A Unique Solution to \( n \)-Person Sequential Bargaining

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Rubinstein's two-person sequential bargaining model yields a unique subgame-perfect equilibrium; this uniqueness result does not hold with three or more persons. Based on Greenberg's theory of social situations uniqueness is here offered as follows: Assume a player can profitably reject a suggested path (at one of his decision nodes) by suggesting a new one to later players, if by doing so he gains more than \( \epsilon \). A path is acceptable if and only if no player can profitably reject it by suggesting another acceptable path. It is shown that only the "stationary" division is acceptable for any \( \epsilon > 0 \). Journal of Economic Literature Classification Number: C72, C78.

1. INTRODUCTION

Rubinstein (1982) models two-person bargaining as an extensive game by making use of a game structure in which two players take turns in making offers. In each round one player makes an offer; then the other player either accepts or rejects this offer; in case of rejection the rejector makes the next offer, etc.

The most attractive feature of the Rubinstein model lies in the fact that the natural restriction of subgame-perfectness suffices to ensure the uniqueness of equilibrium. This result relies on a positive time cost due to discounting. Unfortunately, uniqueness does not carry over when the

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Rubinstein model is generalized in a natural way to \( n \)-person bargaining.\(^1\) In particular, a well-known counterexample attributed to Shaked by Herrero (1985, Proposition 4.1) and Sutton (1986) shows that with \( n \geq 3 \) and with a common discount factor \( \delta \geq 1/(n - 1) \), any division can be sustained as a subgame-perfect equilibrium (SPE).

This is a disturbing result. The Rubinstein model is most appealing for discount factors close to one (corresponding to a short time interval between bargaining rounds) because then the first player advantage to the initial offerer is small. However, it is just in this case that the Rubinstein model loses its ability to predict the outcome of the \( n \)-person bargaining process.

The above observation has lead to a search for suitable refinements of the concept of an SPE in the current application. In particular, one has wanted to demonstrate under what alternative concepts the unique subgame-perfect division for \( \delta < 1/(n - 1) \), viz., that player \( i \) receives \( \delta^{i-1} \cdot (1 - \delta)/(1 - \delta^n) \) (see Herrero, 1985, Proposition 4.3), remains the unique solution also for \( \delta \geq 1/(n - 1) \).

Important results of this research program are as follows:

- Restriction to stationary ("history-free") strategies yields a unique SPE in which each player \( i \) receives \( \delta^{i-1} \cdot (1 - \delta)/(1 - \delta^n) \).
- If the game is truncated after \( T \) rounds, then there is a unique SPE division, which approaches that specified by the stationary equilibrium as \( T \to \infty \).
- There is a unique strong perfect equilibrium yielding the stationary division.\(^2\)
- If the players are restricted to strategies varying continuously with the preceding history, then again the only SPE is the stationary equilibrium.\(^3\)

Here we offer an alternative route to the uniqueness of the stationary division. Before outlining the present approach, some discussion of the plausibility of Shaked's ingeniously constructed equilibria may be warranted.\(^4\) In particular, consider an equilibrium where the initial offerer (player 1) is to offer the entire cake to player 2 (see Fig. 3 of Sutton, 1986).

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\(^1\) In Herrero's (1985) generalization of the Rubinstein model, the \( n \) players rotate in making offers, each offer is voted on in sequence, and acceptance requires unanimity. See Section 2 and Fig. 1 for a detailed description of the game tree. Other generalizations are due to Haller (1986) and Chae and Yang (1988); the former yielding uniqueness for no discount factor, the latter for any discount factor.

\(^2\) Herrero (1985, Proposition 4.4).

\(^3\) This result is attributed to Binmore by Sutton (1986).

\(^4\) For a more thorough discussion of Shaked's construction, see Herrero (1985, Proposition 4.1) and Sutton (1986).
The reason player 1 chooses to make this offer is that—according to the equilibrium—any alternative offer will leave him no part of the cake. This again hinges on the fact that player 1 is not able to influence the strategies of the other players in the subgame starting with the voting on some alternative offer of his. The consequence is that the actual bargaining does not occur in the extensive form of the game; it is over once the players (through preplay communication?) have coordinated on an equilibrium.

Hence, the multiplicity of equilibria in the multiperson case in this sense seems to undermine the Rubinstein model as a model of the bargaining process. The real bargaining occurs through the (unmodeled) selection of the equilibrium to be played.

