real value of net investments:

$$Y(t) := \mathbf{P}(t)\mathbf{C}^*(t) + \mathbf{Q}(t)\mathbf{I}^*(t)\,.$$

Define likewise real prices for utility, consumption spread, and population growth:

$$M(t) = \mu(t)/\pi(t)$$
$$V(t) = \upsilon(t)/\pi(t)$$
$$\Psi(t) = \psi(t)/\pi(t)\,.$$

Since

$$\dot{\mathbf{Q}}(t) = \dot{\mathbf{q}}(t)/\pi(t) + R(t)\mathbf{Q}(t)$$
$$\dot{\Psi}(t) = \dot{\psi}(t)/\pi(t) + R(t)\Psi(t)\,,$$

it follows from (9) that

$$M\dot{U}^* + \tfrac{d}{dt}\big(\mathbf{QI}^* + \Psi\phi(N)\big) = R\big(\mathbf{QI}^* + \Psi\phi(N)\big)\,. \tag{15}$$

Moreover, keeping in mind that

$$U^* = Nu(\mathbf{c}^*), V = M\big(u(\mathbf{c}^*) - \nabla_{\mathbf{c}}u(\mathbf{c}^*)\mathbf{c}^*\big),$$

$\mathbf{P} = M\nabla_{\mathbf{c}}u(\mathbf{c}^*)$, and $\dot{\mathbf{P}}\mathbf{C}^* = 0$, one obtains

$$\begin{aligned} M\dot{U}^* &= M\Big(\big(u(\mathbf{c}^*) - \nabla_{\mathbf{c}}u(\mathbf{c}^*)\mathbf{c}^*\big)\phi(N) + \nabla_{\mathbf{c}}u(\mathbf{c}^*)\dot{\mathbf{C}}^*\Big) \\ &= V\phi(N) + \tfrac{d}{dt}\big(\mathbf{PC}^*\big)\,. \end{aligned} \tag{16}$$

Hence, by combining (15) and (16), it follows that

$$\dot{Y} + V\phi(N) + \tfrac{d}{dt}\big(\Psi\phi(N)\big) = R\big(\mathbf{QI}^* + \Psi\phi(N)\big)\,.$$

In view of Proposition 1, this leads to the following result.

Proposition 2. *Dynamic welfare is increasing at time t if and only if*

$$\dot{Y}(t) + V(t)\phi(N(t)) + \tfrac{d}{dt}\big(\Psi(t)\phi(N(t))\big) > 0\,,$$

provided that the real interest rate, $R(t)$, is positive.

I end this section by discussing the following question: If national accountants can estimate the "genuine savings indicator", $\mathbf{QI}^*$, and real growth in comprehensive NNP, $\dot{Y}$, but not the terms that capture the welfare effects of population change, which of $\mathbf{QI}^*$ and $\dot{Y}$ is the better indicator of welfare improvement?

If one assumes that, in a more developed society,

- absolute population growth, $\phi(N)$, is positive but decreasing towards zero, and
- the marginal value of consumption spreading, V, is positive, entailing that also the marginal value of population growth, Ψ, is positive,

then it follows that both $V\phi(N)$ and $\Psi\phi(N)$ are positive, but eventually decreasing. As the society is getting near to having population saturated at N^* (if a logistic growth function is followed), then $V\phi(N) > 0$ due to a positive value of consumption spread, while $\frac{d}{dt}\big(\Psi\phi(N)\big) < 0$ since population growth is decreasing towards zero. Hence, when using Proposition 2 and approximating $\dot{Y} + V\phi(N) + \frac{d}{dt}\big(\Psi\phi(N)\big)$ by $\dot{Y}$, one would be missing two terms with opposite signs. On the other hand, when using Proposition 1 and approximating $\mathbf{QI}^* + \Psi\phi(N)$ by $\mathbf{QI}^*$, one would be missing one term with a positive sign. Thus, if it is impractical or impossible to calculate terms involving the value of population change, then real NNP growth, $\dot{Y}$, may be an interesting alternative to the "genuine savings indicator", $\mathbf{QI}^*$, as an approximate indicator of welfare improvement.

