Supplementary material related to "Characterizing the sustainability problem in an exhaustible resource model"

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Tapan Mitra[a]
Geir B. Asheim[b]
Wolfgang Buchholz[c]
Cees Withagen[d]

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[a]Department of Economics, 418 Uris Hall, Cornell University, Ithaca, New York 14853, USA (email: tm19@cornell.edu)

[b]Department of Economics, University of Oslo, P.O. Box 1095 Blindern, 0317 Oslo, Norway (email: g.b.asheim@econ.uio.no) *Corresponding author* (tel: 47 45505136)

[c]Department of Economics, University of Regensburg, 93040 Regensburg, Germany. (email: wolfgang.buchholz@wiwi.uni-regensburg.de)

[d]Department of Economics, VU University Amsterdam, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands (email: cwithagen@feweb.vu.nl)

Online appendix A: Proofs of Lemmas

Proof of Lemma 1. We break up the proof of Lemma 1 into several steps.

Step 1: There exists a solution to (4). Since $(c, k) \in D$, there is some $r_0 > 0$ such that $F(k, r_0) > c$. Since $F(k, r)$ is continuous and increasing in r with $F(k, 0) - c < 0$ and $F(k, r_0) - c > 0$, there is a unique $r' \in (0, r_0)$ such that $F(k, r') = c$. Define:

$$R' \equiv \left\{ r > r' : \frac{r}{F(k,r) - c} \leq \frac{r_0}{F(k, r_0) - c} \right\} .$$

Then $r_0 \in R'$, so that R' is non-empty. By the definition of r' and the continuity of F,

$$\lim_{r \downarrow r'} \frac{F(k,r) - c}{r} = 0 < \frac{F(k, r_0) - c}{r_0} ,$$

and there is $\underline{r} > r'$ such that $r \notin R'$ for $r \in (r', \underline{r})$. It follows from $F(k, 0) = 0$ and the concavity of F that $F(k, r) \leq F(k/r, 1)r$ if $r > 1$. By $F(0, r) = 0$ and the continuity of F,

$$\lim_{r \to \infty} \frac{F(k,r) - c}{r} \leq \lim_{r \to \infty} \frac{F(k,r)}{r} \leq \lim_{r \to \infty} F\left(\tfrac{k}{r}, 1\right) = 0 < \frac{F(k, r_0) - c}{r_0} ,$$

and there is $\bar{r} > 0$ such that $r \notin R'$ for $r \in (\bar{r}, \infty)$. Hence, $R' \subset [\underline{r}, \bar{r}]$, so that R' is bounded. Since $r/(F(k, r) - c)$ is a continuous function of r on $[\underline{r}, \bar{r}]$, the set R' is closed. As $r/(F(k, r) - c)$ is a continuous function of r on R', there exists a solution to $\min_{r \in R'} r/(F(k, r) - c)$. By the definition of $\underline{r}$, this is also a solution to (4).

Step 2: There is at most one solution to (HaR). Define:

$$V(r) \equiv F(k, r) - c - F_2(k, r)r \quad \text{for all } r > 0 \tag{A1}$$

Note that V is a C^1 function on $\mathbb{R}_{++}$.

Suppose r' and r'' were both solutions to (HaR), with $0 < r' < r''$. Then $V(r') = V(r'') = 0$. We can find $0 < a < r'$ and $b > r''$, and define

$$U(r) \equiv \int_a^r V(s) ds \quad \text{for } r \in [a, b] .$$

Then, we have $U'(r) = V(r)$ for all $r \in (a, b)$.[1] By using (A1), we can infer that U is a C^2 function on (a, b), and for all $r \in (a, b)$, $U''(r) = V'(r) = -F_{22}(k, r)r > 0$. Thus, U is a strictly convex C^2 function on (a, b), and we get the contradiction:

$$0 = U'(r')(r'' - r') < U(r'') - U(r') < U'(r'')(r'' - r') = 0 .$$

Step 3: There is a unique solution to (4), and this uniquely solves (HaR). Since $r/(F(k, r) - c)$ is a continuously differentiable function of r on $\{r : F(k, r) > c\}$, any solution r to (4) satisfies the first-order condition

$$\frac{1}{F(k,r) - c} - \frac{F_2(k,r)r}{(F(k,r) - c)^2} = 0 ,$$

[1] See Rudin [4, Theorem 6.20, p. 133].

and therefore is also a solution to (HaR). By Steps 1 and 2, there is a unique solution to (4). This implies that $\mathbf{p} : D \to \mathbb{R}_{++}$ and $\mathbf{r} : D \to \mathbb{R}_{++}$, as defined in the statement of Lemma 1, are single-valued functions.

Step 4: The functions $\mathbf{p} : D \to \mathbb{R}_{++}$ and $\mathbf{r} : D \to \mathbb{R}_{++}$ are continuously differentiable on D. Note that by continuity of F, the set D is open in $\mathbb{R}^2$. Define $Y \equiv D \times \mathbb{R}_{++}$. Clearly, Y is an open set in $\mathbb{R}^3$, and we can define:

$$H(c,k,r) = F(k,r) - c - F_2(k,r)r \quad \text{for } (c,k,r) \in Y\,.$$

Then, H is continuously differentiable on Y, and by Step 3, we have:

$$H(c,k,r) = 0 \quad \text{for } r = \mathbf{r}(c,k)\,.$$

Furthermore, for $r = \mathbf{r}(c,k)$,

$$\frac{\partial H(c,k,r)}{\partial r} = -F_{22}(k,r)r > 0\,.$$

Thus, by the implicit function theorem,[2] there is an open set $N \subset X \equiv (0,c') \times (k',\infty)$ containing (c,k) and an open set $M \subset Y$, containing $(c,k,\mathbf{r}(c,k))$, and a *unique* function $g : N \to \mathbb{R}_{++}$, such that

(i) for all $(\tilde{c},\tilde{k}) \in N$, we have $H(\tilde{c},\tilde{k},g(\tilde{c},\tilde{k})) = 0$, and

(ii) $g(c,k) = \mathbf{r}(c,k)$.