The present approach is based on Greenberg's (1990) theory of social situations as applied in (Greenberg, 1990, Chap. 8) to the analysis of extensive form games with perfect information. It captures n-person sequential bargaining by assuming that a player can reject a suggested path by departing from the path at one of his decision nodes and suggesting a new path to be followed by later players. However, the rejection is profitable only if the player gains more than \( \varepsilon \). The nonnegative \( \varepsilon \) may be looked upon as a cost of raising one's voice in the bargaining process. Alternatively, \( \varepsilon \) may be interpreted as a kind of bounded rationality; i.e., the players do not care to act if their utility is within \( \varepsilon \) from what could be attained through a different line of action. Consider a standard of behavior (SB) that for each node of the game tree admits a subset of the paths originating at this node. A path originating at some node is called acceptable at the node if it is admitted by the SB. Impose optimistic internal stability on an SB by requiring that no player can profitably reject an acceptable path if constrained to suggest another acceptable path to later players. Impose optimistic external stability on an SB by requiring that some player can profitably reject an unacceptable path even if constrained to suggest an acceptable path to later players. An optimistic internally and externally stable SB is called optimistic stable. No other requirement (e.g., stationarity) is imposed on an SB.

Section 2 formally introduces the model and concepts described above. Section 3 states as Proposition 1 (due to Greenberg, 1990) that, for the game considered, there exists a unique optimistic stable SB for any positive \( \varepsilon \). This allows for the characterization for each \( \varepsilon > 0 \) of the set of divisions that are acceptable given this unique optimistic stable SB, and leads to the result established through Proposition 2, viz., that only the stationary division is acceptable for any \( \varepsilon > 0 \). In the subsequent discussion in Section 4, it is pointed out that this uniqueness result is driven by the ability of a player to suggest any path to later players, of course taking into account that these may in turn reject the suggested path by suggesting another path from then on. Hence, in contrast to the situation entailed in an SPE, a deviating player is able to influence the actions taken by later
players, this influence being only limited by the ability of the later players
to do so in turn. All proofs are contained in Section 5.

2. The Model and Concepts

Consider the following game of perfect information, where \( N = \{1, 2, \ldots, n\} \) is the set of players. Player 1 at time 0 offers a division \( z \in \mathbb{R}^n_+ \) of a cake of size 1 between the \( n \) players (allowing for the possibility that some part of the cake is not allocated), with players 2, 3, \ldots, \( n \) voting in sequence to accept (A) to reject (R) player 1’s offer. If they all accept, then \( z \) is implemented at time 0 and the game ends. If one player rejects, then at time 1 player 2 becomes the offerer, with players 3, 4, \ldots, \( n \), 1 voting in sequence. Unanimous acceptance ends the game, while a rejection makes player 3 the offerer at time 2. The game continues in this manner until an offer has been accepted by all players but the offerer. The corresponding game tree, denoted \( G \), is partially illustrated in Fig. 1 and is described in Herrero (1985).

Let \( H \) denote the set of nodes in this game tree. This set can be split into sets of terminal nodes, \( H_0 \), and decision nodes, \( H \setminus H_0 = \bigcup_{i \in N} H_i \), where \( H_i \) is the set of decision nodes of player \( i \). Denote the initial node by 0. At any node \( h \in H \), let \( t^h \) denote the number of preceding rejected offers. At a terminal node \( h \in H_0 \), let \( z^h \) denote the offer that was eventually unanimously accepted. If \( h \in H_0 \) is reached, the payoff of each player \( i \) is given by \( n_i(z, t) \), where \( n_i : \mathbb{Z} \times \{0, 1, 2, \ldots\} \to \mathbb{R}^+ \) with \( \mathbb{Z} := \{z \in \mathbb{R}^n_+ \mid z_i \geq 0 \text{ for } i \in N, \text{ and } \Sigma_{i \in N} z_i \leq 1\} \). Consider the following assumptions.

(A1) For each \( i \), \( n_i(z, t) \) converges uniformly to 0 as \( t \to \infty \).

(A2) For each \( i \), \( n_i(z, t) = \delta^t \cdot z_i \), where \( \delta \in (0, 1) \).

Clearly, (A2) implies (A1).

If \( h \in H \), then a feasible path at \( h \), denoted \( x = (x_1, x_2, \ldots) \), is either a finite sequence of feasible actions leading up to a terminal node (i.e., \( k = (h, x) \in H_0 \)) or an infinite sequence of feasible actions. Define \( X^h \) as the set of feasible paths at \( h \). For notational convenience, write \( X^h = \{0\} \) if \( h \in H_0 \), understanding that \((h, 0) = h \). Define \( u^h_i : X^h \to \mathbb{R} \), the utility function of player \( i \) at \( h \), by

if \( x \) is finite, then \( u^h_i(x) = \pi_i(z^k, t^k) \) with \( k = (h, x) \)

if \( x \) is infinite, then \( u^h_i(x) = 0 \).