In the present section I have considered real growth in total NNP, not real growth in per capita NNP. To be able to analyze per capita measures – as I will do in the next section – one needs to impose an assumption of constant returns to scale. One can, however, use (16) to make the following observation:

$$M\dot{U}^* = Mu(\mathbf{c}^*)\phi(N) \; + \; \mathbf{PC}^* \cdot \Big(\tfrac{d(\mathbf{PC}^*)/dt}{\mathbf{PC}^*} - \nu(N) \Big) \, .$$

Hence, if $u(\mathbf{c}^*)$ is non-negative – so that instantaneous well-being is not decreased if an additional person is brought into society and offered the existing per capita consumption flows – and real per capita consumption increases throughout, then it follows from (5) that dynamic welfare is improving.

6 Per capita measures

In the present section I consider two per capita measures: ‘value of net changes in per capita stocks’ and ‘real growth in per capita NNP’. These will be considered in turn in each of the two following subsections. In both subsections I impose the additional assumption of constant returns to scale.

6.1 Value of net changes in per capita stocks

Denote by $\mathbf{k}^*(t) := \mathbf{K}^*(t)/N(t)$ the vector of per capita capital stocks. Since

$$\dot{\mathbf{k}}^* = \tfrac{\dot{\mathbf{K}}^*}{N} - \tfrac{\dot{N}}{N}\tfrac{\mathbf{K}^*}{N} = \tfrac{\mathbf{I}^*}{N} - \nu(N)\mathbf{k}^*$$

and $\phi(N) = \nu(N)N$, it follows that

$$\frac{\mathbf{qI}^* + \psi\phi(N)}{N} \; = \; \mathbf{q}\dot{\mathbf{k}}^* + \nu(N)\big(\mathbf{qk}^* + \psi\big)$$

(cf. Arrow et al., 2003, p. 222). Hence, by Proposition 1, welfare is increasing at time t if and only if $\mathbf{q}\dot{\mathbf{k}}^* + \nu(N)\big(\mathbf{qk}^* + \psi\big) > 0$. This means that dynamic welfare can be improving even if the *value net changes in per capita stocks*, $\mathbf{q}\dot{\mathbf{k}}^*$, is negative, provided that the term $\nu(N)\big(\mathbf{qk}^* + \psi\big)$ is sufficiently positive. How can $\mathbf{qk}^* + \psi$ be calculated?

To allow analysis of this question – in particular, to derive a generalized version of Arrow et al.’s (2003) Theorem 2 – one must impose constant returns to scale by assuming that $\mathcal{C}$ is a convex cone. Then it follows directly from C2 that, at each t,

$$\mathbf{p}(t)\mathbf{C}^*(t) - w(t)N(t) + \mathbf{q}(t)\mathbf{I}^*(t) + \dot{\mathbf{q}}(t)\mathbf{K}^*(t) = 0 \, , \tag{17}$$

or, equivalently,

$$-\frac{d\big(\mathbf{q}(t)\mathbf{K}^*(t)\big)}{dt} = \mathbf{p}(t)\mathbf{C}^*(t) - w(t)N(t) \, . \tag{18}$$

This means that the value of capital equals the present value of the difference between the value of consumption and the functional share of labor,

$$\mathbf{q}(t)\mathbf{K}^*(t) = \int_t^\infty \big(\mathbf{p}(s)\mathbf{C}^*(s) - w(s)N(s)\big)ds\,,$$

provided that (1) holds as a capital value transversality condition.

