Mathematical appendix to "Paradoxical consumption behavior when economic activity has environmental effects"

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Sufficient conditions for optimality

Conditions 4–6 are necessary for optimality if one can justify the usual "normalization procedure" (i.e. setting equal to unity the constant multiplier, say λ_0, associated with the integrand of the objective function) when formulating the Hamiltonian. One can set $\lambda_0 = 1$ if certain growth conditions are satisfied (cf. Sydsæter et al., 2005, Theorem 9.11.2, and its more general version in Seierstad and Sydsæter, 1987, Theorem 16). This result cannot be applied here without modifying the optimization problem slightly.

In this appendix we show that paths satisfying Conditions 4–6 as well as the additional properties (A1)–(A3) are indeed optimal, and that these conditions and properties are satisfied for the paths considered in Lemmas 1 and Lemma 2. None of the results of this paper depends on Conditions 4–6 being necessary.

Proposition 1 *Let* $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ *be an interior path feasible from* $(k(0), x(0))$*, which satisfies* $\ell(t) = 1$ *(i.e., full utilization of labor) for all* t*. If there exist continuously differentiable functions* $p(t)$ *and* $q(t)$ *satisfying Conditions 4–6 and the*

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following additional properties,

$$p(t) > 0 \quad and \quad q(t) > 0 \quad for\ all\ t\,, \tag{A1}$$

$$\lim_{t\to\infty} p(t)e^{-\rho t} = 0 \quad and \quad \lim_{t\to\infty} q(t)e^{-\rho t} = 0 \tag{A2}$$

$$\frac{d\left((p(t)-q(t))\,x_\rho(t)e^{-\rho t}\right)}{dt} \leq 0 \quad for\ all\ t\,, \tag{A3}$$

then $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^{\infty}$ *is the unique optimal path from* $(k(0), x(0))$.

Proof. Showing sufficiency is complex because $xf(k)$ is not a concave function. Note that (A2) and (A3) imply $(p-q)x_\rho \geq 0$ and $p-q \geq 0$ for all $t \geq 0$ since the path is interior. Let $\{c(t), k(t), x(t)\}_{t=0}^{\infty}$ be any path feasible from $(k(0), x(0))$. This alternative path may have partial utilization of labor; hence, we only require that $\{\ell(t)\}_{t=0}^{\infty}$ is a piecewise continuous path satisfying $0 \leq \ell(t) \leq 1$ for all t. Then

$$\int_0^\infty \big[u(c)-u(c_\rho)\big]e^{-\rho t}dt$$

$$= \int_0^\infty \Big[u(c)-u(c_\rho) + p\Big(\big(x\tilde{f}(k,\ell)-c-\delta k-\dot{k}\big) - \big(x_\rho f(k_\rho)-c_\rho-\delta k_\rho-\dot{k}_\rho\big)\Big)$$
$$+ q\Big(\big(g(x)-x\tilde{f}(k,\ell)-\dot{x}\big) - \big(g(x_\rho)-x_\rho f(k_\rho)-\dot{x}_\rho\big)\Big)\Big]e^{-\rho t}dt$$

(by Conditions 1 and 2)

$$\leq \int_0^\infty \Big[(p-q)\big(x-x_\rho\big)\big(\tilde{f}(k,\ell)-f(k_\rho)\big)$$
$$- p\big((\dot{k}+\delta k)-(\dot{k}_\rho+\delta k_\rho)\big) + (p-q)x_\rho\big(\tilde{f}(k,\ell)-f(k_\rho)\big)$$
$$- q\big((\dot{x}-g(x))-(\dot{x}_\rho-g(x_\rho))\big) + (p-q)\big(x-x_\rho\big)f(k_\rho)\Big]e^{-\rho t}dt$$

(by Condition 6 and by rearranging terms). Conditions 4 and 5 imply that the four last terms are non-positive since

(a) $(p-q)x_\rho\big(\tilde{f}(k,\ell)-f(k_\rho)\big) \leq \big(k-k_\rho\big)(p-q)x_\rho f'(k_\rho)$
by $f'' < 0$ and $(p-q)x_\rho \geq 0$ since $\tilde{f}(k,l) \leq f(k)$

(b) $q\big(g(x)-g(x_\rho)\big) \leq \big(x-x_\rho\big)qg'(x_\rho)$ by $g'' < 0$ and $q > 0$,