A subtree of \( G \) with root \( h \) is described by \((N; X^h; u^h_i(\cdot), i \in N)\). Define \( P : H \setminus \{0\} \sim H \) by the property that \( P(k) \) is the immediate predecessor of the node \( k \). Let \( x = (x_1, \ldots, x_j, \ldots) \in X^h \). Player \( i \) is said to be able to induce \( k \) from \( h \) through \( x \) if \( P(k) \in H_i \) and \( P(k) \) is reachable from \( h \) through \( x \) (i.e., \( P(k) = (h, x_1, \ldots, x_j) \) for some \( j \)).
Fig. 1. The game tree $G$.

A standard of behavior (SB) for $G$ is a correspondence $\sigma$ assigning to each subtree $h \in H$ a subset $\sigma(h)$ of $X^h$. An SB $\sigma$ for $G$ is said to be optimistic internally stable against rejection at $\varepsilon$-cost if

$$\text{IS } h \in H \text{ and } x \in \sigma(h) \text{ imply that for any } i \in N \text{ there do not exist } k \in H \text{ and } y \in \sigma(k) \text{ such that } i \text{ can induce } k \text{ from } h \text{ through } x \text{ and } u_i^h(x) < u_i^k(y) - \varepsilon.$$  

An SB $\sigma$ for $G$ is said to be optimistic externally stable against rejection at $\varepsilon$-cost if

$$\text{ES } h \in H \text{ and } x \in X^h \setminus \sigma(h) \text{ imply that for some } i \in N \text{ there exist } k \in H \text{ and } y \in \sigma(k) \text{ such that } i \text{ can induce } k \text{ from } h \text{ through } x \text{ and } u_i^h(x) < u_i^k(y) - \varepsilon.$$  

An SB for $G$ is said to be optimistic stable against rejection at $\varepsilon$-cost if it satisfies both IS and ES. Note that stationarity is not imposed; the sets of paths assigned by an SB may differ even for isomorphic game trees. The only requirement is that it be stable.

An SB that is optimistic stable against rejection at $\varepsilon$-cost is called an $\varepsilon$-optimistic stable SB for the tree situation in Greenberg's (1990, Chap. 8) treatment of extensive form games. Greenberg's notion is here applied to the $n$-person sequential bargaining game without presenting the general structure and terminology of the theory of social situations.

3. Results

For any $\varepsilon \geq 0$, it would be desirable to establish the existence of a unique SB that is optimistic stable against rejection at $\varepsilon$-cost. Unfortu-
nately, such uniqueness cannot be established for \( \varepsilon = 0 \) even if (A.2) is assumed. In particular, assuming \( n = 3 \) and \( \delta > \frac{1}{2} \) it can be shown\(^5\) that for any strictly positive division in \( Z \) there exists an SB that is optimistic stable against rejection at \( \varepsilon \)-cost such that this division is acceptable at time zero (given the SB).\(^6\) This result can readily be generalized to the case where \( n > 3 \) and \( \delta > 1/(n - 1) \).

It turns out, however, that if \( \varepsilon > 0 \), there exists a unique SB that is optimistic stable against rejection at \( \varepsilon \)-cost, provided only that (A.1) is satisfied.

**Proposition 1** (Greenberg, 1990, Theorem 8.3.9). Consider the game tree \( G \) associated with the \( n \)-person sequential bargaining game. Assume (A.1) and \( \varepsilon > 0 \). Then there exists a unique standard of behavior for \( G \), denoted \( \sigma_\varepsilon \), that is optimistic stable against rejection at \( \varepsilon \)-cost.

By interpreting the time discounting as due to a cake shrinking exponentially in a physical sense,\(^7\) we can explain this result as follows: After sufficiently many bargaining rounds the available cake is smaller than \( \varepsilon \). At this point, no player can profitably reject a suggested path; i.e., by ES all paths are acceptable. An inductive argument uniquely determines the set of acceptable paths at earlier stages of the game.

Under the stronger assumption of (A.2), the inductive argument above is used in Lemmas 2 and 3 of Section 5 to characterize for each \( \varepsilon > 0 \) the set of acceptable divisions given \( \sigma_\varepsilon \). Based on this characterization, the main uniqueness result of the present paper can be established.

**Proposition 2.** Consider the game tree \( G \) associated with the \( n \)-person sequential bargaining game. Assume (A.2) and consider, for any \( \varepsilon > 0 \), the standard of behavior, \( \sigma_\varepsilon \), of Proposition 1. The set of divisions acceptable at time 0 (given \( \sigma_\varepsilon \)) is nested as a function of \( \varepsilon > 0 \) and contains in the limit, as \( \varepsilon \downarrow 0 \), only the stationary ("history-free") division \([((1 - \delta)/(1 - \delta^n)), \delta \cdot (1 - \delta)/(1 - \delta^n), \ldots, \delta^{n-1} \cdot (1 - \delta)/(1 - \delta^n)]\).