It now follows from (2), (11), and (18) that

$$\begin{aligned}-\frac{d(\psi N)}{dt} &= -\big(\mathbf{p}\mathbf{C}^* - wN\big) \ + \ \mu u(\mathbf{c}^*)N \ + \ \nu'(N)N\psi N \\ &= \frac{d(\mathbf{q}\mathbf{K}^*)}{dt} \ + \ N\big(\mu u(\mathbf{c}^*) + \nu'(N)\psi N\big)\,.\end{aligned}$$

or, equivalently,

$$-\frac{d\big(\mathbf{q}\mathbf{K}^* + \psi N\big)}{dt} = N\big(\mu u(\mathbf{c}^*) + \nu'(N)\psi N\big)\,.$$

If (1) and (13) hold as transversality conditions, then

$$\begin{aligned}&\mathbf{q}(t)\mathbf{K}^*(t) + \psi(t)N(t) \\ &= \int_t^\infty N(s)\big(\mu(s)u(\mathbf{c}^*(s)) + \nu'(N(s))\psi(s)N(s)\big)ds\,,\end{aligned}$$

and, by dividing by $N(t)$,

$$\mathbf{q}(t)\mathbf{k}^*(t) + \psi(t) = \int_t^\infty \tfrac{N(s)}{N(t)}\big(\mu(s)u(\mathbf{c}^*(s)) + \nu'(N(s))\psi(s)N(s)\big)ds\,. \quad (19)$$

By differentiating both sides of (19) w.r.t. time, one obtains

$$\begin{aligned}-\frac{d\big(\mathbf{q}\mathbf{k}^* + \psi\big)}{dt} &= \mu u(\mathbf{c}^*) \ + \ \nu'(N)\psi N \ + \ \nu(N)\mathbf{q}\mathbf{k}^* \ + \ \nu(N)\psi \\ &= \mu u(\mathbf{c}^*) \ - \ \nu'(N)\mathbf{q}\mathbf{K}^* \ + \ \phi'(N)\big(\mathbf{q}\mathbf{k}^* + \psi\big)\,,\end{aligned}$$

where I have followed Arrow et al. (2003) by using $\phi'(N) = \nu'(N)N + \nu(N)$ to establish the last equality. Integrating this yields

$$\begin{aligned}&\mathbf{q}(t)\mathbf{k}^*(t) + \psi(t) \\ &= \int_t^\infty \tfrac{\phi(N(s))}{\phi(N(t))}\big(\mu(s)u(\mathbf{c}^*(s)) - \nu'(N(s))\mathbf{q}(s)\mathbf{K}^*(s)\big)ds\,.\end{aligned} \quad (20)$$

In light of Proposition 1, (19) and (20) lead to the following result:

Proposition 3 (Arrow et al., 2003, Theorem 2). *Assuming constant returns to scale, dynamic welfare is increasing at time t if and only if*

$$\mathbf{q}(t)\dot{\mathbf{k}}^*(t) + \nu(N(t))\big(\mathbf{q}(t)\mathbf{k}^*(t) + \psi(t)\big) > 0\,,$$

where

$$\begin{aligned}
&\mathbf{q}(t)\mathbf{k}^*(t) + \psi(t) \\
&= \int_t^{\infty} \tfrac{N(s)}{N(t)} \big(\mu(s)u(\mathbf{c}^*(s)) + \nu'(N(s))\psi(s)N(s)\big)ds \\
&= \int_t^{\infty} \tfrac{\phi(N(s))}{\phi(N(t))} \big(\mu(s)u(\mathbf{c}^*(s)) - \nu'(N(s))\mathbf{q}(s)\mathbf{K}^*(s)\big)ds\,.
\end{aligned}$$

Hence, if $u(\mathbf{c}^*)$ is non-negative – so that instantaneous well-being is not decreased if an additional person is brought into society and offered the existing per capita consumption flows – and $\nu'(N) < 0$ – so that the rate of growth of population decreases as population increases, then $\mathbf{qk}^* + \psi > 0$, meaning that welfare improvement is possible even if the value of net changes in per capita stocks is negative.