(c) $-\int_0^\infty p\big(\dot{k}-\dot{k}_\rho\big)e^{-\rho t}dt = \int_0^\infty \big(k-k_\rho\big)(\dot{p}-p\rho)e^{-\rho t}dt$,

(d) $-\int_0^\infty q\big(\dot{x}-\dot{x}_\rho\big)e^{-\rho t}dt = \int_0^\infty \big(x-x_\rho\big)(\dot{q}-q\rho)e^{-\rho t}dt$, where (c) and (d) follows from (A2) using integration by parts. Concerning the first term, Assumption 1 and Condition 1 imply that

$$\big(x - x_\rho\big)\big(\tilde{f}(k,\ell) - f(k_\rho)\big) \le \big(x - x_\rho\big)\left(\frac{\dot{x}_\rho}{x_\rho} - \frac{\dot{x}}{x}\right),$$

because $\operatorname{sgn}\big(x - x_\rho\big) = \operatorname{sgn}\big(g(k_\rho)/x_\rho - g(x)/x\big) = \operatorname{sgn}\big((\dot{x}_\rho + x_\rho f(k_\rho))/x_\rho - (\dot{x} + \tilde{f}(k,\ell))/x\big) = \operatorname{sgn}\big((\dot{x}_\rho/x_\rho - \dot{x}/x) - (\tilde{f}(k,\ell) - f(k_\rho))\big)$. Since $p - q \ge 0$, it therefore follows that

$$\int_0^\infty \big[u(c) - u(c_\rho)\big]e^{-\rho t}dt \le \int_0^\infty (p-q)\big(x - x_\rho\big)\left(\frac{\dot{x}_\rho}{x_\rho} - \frac{\dot{x}}{x}\right)e^{-\rho t}dt$$
$$= \int_0^\infty (p-q)x_\rho\left(\frac{x}{x_\rho} - 1\right)\tfrac{d}{dt}\big(\ln x_\rho - \ln x\big)e^{-\rho t}dt = \int_0^\infty (p-q)x_\rho e^{-\rho t}\big(e^{-m} - 1\big)\dot{m}dt\,,$$

where $m(t)$ is defined by $m(t) := \ln x_\rho - \ln x$. Note that $m(0) = 0$. Let T be an arbitrary positive number. Then, using integration by parts,

$$\int_0^T (p-q)x_\rho e^{-\rho t}\big(e^{-m} - 1\big)\dot{m}dt$$
$$= \big(p(0) - q(0)\big)\,x_\rho(0) - \big(p(T) - q(T)\big)\,x_\rho(T)e^{-\rho T}\big(e^{-m(T)} + m(t)\big)$$
$$+ \int_0^T \tfrac{d}{dt}\big((p-q)x_\rho e^{-\rho t}\big)\big(e^{-m} + m\big)dt$$
$$\le \big(p(0) - q(0)\big)\,x_\rho(0) - \big(p(T) - q(T)\big)\,x_\rho(T)e^{-\rho T} + \int_0^T \tfrac{d}{dt}\big((p-q)x_\rho e^{-\rho t}\big)dt\,,$$

since $\big(p(T) - q(T)\big)\,x_\rho(T) \ge 0$, $d\big((p-q)x_\rho e^{-\rho t}\big)/dt \le 0$ for all $0 < t < T$ by (A3), and $e^{-m} + m \ge 1$ for all m. The last expression is equal to zero, and the optimality of $\{c_\rho(t), k_\rho(t), x_\rho(t)\}_{t=0}^\infty$ follows by letting T approach infinity. Uniqueness is implied by $g'' < 0$ and $q > 0$. ∎

The existence, uniqueness, and saddle point characteristics of (x_ρ^*, z^*).