This uniqueness result is caused by the game being effectively, if not formally, truncated at the time when the cake has shrunk to a size smaller than \( \varepsilon \). In particular, it does not apply if \( \varepsilon = 0 \). Hence, one may claim that Proposition 2 is not surprising given result (b) of the introduction. However, two lines of defense can be made.

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\(^5\) See Proposition 1 of Asheim (1989). The proof is inspired by Shaked's nonuniqueness demonstration.

\(^6\) Say that \( z \in Z \) is acceptable at time \( t \) (given \( \sigma \)) if there exists \( h \in H \) corresponding to an offer at time \( t \) such that \( x \in \sigma(h) \) and \( u_i(x) = \pi_i(z, t) \) for all \( i \in N \).

\(^7\) The interpretation of \( \delta \) as the players' rate of impatience does not as easily fit the model. In that case, it would have seemed more appropriate to have measured \( \varepsilon \) as a fixed relative share of the cake, thereby diminishing over time in terms of utility. This contrasts the assumption made here, viz., that \( \varepsilon \) is constant in terms of utility.
(1) Here, the stationary division is acceptable for any \( \varepsilon > 0 \). In contrast, the stationary division is not sustained as an SPE of any bargaining game with an exogenously given last bargaining round (although approached in the limit as \( T \to \infty \)). Hence, with such an exogenous truncation the plausible division is not included, while in the present formulation any other division is excluded.

(2) Here, the truncation of the game is endogenous. This is of particular consequence when letting the discount factor tend to 1: With a fixed exogenous truncation, an extreme last offerer advantage results. Here, as the following proposition shows, the endogenously determined number of bargaining rounds increases such that, in the limit, there is no advantage for any player.

**Proposition 3.** Consider the game tree \( G \) associated with the \( n \)-person sequential bargaining game. Assume (A.2) and consider, for some \( 0 < \varepsilon < 1 \), the standard of behavior, \( \sigma_\varepsilon \), of Proposition 1. As \( \delta \) tends to 1, the set of divisions acceptable at time 0 (given \( \sigma_\varepsilon \)) converges to the set \( \{ z | z_i \geq (1 - \varepsilon)/n \text{ for all } i \in N \} \).

4. Discussion

Following Radner (1980) one may define that a strategy profile of a particular game is an \( \varepsilon \)-equilibrium if any player by choosing his best reply (rather than playing his component of the strategy profile) can gain at most \( \varepsilon > 0 \). An \( \varepsilon \)-equilibrium is \( \varepsilon \)-subgame-perfect if it induces an \( \varepsilon \)-equilibrium on each subgame. Since any SPE is an \( \varepsilon \)-SPE, for any particular game the set of \( \varepsilon \)-SPE contains the set of SPE. E.g., Radner (1980) shows that in a finitely repeated oligopoly game, the set of \( \varepsilon \)-SPE contains collusive behavior, even though the unique SPE repeats static Cournot behavior.

In the present analysis of the sequential bargaining game in Sections 2 and 3, \( \varepsilon > 0 \) is used in an analogous way: A player can profitably raise his voice in the bargaining process only if by doing so he will gain more than \( \varepsilon \). However, in the present application, introducing such an \( \varepsilon \) effectively limits the set of acceptable outcomes. The following discussion is intended to explain why this is so.

Pick any \( \varepsilon \)-SPE of the \( n \)-player sequential bargaining game. This induces a path from each node in the game tree. Thus, it determines an SB which for each node admits only the path induced by it. Being an \( \varepsilon \)-equilibrium of every subgame, there is no node at which a player can gain more than \( \varepsilon \) by doing a single deviation at this node. Therefore, the SB so determined by an \( \varepsilon \)-SPE is optimistic internally stable against rejection at \( \varepsilon \)-cost. As mentioned in the introduction, any division of the cake
can be sustained as an SPE for $\delta \geq 1/(n - 1)$; hence, for $\delta \geq 1/(n - 1)$, any division is acceptable given some SB that is optimistic internally stable against rejection at $\varepsilon$-cost.

It is therefore the optimistic external stability that enables us to exclude (almost) every division but the stationary ("history-free") one. Why is this so?

By imposing optimistic external stability, the SB permits each player to suggest to later players any path which is not in conflict with the optimistic internal stability of the SB. Hence, each player is free to suggest a path if no later player can profitably reject it by suggesting another path admitted by the SB. In this sense, each player is given a reason why he cannot suggest a path which is not admitted by an optimistic externally stable SB; viz., that such a path would be profitably rejected by some later player suggesting an acceptable path. In contrast, consider the optimistic internally stable SB determined by an $\varepsilon$-SPE of the sequential bargaining game. Such an SB does not survive when optimistic external stability is imposed, because it is left completely unexplained why a path different from the one induced by the $\varepsilon$-SPE may not be suggested at some decision node in the game tree.