In the case of exponential population growth – so that the rate of growth of population is constant – it follows that $\nu'(N) = 0$, $N(s)/N(t) = \phi(N(s))/\phi(N(t))$, and

$$\mathbf{q}(t)\mathbf{k}^*(t) + \psi(t) = \int_t^{\infty} \tfrac{N(s)}{N(t)} \mu(s)u(\mathbf{c}^*(s))ds\,.$$

6.2 Real growth in per capita NNP

In the previous subsection I have translated the result on the generalized "genuine savings indicator" in a setting where there is population growth, Proposition 1, into a finding, Proposition 3, stated in per capita terms. In this subsection I do the same for Proposition 2 by translating a result on real growth in total NNP into a finding stated in terms of *real growth in per capita NNP*. The obtained result will be reported below as Proposition 4. Also in this subsection I impose the additional assumption of constant returns to scale.

Write $y(t) := Y(t)/N(t)$ for real per capita NNP and $\mathbf{i}^*(t) := \mathbf{I}^*(t)/N(t)$ for the vector of per capita investment flows. It follows from (17) that

$$y = \mathbf{Pc}^* + \mathbf{Qi}^* = W + \big(R\mathbf{Q} - \dot{\mathbf{Q}}\big)\mathbf{k}^*$$

since $-\dot{\mathbf{q}}/\pi = R\mathbf{Q} - \dot{\mathbf{Q}}$, with $W(t) = w(t)/\pi(t)$ denoting the real wage rate. Moreover,

$$\dot{y} = \tfrac{\dot{Y}}{N} - \tfrac{\dot{N}}{N}\tfrac{Y}{N} = \tfrac{\dot{Y}}{N} - \nu(N)y$$

Hence, by combining these two observations it follows that

$$\tfrac{\dot{Y}}{N} = \dot{y} + \nu(N)W + \nu(N)\big(R\mathbf{Q} - \dot{\mathbf{Q}}\big)\mathbf{k}^*\,. \tag{21}$$

Likewise, since $-\dot{\psi}/\pi = V + W + \Psi\phi'(N)$ and $-\dot{\psi}/\pi = R\Psi - \dot{\Psi}$, one obtains

$$V + W + \Psi\phi'(N) + \dot{\Psi} = R\Psi\,. \tag{22}$$

The stage is now set for expressing $\dot{Y} + V\phi(N) + \frac{d}{dt}\big(\Psi\phi(N)\big)$ in per capita terms and thus, derive a fourth expression that indicates welfare improvement: Using (21) and (22) it follows through tedious but straightforward calculations that

$$\tfrac{1}{N}\big(\dot{Y} + V\phi(N) + \tfrac{d}{dt}(\Psi\phi(N))\big) \;=\; \dot{y} \;-\; \nu(N)\dot{\mathbf{Q}}\mathbf{k}^* \;+\; \nu(N)R\big(\mathbf{Q}\mathbf{k}^* + \Psi\big)\,.$$

On the basis of Propositions 2 and 3, this leads to the following result.

Proposition 4. *Assuming constant returns to scale, dynamic welfare is increasing at time t if and only if*

$$\dot{y}(t) \;-\; \nu(N(t))\dot{\mathbf{Q}}(t)\mathbf{k}^*(t) \;+\; \nu(N(t))R(t)\big(\mathbf{Q}(t)\mathbf{k}^*(t) + \Psi(t)\big) \;>\; 0\,,$$

provided that the real interest rate, $R(t)$, is positive, where

$$\begin{aligned}&\mathbf{Q}(t)\mathbf{k}^*(t) + \Psi(t)\\ &= \int_t^\infty \tfrac{N(s)}{N(t)}\big(M(s)u(\mathbf{c}^*(s)) + \nu'(N(s))\Psi(s)N(s)\big)ds\\ &= \int_t^\infty \tfrac{\phi(N(s))}{\phi(N(t))}\big(M(s)u(\mathbf{c}^*(s)) - \nu'(N(s))\mathbf{Q}(s)\mathbf{K}^*(s)\big)ds\,.\end{aligned}$$

In the setting of a one-sector model like the one considered by Arrow et al. (2003) – where consumption, investment, and capital are all one-dimensional, and output is split between consumption and investment – the anticipated capital gains are zero: $\dot{Q} = 0$. Under this simplifying assumption one can draw the following conclusion from Proposition 4: If a positive interest rate R, a non-negative $u(c^*)$, and a positive and decreasing $\nu(N)$ lead to $\nu(N)R\big(Qk^* + \Psi\big)$ being positive, then welfare improvement is possible even if real per capita NNP is decreasing.