Consider equations (7) and (8). Extend the domain of $\kappa(\cdot)$ to non-positive values of z as follows: $\kappa(z) = 0$ for $z \in (-\infty, 0]$. Then $\xi(\kappa(\cdot))$ is a continuous function from $\mathbb{R}$ into $[0, \bar{x}]$ with the following properties:

$$x_\rho = \xi(\kappa(z)) \quad \text{implies} \quad \dot{x}_\rho = 0$$
$$\xi(\kappa(z)) = \bar{x} \;\text{ for } z \in (-\infty, 0] \quad \text{and} \quad \xi(\kappa(z)) \in [0, \bar{x}) \;\text{ for } z \in (0, \infty)\,.$$

Since $\partial\dot{z}/\partial z = g(x_\rho)/x_\rho - g'(x_\rho) + \rho \ge \rho > 0$ for all z and x_ρ and $\dot{z} = g(x_\rho)$ if $z = x_\rho$, there exists a continuous function from $\zeta(\cdot)$ from $[0, \bar{x}]$ into $\mathbb{R}$ with the

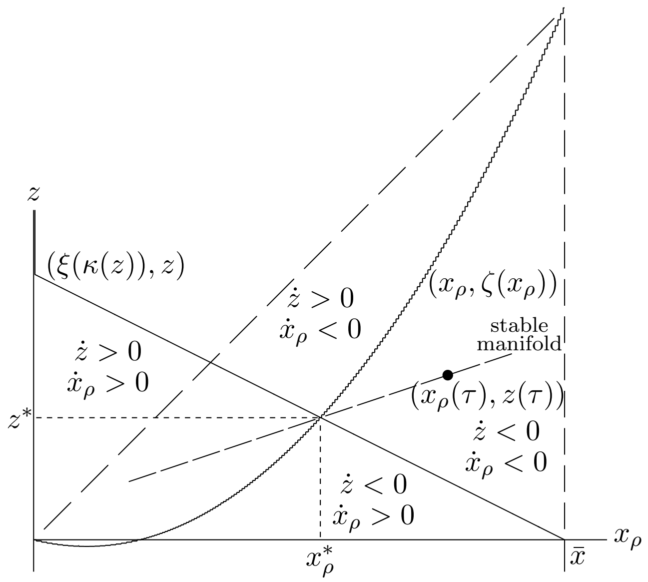

Figure 1: A phase diagram.

following properties

$$z = \zeta(x_\rho) \quad \text{if and only if} \quad \dot{z} = 0$$
$$\zeta(0) = 0\,, \quad \zeta(x_\rho) < x_\rho \ \text{ for } x_\rho \in (0, \bar{x})\,, \quad \text{and} \quad \zeta(\bar{x}) = \bar{x}\,.$$

By Brouwer's fixed point theorem, there exists $x_\rho^* \in [0, \bar{x}]$ such that $x_\rho^* = \xi(\kappa(\zeta(x_\rho^*)))$. It follows that $0 < x_\rho^* < \bar{x}$ and $z^* = \zeta(x_\rho^*) > 0$ since

$$x_\rho^* = 0 \quad \Rightarrow \quad z^* = \zeta(x_\rho^*) \leq 0 \quad \Rightarrow \quad x_\rho^* = \xi(\kappa(z^*)) = \bar{x}$$
$$\Rightarrow \quad z^* = \zeta(x_\rho^*) > 0 \quad \Rightarrow \quad x_\rho^* = \xi(\kappa(z)) < \bar{x}\,.$$

Furthermore, $z^* < x_\rho^*$, since $z^* = \zeta(x_\rho^*) \geq x_\rho^*$ implies $x_\rho^* = 0$ or $x_\rho^* = \bar{x}$. Finally, (x_ρ^*, z^*) is unique and has saddle point characteristics since

$$\frac{\partial \dot{x}_\rho}{\partial x_\rho} < 0 \quad \text{and} \quad \frac{\partial \dot{x}_\rho}{\partial z} < 0 \quad \text{when} \quad x_\rho = \xi(\kappa(z)) \in (0, \bar{x})$$
$$\frac{\partial \dot{z}}{\partial x_\rho} < 0 \quad \text{and} \quad \frac{\partial \dot{z}}{\partial z} > 0 \quad \text{when} \quad z = \zeta(x_\rho) \in (0, x_\rho)\,.$$

These results are illustrated by a phase diagram in Figure 1.

Lemma 1 and sufficient optimality conditions.

Properties (A1) and (A2) are satisfied since both p and q are positive and constant. Property (A3) is satisfied since $(p - q)x_\rho \equiv p^*(1 - v^*)x_\rho^* \equiv p^* z^*$ is positive and constant.

Lemma 2 and sufficient optimality conditions.