In his treatment of extensive form games, Greenberg (1990, Chap. 8) considers conservative stability as an alternative to optimistic stability. In fact, he shows that the concept of an $\varepsilon$-conservative stable SB for the tree situation is closely related to the notion of subgame-perfectness for $\varepsilon = 0$. Conservative stability does not allow each player to suggest to later players a particular path among those admitted by the SB; rather, if a suggested path is rejected, an acceptable path that minimizes the utility of the rejector will be followed. Since each player is thus unable to influence the subsequent play, an SB that is conservative stable against rejection at $\varepsilon$-cost does not yield the refinement of subgame-perfectness that is here obtained by optimistic stability for $\varepsilon > 0$.

5. PROOFS

For the proof of Proposition 1, consider the following preliminaries. Following Greenberg (1990, Chap. 4), an (abstract) system is a pair $(D, <)$ where $D$ is a set and $<$ is a dominance relation on $D$. Say that $<$ is strictly acyclic if there is no infinite sequence $d_1, d_2, \ldots$ satisfying $d_i \in D$ and $d_i < d_{i+1}$ for $i \in \mathbb{N}$. Let $\Delta(F) := \{d \in D \mid \exists f \in F \text{ such that } d < f\}$. Say that $F$ is a vN&M (abstract) stable set if $F = D \setminus \Delta(F)$.

**Lemma 1** (von Neumann and Morgenstern, 1953, p. 601; Greenberg, 1990, Theorem 4.7). Let $(D, <)$ be a system, where the dominance relation is strictly acyclic. Then there exists a unique vN&M stable set for the system $(D, <)$.
Let $D := \{(h, x) \mid h \in H, x \in X^h\}$ and define $\prec_\varepsilon$ by $(h, x) \prec_\varepsilon (k, y)$ if there exists $i \in N$ such that $i$ can induce $k$ from $h$ through $x$ and $u^h_i(x) < u^k_i(y) - \varepsilon$. By a result of Shitovitz (Greenberg, 1990, Theorem 4.5), $\sigma_\varepsilon$ is optimistic stable against rejection at $\varepsilon$-cost if and only if $F_\varepsilon := \{(h, x) \mid h \in H, x \in \sigma_\varepsilon(h)\}$ is a stable set for $(D, \prec_\varepsilon)$.

**Proof of Proposition 1.** By Lemma 1 it suffices to show that $\prec_\varepsilon$ is strictly acyclic. By (A.1), there exists $T > 0$ such that for any $i \in N, |\pi_i(z, t) - \pi_i(z', t')| \leq \varepsilon$ if $z, z' \in Z$ and $t, t' \geq T$. Since $(h, x) \prec_\varepsilon (k, y)$ implies that $h$ precedes $k$ in time, it follows that $\prec_\varepsilon$ is strictly acyclic.

For the following, consider the game tree $G$ associated with the $n$-player sequential bargaining game. Assume (A.2) and $\varepsilon > 0$. By Proposition 1 there exists a unique SB, $\sigma_\varepsilon$, that is optimistic stable against rejection at $\varepsilon$-cost. Define $T$ by the property that $\varepsilon/\delta T - 1 < 1$ and $\varepsilon/\delta T \geq 1$. Also, let $V^t$ denote the set of voters at time $t$; i.e., $i \in V^t$ if an only if $(t) \text{mod}(n) \neq i - 1$. Hence, $i \not\in V^t$ if and only if $i$ is the offerer at time $t$.

The subsequent lemma characterizes $\sigma_\varepsilon$ by using times at which the available cake is smaller than $\varepsilon$ to establish an inductive argument.

**Lemma 2.** The standard of behavior $\sigma_\varepsilon$ is characterized as follows:

1. For $t \geq T$, $z \in Z$ is acceptable at time $t$ (given $\sigma_\varepsilon$) if and only if $z_i \geq 0$ for all $i \in N$.

2. Assume that for all $s \geq t$ with $t \geq T$, $z \in Z$ is acceptable at time $s$ (given $\sigma_\varepsilon$) if and only if $z_i \geq \bar{z}^i_{s-1} \geq 0$ for all $i \in N$, $\sum_{i \in N} \bar{z}^i_s = \max\{0, 1 - \varepsilon/\delta^s\}$, and $\bar{z}^i_s = \delta \cdot \bar{z}^i_{s-1}$ for all $i \in V^s$. Then $z \in Z$ is acceptable at time $t - 1$ (given $\sigma_\varepsilon$) if and only if

$$z_i \geq \bar{z}^i_{t-1} = \delta \cdot \bar{z}^i_{t-1} \quad \text{for } i \in V^{t-1}$$

and

$$z_i \geq \bar{z}^i_{t-1} = 1 - \sum_{i' \in V^{t-1}} \bar{z}^i_{t-1} - \varepsilon/\delta^{t-1} \quad \text{for } i \not\in V^{t-1}.$$