7 Welfare when only per capita consumption matters

In this section I investigate how the welfare analysis will change if I – instead of letting total utility, $N(t)u(\mathbf{c}^*(t))$, constitute the instantaneous well-being at time t – let instantaneous well-being at time t depend only on the utility derived from the vector of per capita consumption flows, $u(\mathbf{c}^*(t))$, but not on the size of the population, $N(t)$. This is the underlying assumption made by, e.g., Hamilton (2002).

The following is a straightforward adaptation of the analysis of Section 3. Write $\tilde{U}(K,N) := u(\mathbf{C}(K,N)/N)$ and $\tilde{U}^*(t) := \tilde{U}(K^*(t), N(t))$ for the flow of per capita utility. In the alternative welfare analysis, $\tilde{U}^*(t)$ measures the social level of instantaneous well-being at time t, and dynamic welfare is increasing at time t if and only if

$$\int_t^\infty \tilde{\mu}(s)\dot{\tilde{U}}^*(s)ds > 0\,, \tag{23}$$

where, for each t, $\tilde{\mu}(t) = \mu(t)N(t)$.

In analogy to the demonstration in Section 3, it can be shown that this welfare analysis includes discounted utilitarianism, where society through its implemented program maximizes the sum of per capita utilities discounted at a constant rate ρ. However, as pointed out by Dasgupta (2001b, Section 6.4), it is hard to offer an ethical defense for such discounted average utilitarianism.

The framework of Asheim and Buchholz (2002) can, however, be used show how the welfare analysis of Section 3 applies also to cases like maximin, where society through its implemented program maximizes infimum of per capita utilities, and welfare criteria that impose sustainability as a constraint. In such cases, it seems appropriate to adopt the assumption of the present section and let instantaneous well-being depend only on per capita consumption, and not on population size (see also Pezzey 2003, Section 4).

Since u is concave and differentiable, and $\mathcal{C}$ is a convex and smooth set, with free disposal of consumption flows, it follows that, at each t,

$$\tilde{\mathcal{U}}(t) := \{(\tilde{U}, \mathbf{I}, \mathbf{K}) |\ \tilde{U} = u(\mathbf{C}/N(t)) \text{ and } (\mathbf{C}, \mathbf{I}, \mathbf{K}, N(t)) \in \mathcal{C}\}$$

is a convex and smooth set. Furthermore, it follows from C1 and C2 that, at each t, $(\tilde{U}^*(t), \mathbf{I}^*(t), \mathbf{K}^*(t))$ maximizes

$$\tilde{\mu}(t)\tilde{U} + \mathbf{q}(t)\mathbf{I} + \dot{\mathbf{q}}(t)\mathbf{K}$$

over all $(\tilde{U}, \mathbf{I}, \mathbf{K}) \in \tilde{\mathcal{U}}(t)$. In particular,

$$-\dot{\mathbf{q}}(t) = \tilde{\mu}(t)\nabla_{\mathbf{K}}\tilde{U}(\mathbf{K}^*(t), N(t)) + \mathbf{q}(t)\nabla_{\mathbf{K}}\mathbf{I}(\mathbf{K}^*(t), N(t))\,. \tag{24}$$