If $p > 0$, then it follows from Conditions 1, 4, and 5 that

$$\dot{z} = \tfrac{z}{x_\rho} g(x_\rho) - (x_\rho - z)\big(z f'(k_\rho) - \delta - g'(x_\rho)\big)\,, \tag{A4}$$

which becomes (8) under the additional assumption that $\dot{p} = 0$. Since $\{k_\rho\}_{t=0}^\infty$ and $\{x_\rho\}_{t=0}^\infty$ are absolutely continuous, it follows that z is continuously differentiable.

The convergence phase. By the linearity of u, Condition 6 is satisfied by letting $p = p^* > 0$ for all $t \geq \tau$. Since $p > 0$ and $\dot{p} = 0$ for $t \geq \tau$, $\{x_\rho(t), z(t)\}_{t=\tau}^\infty$ moves along the stable manifold of Figure 1 leading to the saddle point (x_ρ^*, z^*). The properties of the converging path from $(x_\rho(\tau), z(\tau))$ with $x_\rho(\tau) > x_\rho^*$ imply that $z > 0$ and $\dot{z} < 0$ for all $t \geq \tau$. It follows from (6), (7), and (8) that Conditions 4 and 5 are satisfied by letting $q = p^* v \equiv p^*\big(1 - z/x_\rho\big)$ for all $t \geq \tau$.

The investment phase. Turn now to the properties of p and q for $0 < t < \tau$ when Conditions 4 and 5 are imposed. Since $p(\tau) = p^* > 0$ and p is continuous, we must have $p > 0$ for all $t > \tau - \epsilon$, for some $\epsilon > 0$. Hence, (A4) holds for $\tau - \epsilon < t < \tau$. Furthermore, for $\tau - \epsilon < t < \tau$,

(a) $z = x_\rho$ implies $\dot{z} > 0$ by (A4) since $0 < x_\rho < 1$,

(b) $z \in (0, x_\rho)$ and $\dot{z} = 0$ imply that $d\dot{z}/dt$ exists and is positive since (A4) holds, and $\dot{k}_\rho > 0$ and $\dot{x}_\rho \leq 0$ (by the proof of Lemma 2).

Hence, $z(\tau) > 0$ and $\dot{z}(\tau) < 0$ imply that $z < x$ for $\tau - \epsilon < t < \tau$ by (a) and $z > 0$ and $\dot{z} < 0$ for $\tau - \epsilon < t < \tau$ by (b). Therefore, $0 < z < x_\rho$ and $\dot{z} < 0$ for $\tau - \epsilon < t < \tau$.

It follows from the previous paragraph that $k_\rho < k_\rho(\tau) = \kappa(z(\tau)) < \kappa(z)$ for $\tau - \epsilon < t < \tau$, since $\dot{k}_\rho > 0$ and $\dot{z} < 0$ for $\tau - \epsilon < t < \tau$ and κ is strictly increasing. By (5) (which follows from Condition 5) and (6) this means that

$$-\tfrac{\dot{p}}{p} = z f'(k) - (\rho + \delta) > z f'(\kappa(z)) - (\rho + \delta) \equiv 0$$

for $\tau - \epsilon < t < \tau$ since $z > 0$, implying that $\dot{p} < 0$ for $\tau - \epsilon < t < \tau$. This implies that we can set $\epsilon = \tau$, so that $p > p^*$ and $\dot{p} < 0$ for $0 < t < \tau$. Therefore, $0 < z < x_\rho$ and $\dot{z} < 0$ for $0 < t < \tau$.

It follows from the above analysis that, for $0 < t < \tau$, Conditions 4 and 5 are satisfied by construction, while Condition 6 is satisfied since $p > p^*$ and $c_\rho = 0$.

The analysis of the convergence and investment phases entails that $p > 0$, $\dot{p} \leq 0$, $0 < z < x_\rho$, and $\dot{z} < 0$ for all $t > 0$. Since thus $p > 0$ and $q = pv = p(1 - z/x_\rho) > 0$ for all $t > 0$, property (A1) is satisfied. Furthermore, both p and q converge to the positive constants p^* and $q^* := p^*\big(1 - z^*/x_\rho^*\big)$, implying that property (A2) is satisfied. Finally, $(p - q)x_\rho \equiv pz$ is positive and decreasing, implying that property (A3) is satisfied.

References

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