**Proof.** (1) Consider any $h$ corresponding to an offering at time $t^h \geq T$. For any $z \in Z$, there exists $x \in \sigma_\varepsilon(h)$ such that $x = (z, A, \ldots, A)$ and $u^h_i(x) = \delta^{i^h} \cdot z_i$ for all $i \in N$. Since $\varepsilon \leq \delta^{i^h}$, for any $i \in N$ there do not exist $k$ and $y \in X^k \supseteq \sigma_\varepsilon(k)$ such that $i$ can induce $k$ from $h$ through $x$ and $u^h_i(x) < u^k_i(y) - \varepsilon$. By ES of $\sigma_\varepsilon$, $x \in \sigma_\varepsilon(h)$.

(2) Part A. Let $h$ correspond to an offering at time $t^h = t - 1 < T$. Let $x \in X^h$ yield some player $i \in V^t$ a payoff of $u^h_i(x) < \delta^{-1} \cdot \bar{z}^i_{t-1} = \delta^{i^h} \cdot \bar{z}^i_{t-1}$. (Hence, $\bar{z}^i_t > 0$.) By the structure of the game tree there exists some node $h \in H_i$ reachable from $h$ through $x$ such that $i \in V^s$ for all $s \in \{t - 1, \ldots, t^h\}$. (I.e., sooner or later $i$ will be asked to vote on an offer.) Let $k := (h, R)$. By the premise (and since $\bar{z}^i_t > 0$) there exists $y \in \sigma_\varepsilon(k)$ such
that \( u^k_i(y) = \delta^k \cdot [\tilde{z}^{k}_i + \varepsilon/\delta^k] = \delta^k \cdot \tilde{z}^{k}_i + \varepsilon (> u^k_i(x) + \varepsilon) \). Since \( i \) can induce \( k \) from \( h \) through \( x \), it follows by IS of \( \sigma \) that \( x \in X^h \setminus \sigma(h) \).

**Part B.** Let \( h \) correspond to an offering at time \( t^h = t - 1 < T \). Let \( x \in X^h \) yield player \( i \notin V^{t-1} \) a payoff of \( u^k_i(x) > \delta^{t-1} \cdot (1 - \sum_{i \in V^{t-1}} \tilde{z}_i^{t-1}) \). This implies that there exists \( i' \in V^{t-1} \) such that \( u^k_i(x) < \delta^{t-1} \cdot \tilde{z}_i^{t-1} \). By part A it follows that \( x \in X^h \setminus \sigma(h) \).

**Part C.** Let \( h \) correspond to an offering at time \( t^h = t - 1 < T \). For any \( z \in Z \) such that \( z_i \equiv \tilde{z}_i^{t-1} = \delta \cdot \tilde{z}_i^t \) for \( i \in V^{t-1} \) and \( z_i \equiv \tilde{z}_i^{t-1} = 1 - \sum_{i' \in V^{t-1}} \tilde{z}_{i'}^{t-1} - \varepsilon/\delta^{t-1} \) for \( i \notin V^{t-1} \), there exists \( x \in X^h \) such that \( x = (z, A, \ldots, A) \) and \( u^k_i(x) = \delta^{t-1} \cdot z_i \) for all \( i \in N \). By the premise and part B for any \( i \in N \) there do not exist \( k \in H \) and \( y \in \sigma(k) \) such that \( i \) can induce \( k \) from \( h \) through \( x \) and \( u^k_i(x) < u^k_i(y) - \varepsilon \). By ES of \( \sigma \), \( x \in \sigma(h) \).

**Part D.** Let \( h \) correspond to an offering at time \( t^h = t - 1 < T \). Let \( x \in X^h \) yield player \( i \notin V^{t-1} \) a payoff of \( u^k_i(x) < \delta^{t-1} \cdot \tilde{z}_i^{t-1} - \varepsilon \). Consider \( k = (h, z) \) where \( z_i = \tilde{z}_i^{t-1} \) for \( i' \in V^{t-1} \) and \( z_i = 1 - \sum_{i' \in V^{t-1}} \tilde{z}_{i'}^{t-1} \) for \( i \notin V^{t-1} \), and let \( y = (A, \ldots, A) \). By the proof of part C we have that \( y \in \sigma(k) \). Since \( i \) can induce \( k \) from \( h \) through \( x \) and \( u^k_i(x) < u^k_i(y) - \varepsilon \) it follows by IS of \( \sigma \) that \( x \in X^h \setminus \sigma(h) \).

It is useful to restate the result of Lemma 2 in the following manner.