Denote by $\tilde{\psi}(t)$ is the marginal value of population growth, measured in present value terms, under the alternative welfare analysis. It follows that

$$\begin{aligned} -\dot{\tilde{\psi}}(t) &= \tilde{\mu}(t)\frac{\partial\tilde{U}(\mathbf{K}^*(t), N(t))}{\partial N} + \mathbf{q}(t)\frac{\partial\mathbf{I}(\mathbf{K}^*(t), N(t))}{\partial N} + \tilde{\psi}(t)\phi'(N(t)) \\ &= \tilde{v}(t) + w(t) + \tilde{\psi}(t)\phi'(N(t))\,, \end{aligned} \tag{25}$$

where the second equality follows from (2), (3), and the definition of $\tilde{U}\ (K, N)$, with $\tilde{v}(t) := -\mathbf{p}(t)\mathbf{c}^*(t)$ denoting the marginal value of consumption spread, measured in present value terms, under the alternative welfare analysis. Hence, increasing N leads to three different kinds of marginal contributions:

1. *Consumption is spread on more people*: $\tilde{v}(t)$,
2. *Output increases*: $w(t)$,
3. *Population growth increases*: $\tilde{\psi}(t)\phi'(N(t))$.

By combining (24) and (25), one obtains

$$\begin{aligned} \tilde{\mu}\dot{\tilde{U}}^* &= \tilde{\mu}\big(\nabla_{\mathbf{K}}\tilde{U} \cdot \mathbf{I}^* + \tfrac{\partial\tilde{U}}{\partial N} \cdot \phi(N)\big) \\ &= -\big(\dot{\mathbf{q}}\mathbf{I}^* + \mathbf{q}\dot{\mathbf{I}}^* + \dot{\tilde{\psi}}\phi(N) + \tilde{\psi}\tfrac{d}{dt}\big(\phi(N)\big)\big) \\ &= -\tfrac{d}{dt}\big(\mathbf{q}\mathbf{I}^* + \tilde{\psi}\phi(N)\big)\,. \end{aligned} \tag{26}$$

Assuming that

$$\lim_{t\to\infty}\big(\mathbf{q}(t)\mathbf{I}^*(t)+\tilde{\psi}(t)\phi(N(t))\big)=0$$

holds as an investment value/population growth value transversality condition, one arrives at the following result by integrating (26), and using (23) as an indicator of welfare improvement.

Proposition 5. *Dynamic welfare is increasing at time* t *if and only if*

$$\mathbf{q}(t)\mathbf{I}^*(t)+\tilde{\psi}(t)\phi(N(t))>0\,.$$

Solving (25) and imposing

$$\lim_{t\to\infty}\tilde{\psi}(t)=0$$

as a terminal condition yields

$$\tilde{\psi}(t)=-\int_t^\infty \tfrac{\phi(N(s))}{\phi(N(t))}\big(\mathbf{p}(s)\mathbf{c}^*(s)-w(s)\big)ds\,.$$

It follows that the marginal value of population growth is negative, provided that the value of consumption, $\mathbf{pC}^*$, exceeds the total functional share of labor, wN. This reflects that population growth means that society's assets must be shared among more people, while – under this alternative welfare analysis – there is no countervailing effect. Note that $\tilde{\psi}$ *is* invariant under an additive shift in the utility function.

By repeating the analysis of Section 5, it follows that dynamic welfare is increasing at time t if and only if

$$\dot{Y}(t)+\tilde{V}(t)\phi(N(t))+\tfrac{d}{dt}\big(\tilde{\Psi}(t)\phi(N(t))\big)>0\,,$$

provided that the real interest rate, $R(t)$, is positive, where, under the alternative welfare analysis, $\tilde{V}(t)=\tilde{v}(t)/\pi(t)$ is the marginal value of consumption spread, and $\tilde{\Psi}(t)=\tilde{\psi}(t)/\pi(t)$ is the marginal value of population growth, measured in real terms.