**Corollary.** A division \( z \) is acceptable at time \( t \) (given \( \sigma \)) if and only if \( z \equiv \tilde{z}^t := (z_1^t, \ldots, z_n^t) \), where the sequence \( \tilde{z}^0, \tilde{z}^1, \ldots, \tilde{z}^t, \ldots \) is uniquely determined by

1. For \( t \geq T \), \( \tilde{z}_i^t = 0 \) for all \( i \in N \).
2. For \( t < T \), \( \tilde{z}_i^t \geq 0 \) for all \( i \in N \) and \( \sum_{i \in N} \tilde{z}_i^t = 1 - \varepsilon/\delta^t \).
3. For all \( t, i \in V^t \) implies \( \tilde{z}_i^t = \delta \cdot \tilde{z}_i^{t+1} \).

**Proof.** The induction of Lemma 2 clearly implies the corollary. Conversely, the corollary is sufficient to establish the induction of Lemma 2.

**Lemma 3.** A division \( z \) is acceptable at time \( t \) (given \( \sigma \)) if and only if \( z \equiv \tilde{z}^t := (z_1^t, \ldots, z_n^t), \) where, for each \( i \in N \), \( \tilde{z}_i^t \) is calculated as follows: Determine \( v \in \{0, 1, \ldots, n - 1\} \) by \((t + v) \mod(n) = i - 1 \); i.e., \( i \) is the offerer if \( v = 0 \), \( i \) is the \( v \)th voter if \( v > 0 \).

(a) If \( t \) satisfies \( \varepsilon/\delta^t \geq \delta^v \) (i.e., \( t \geq T - v \)), then \( \tilde{z}_i^t = 0 \).

(b) If \( t \) satisfies \( \varepsilon/\delta^t \in [\delta^{bn+1+v}, \delta^{bn+v}) \) for some \( b \geq 0 \), then

\[
\tilde{z}_i^t = \delta^v \cdot \left( (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} + \delta^{bn+1} \right) - \frac{\varepsilon}{\delta^t}
\]
\[ = \delta^v \cdot \frac{1 - \delta}{1 - \delta^n} \cdot (1 - \delta^{(b+1)n}) + \delta^{bn+1/v} \cdot \frac{\varepsilon}{\delta^t}. \]

(c) If \( t \) satisfies \( \varepsilon/\delta^t \in [\delta^{(b+1)n+u}, \delta^{bn+1+u}) \) for some \( b \geq 0 \), then

\[ \bar{z}'_t = \delta^v \cdot \left( (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} \right) = \delta^v \cdot \frac{1 - \delta}{1 - \delta^n} \cdot (1 - \delta^{(b+1)n}). \]

**Proof.** We need to show that the sequence \( \bar{z}^0, \bar{z}^1, \ldots, \bar{z}', \ldots \) as calculated through (a), (b), and (c) satisfies (1), (2), and (3) of the corollary. It is evident that (1) and (3) are satisfied. It remains to be shown that (2) is satisfied; i.e., that \( \sum_{i \in N} \bar{z}^i_t = 1 - \varepsilon/\delta^t \) for any \( t < T \).

It follows from \( t < T \) that \( \varepsilon/\delta^t < 1 \). First consider the case where \( \varepsilon/\delta^t \in [\delta^n, 1) \). Determine \( v' \in \{0, 1, \ldots, n - 1\} \) by \( \varepsilon/\delta^t \in [\delta^{1+v'}, \delta^{v'}) \). Then

\[ \sum_{i \in N} \bar{z}^i_t = \sum_{v=0}^{v'-1} \delta^n \cdot (1 - \delta) + \delta^{v} \cdot (1 - \delta) + \delta^{v'} \cdot \delta - \frac{\varepsilon}{\delta^t} = 1 - \varepsilon/\delta^t. \]

Then consider the case where \( \varepsilon/\delta^t \in [\delta^{(b+1)n}, \delta^{bn}) \) for some \( b \geq 1 \). Determine \( v' \in \{0, 1, \ldots, n - 1\} \) by \( \varepsilon/\delta^t \in [\delta^{bn+1+v'}, \delta^{bn+v'}) \). In this case we have that

\[
\sum_{i \in N} \bar{z}^i_t = \sum_{v=0}^{v'-1} \delta^v \cdot \left( (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} \right) + \delta^{v} \cdot \left( (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} + \delta^{bn+1} \right)
- \frac{\varepsilon}{\delta^t} + \sum_{v=v'-1}^{n-1} \delta^v \cdot \left( (1 - \delta) \cdot \sum_{a=0}^{b-1} \delta^{an} \right)
= \sum_{a=0}^{b} \delta^{an} - \delta^{v'+1} \cdot \sum_{a=0}^{b} \delta^{an} + \delta^{v} \cdot \delta^{bn+1}
- \frac{\varepsilon}{\delta^t} + \delta^{v'+1} \cdot \sum_{a=0}^{b-1} \delta^{an} - \delta^n \cdot \sum_{a=0}^{b-1} \delta^{an}
= 1 + \sum_{a=1}^{b} \delta^{an} - \delta^{v'+1} \cdot \sum_{a=0}^{b-1} \delta^{an} - \delta^{v'+1} \cdot \delta^{bn} + \delta^{v} \cdot \delta^{bn+1}
- \frac{\varepsilon}{\delta^t} + \delta^{v'+1} \cdot \sum_{a=0}^{b-1} \delta^{an} - \sum_{a=1}^{b} \delta^{an}
= 1 - \frac{\varepsilon}{\delta^t}. \]

From Lemma 3 the proofs of Propositions 2 and 3 follow.
Proof of Proposition 2. Consider player 1. By Lemma 3 (provided $\varepsilon < 1$)

$$z_1^0 = (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} + \delta^{bn+1} - \varepsilon = \frac{1 - \delta}{1 - \delta^n} \cdot (1 - \delta^{(b+1)n}) + \delta^{bn+1} - \varepsilon$$

if $\varepsilon \in [\delta^{bn+1}, \delta^{bn})$ for some $b \geq 0$;

$$z_1^0 = (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} = \frac{1 - \delta}{1 - \delta^n} \cdot (1 - \delta^{(b+1)n})$$

if $\varepsilon \in [\delta^{(b+1)n}, \delta^{bn+1})$ for some $b \geq 0$.

Let us first establish that $z_1^0$ is continuous in $\varepsilon$:

$$\lim_{\varepsilon \uparrow \delta^{bn}} z_1^0 = (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} + \delta^{bn+1} - \delta^{bn}$$

$$= (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} - (1 - \delta) \cdot \delta^{bn} = z_1^0|_{\varepsilon = \delta^{bn}}$$

$$\lim_{\varepsilon \uparrow \delta^{bn+1}} z_1^0 = (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an}$$

$$= (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} + \delta^{bn+1} - \delta^{bn+1} = z_1^0|_{\varepsilon = \delta^{bn+1}}.$$

Furthermore, if $\varepsilon \in [\delta^{bn+1}, \delta^{bn})$ for some $b \geq 0$, then $z_1^0$ is a decreasing function of $\varepsilon$, while if $\varepsilon \in [\delta^{(b+1)n}, \delta^{bn+1})$ for some $b \geq 0$, then $z_1^0$ is a constant function of $\varepsilon$; i.e., $z_1^0$ is a nonincreasing function of $\varepsilon$. Similarly for players 2, 3, \ldots, $n$. Hence, the set of acceptable divisions at time 0 (given $\sigma_x$) as a function of $\varepsilon$ is nested.

Finally, $\lim_{\varepsilon \uparrow 0} z_i^0 = \delta^{i-1} \cdot (1 - \delta)/(1 - \delta^n)$. Since $z_i^0$, $i \in N$, are nonincreasing functions of $\varepsilon$, it follows that $z$ is acceptable at time 0 (given $\sigma_x$) for all $\varepsilon > 0$ if and only if $z = (z_1, \ldots, z_i, \ldots, z_n) = ((1 - \delta)/(1 - \delta^n), \ldots, \delta^{i-1} \cdot (1 - \delta)/(1 - \delta^n), \ldots, \delta^{n-1} \cdot (1 - \delta)/(1 - \delta^n))$. □

Proof of Proposition 3. Consider player 1. By Lemma 3 (since $\varepsilon < 1$)

$$z_1^0 = (1 - \delta) \cdot \sum_{a=0}^{b} \delta^{an} + \delta^{bn+1} - \varepsilon = \frac{1 - \delta}{1 - \delta^n} \cdot (1 - \delta^{(b+1)n}) + \delta^{bn+1} - \varepsilon$$

if $\varepsilon \in [\delta^{bn+1}, \delta^{bn})$ for some $b \geq 0$;
$z_1^0 = (1 - \delta) \cdot \sum_{a=0}^{b} \delta^a n = \frac{1 - \delta}{1 - \delta^n} \cdot (1 - \delta^{(b+1)n})$

if $\epsilon \in [\delta^{(b+1)n}, \delta^{bn+1})$ for some $b \geq 0$. Since by construction $b$ satisfies $\epsilon \in [\delta^{(b+1)n}, \delta^{bn})$ it follows that $b \to \infty$, $\delta^{bn+1} \to \epsilon$, $\delta^{(b+1)n} \to \epsilon$, and $(1 - \delta)/(1 - \delta^n) \to 1/n$ as $\delta \uparrow 1$. Hence, $z_1^0 \to (1 - \epsilon)/n$ as $\delta \uparrow 1$. Similarly for players 2, 3, . . . , $n$.

REFERENCES