Adapting the analysis of Section 6.1 to the welfare criterion considered in the present section, yields the following result:

Proposition 6. *Assuming constant returns to scale, dynamic welfare is increasing at time* t *if and only if*

$$\mathbf{q}(t)\dot{\mathbf{k}}^*(t)+\nu(N(t))\big(\mathbf{q}(t)\mathbf{k}^*(t)+\tilde{\psi}(t)\big)>0\,,$$

where

$$\begin{aligned}\mathbf{q}(t)\mathbf{k}^*(t)+\tilde{\psi}(t)&=\int_t^\infty \tfrac{N(s)}{N(t)}\nu'(N(s))\tilde{\psi}(s)N(s)ds\\ &=-\int_t^\infty \tfrac{\phi(N(s))}{\phi(N(t))}\nu'(N(s))\mathbf{q}(s)\mathbf{K}^*(s)ds\,.\end{aligned}$$

If a positive and decreasing $\nu(N)$ leads to $\nu(N)\big(\mathbf{q}\mathbf{k}^* + \tilde{\psi}\big)$ being positive (since $\nu'(N) < 0$, and keeping in mind that $\tilde{\psi} < 0$), then welfare improvement is possible even if the value of net changes in per capita stocks is negative. This result has a clear intuitive interpretation: When the rate of population growth is decreasing, it is not necessary for the current generation to compensate fully for current population growth in order for dynamic welfare to be non-decreasing.

Since $\nu'(N) = 0$ if population growth is exponential, one obtains as a corollary the following result, shown by Hamilton (2002) under discounted average utilitarianism and by Dasgupta (2001b, p. 258) under 'dynamic average utilitarianism'.[6]

Proposition 7. *Assuming constant returns to scale and exponential population growth, dynamic welfare is increasing at time t if and only if*

$$\mathbf{q}(t)\dot{\mathbf{k}}^*(t) > 0\,.$$

Note that this result does *not* entail that dynamic welfare is increasing if and only if real per capita wealth is increasing, since time-differentiating real per capita wealth, $\mathbf{Q}\mathbf{K}^*/N$, yields

$$\frac{d(\mathbf{Q}\mathbf{K}^*/N)}{dt} = \frac{\mathbf{q}\dot{\mathbf{k}}^*}{\pi} + \dot{\mathbf{Q}}\mathbf{k}^*\,,$$

where it does *not* follow from our assumptions that $\dot{\mathbf{Q}}\mathbf{k}^* = 0$. However, since

$$\tfrac{\mathbf{q}\dot{\mathbf{k}}^*}{\pi} = \tfrac{\mathbf{Q}\dot{\mathbf{K}}^*}{N} - \tfrac{\dot{N}}{N}\tfrac{\mathbf{Q}\mathbf{K}^*}{N} = \tfrac{\mathbf{Q}\mathbf{K}^*}{N}\big(\tfrac{\mathbf{Q}\mathbf{I}^*}{\mathbf{Q}\mathbf{K}^*} - \nu\big)\,,$$

it follows that $\mathbf{q}\dot{\mathbf{k}}^*$ can be signed by comparing the ratio of the value of net investments and wealth with the rate of growth of population.[7]

By repeating the analysis of Section 6.2, it follows that

$$\dot{y} \;-\; \nu(N)\dot{\mathbf{Q}}\mathbf{k}^* \;+\; \nu(N)R\big(\mathbf{Q}\mathbf{k}^* + \tilde{\Psi}\big) = R\big(\mathbf{Q}\dot{\mathbf{k}}^* + \nu(N)(\mathbf{Q}\mathbf{k}^* + \tilde{\Psi})\big)\,. \quad (27)$$

Hence, Proposition 6 implies that dynamic welfare is increasing at time t if and only if

$$\dot{y}(t) \;-\; \nu(N(t))\dot{\mathbf{Q}}(t)\mathbf{k}^*(t) \;+\; \nu(N(t))R(t)\big(\mathbf{Q}(t)\mathbf{k}^*(t) + \tilde{\Psi}(t)\big) > 0\,,$$

provided that the real interest rate, $R(t)$, is positive. Define $z(t)$ by

$$z(t) := \mathbf{P}(t)\mathbf{c}^*(t) + \mathbf{Q}(t)\dot{\mathbf{k}}^*(t)\,.$$

Since $y = \mathbf{P}\mathbf{c}^* + \mathbf{Q}\mathbf{i}^*$ and $\dot{\mathbf{k}}^* = \mathbf{i}^* - \nu(N)\mathbf{k}^*$, it follows that

$$\begin{aligned}\dot{z} &= \tfrac{d}{dt}\big(y - \nu(N)\mathbf{Q}\mathbf{k}^*\big) \\ &= \dot{y} - \nu(N)\dot{\mathbf{Q}}\mathbf{k}^* - \nu(N)\mathbf{Q}\dot{\mathbf{k}}^* - \nu'(N)\phi(N)\mathbf{Q}\mathbf{k}^*\,.\end{aligned} \quad (28)$$

By combining (27) and (28) with an assumption of exponential population growth (so that $\nu'(N) = 0$ and Proposition 6 implies $\mathbf{Q}\mathbf{k}^* + \tilde{\Psi} = 0$), it follows that

$$\dot{z}(t) = \big(R(t) - \nu(N(t))\big)\mathbf{Q}(t)\dot{\mathbf{k}}^*(t)\,,$$

which by Proposition 7 means that the following result is obtained.

[6] Cf. footnote 1. 'Dynamic average utilitarianism' (as introduced by Dasgupta 2001b, pp. 100, 258) coincides with discounted average utilitarianism when population growth is exponential.

[7] I am grateful to Partha Dasgupta for making this observation.

Proposition 8. *Assuming constant returns to scale and exponential population growth, dynamic welfare is increasing at time t if and only if*

$$\dot{z}(t) > 0 \, ,$$

provided that the real interest rate net of population growth, $R(t) - \nu(N(t))$*, is positive.*

It is important to observe that $\dot{z}$ is *not* real growth in per capita NNP; rather, $\dot{z}$ is real growth in the sum of the value of per capita consumption and the value of net changes in per capita stocks.

8 Concluding remarks

In this paper, I have followed a standard argument in welfare economics – which was suggested by Samuelson (1961, p. 52) in the current setting – and identified welfare improvement with

$$\int_t^{\infty} \mu(s) \tfrac{d}{ds} \big(N(s) u(\mathbf{c}^*(s))\big) ds > 0 \, ,$$

if both population size and per capita consumption contribute to instantaneous well-being, or,

$$\int_t^{\infty} \tilde{\mu}(s) \tfrac{d}{ds} \big(u(\mathbf{c}^*(s))\big) ds > 0 \, .$$

if only per capita consumption matters. In each case, the analysis encompasses discounted utilitarianism (which, however, seems more defendable in the former case).

Through Propositions 1–8 I have established eight ways to indicate welfare improvement, depending on which welfare criterion is adopted, on whether population growth is exponential, and on whether the technology exhibits constant returns to scale. Thereby, this paper offers a substantial generalization and extension of Arrow et al.' (2003) analysis and results.

These two cases are of interest for different reasons: Following Arrow et al. (2003, p. 221), there are strong arguments in favor of associating instantaneous well-being with *total utility*, $Nu(\mathbf{c}^*)$, when investigating whether utilitarian welfare is increasing over time. However, in line with Pezzey (2003), one might argue that *per capita utility*, $u(\mathbf{c}^*)$, is more relevant in a discussion of sustainability. This would mean that development is said to be sustainable at the current time, if the current level of individual utility derived from per capita consumption flows, $u(\mathbf{c}^*)$, can potentially be sustained indefinitely.

If per capita utility, $u(\mathbf{c}^*)$, is sustained throughout, so that $d(u(\mathbf{c}^*))/dt \geq 0$ at all times, then it follows directly from the criterion for welfare improvement that welfare is non-decreasing, in the case where only per capita consumption matters for instantaneous well-being. However, the converse implication does not hold: Non-decreasing welfare does not imply that current instantaneous well-being can potentially be maintained forever. In fact, it can be shown (cf. Pezzey 2003, Section 4) – under the provision that $\{\tilde{\mu}(t)\}_{t=0}^{\infty}$ is an exponentially decreasing function – that non-decreasing welfare is a necessary, but not sufficient, condition for sustainable per capita utility.

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