Furthermore, g is continuously differentiable on N. Since, by Step 3, we certainly have $H(\tilde{c},\tilde{k},\mathbf{r}(\tilde{c},\tilde{k})) = 0$ for all $(\tilde{c},\tilde{k}) \in N \subset D$, we can infer that $g(\tilde{c},\tilde{k}) = \mathbf{r}(\tilde{c},\tilde{k})$ for all $(\tilde{c},\tilde{k}) \in N$. Thus, $\mathbf{r}$ is continuously differentiable on N. Since $(c,k) \in D$ was arbitrary, $\mathbf{r}$ is continuously differentiable on D. As $\mathbf{p}(c,k) = 1/F_2(k,\mathbf{r}(c,k))$ for $(k,r) \in \mathbb{R}^2_{++}$ and F is twice continuously differentiable on $\mathbb{R}^2_{++}$, also $\mathbf{p}$ is continuously differentiable on D.

Step 5: $\mathbf{p}_1(c,k)$ and $\mathbf{r}_1(c,k)$ are positive. By definition of $\mathbf{r}$ on D, we have:

$$F(k,\mathbf{r}(c,k)) - c - F_2(k,\mathbf{r}(c,k))\mathbf{r}(c,k) = 0 \quad \text{for all } (c,k) \in D\,.$$

Thus, differentiating this equation w.r.t. c yields

$$F_2(k,\mathbf{r}(c,k))\mathbf{r}_1(c,k) - 1 - F_{22}(k,\mathbf{r}(c,k))\mathbf{r}_1(c,k)\mathbf{r}(c,k) - F_2(k,\mathbf{r}(c,k))\mathbf{r}_1(c,k) = 0\,,$$

from which $\mathbf{r}_1(c,k) > 0$ can be inferred. Since $\mathbf{p}(c,k) = 1/F_2(k,\mathbf{r}(c,k))$ and $F_{22}(k,r) < 0$ for $(k,r) \in \mathbb{R}^2_{++}$, we now obtain that $\mathbf{p}_1(c,k) > 0$. ■

Proof of Lemma 2. Since $(c,k_0) \in D$ implies that $F(k_0,\mathbf{r}(c,k_0)) - c > 0$, we have $\dot{k}^c(0) > 0$. Since $\dot{k}^c$ is continuous, there is $\varepsilon \in (0,T)$ such that $\dot{k}^c(t) > 0$ for all $t \in [0,\varepsilon]$, and so by the Mean Value theorem, $k^c(t) > k_0$ for all $t \in (0,\varepsilon]$. We claim

[2] See Rudin [4, Theorem 9.28, p. 224–225].

that $k^c(t) \geq k_0$ for all $t \in [0,T)$. If not, there is $\tau' \in (\varepsilon, T)$ such that $k^c(\tau') < k_0$. Let $\tau \equiv \inf\{t \in (\varepsilon,T) : k^c(t) < k_0\}$. Then, $\varepsilon \leq \tau \leq \tau' < T$, and $k^c(\tau) \leq k_0$ by continuity of k^c. Also, by definition of τ, $k^c(t) \geq k_0$ for all $t \in [0,\tau)$. Thus, $k^c(\tau) \geq k_0$ by continuity of k^c, and consequently $k^c(\tau) = k_0$. Then, using the fact that $k^c(t) \geq k^c(\tau)$ for all $t \in [0,\tau)$, we must have $\dot{k}^c(\tau) \leq 0$. On the other hand, since $k^c(\tau) = k_0$, we must have $\dot{k}^c(\tau) > 0$ by using (6). This contradiction establishes our claim.

Since $(c, k_0) \in D$ and $k^c(t) \geq k_0$ for all $t \in [0,t)$, it follows that $(c, k^c(t)) \in D$ for all $t \in [0,T)$. Thus, by definition of $\mathbf{r}$, we must have $\dot{k}^c(t) > 0$ for all $t \in [0,T)$. ■

Proof of Lemma 3. For every $T' \in (0,T)$, $\mathbf{r}(c, k^c(t))$ is continuous on $[0,T']$ and the Riemann integral $\int_0^{T'} \mathbf{r}(c,k^c(t))dt$ is well-defined. By the change of variable formula,[3]

$$\int_{k_0}^{k^c(T')} \mathbf{p}(c,x)dx = \int_0^{T'} \mathbf{p}(c,k^c(t))\dot{k}^c(t)dt\,.$$

Thus, (10) follows from (6) and the definition of the function $\mathbf{p}$.

Assume now that (11) holds. Write $r^c(t) = \mathbf{r}(c,k^c(t))$ for $t \in [0,T)$ and

$$\lambda = \max\{F(1,1/k_0)/k_0, F(k_0,1), F(k_0,1)/k_0\}\,.$$

For $t \in [0,T)$, either (i) $r^c(t) \leq k^c(t)/k_0$, or (ii) $r^c(t) > k^c(t)/k_0$. Case (i) can be divided into two subcases: (i)(a) $k^c(t) \leq 1$ and (i)(b) $k^c(t) > 1$. In case (i)(a),

$$\dot{k}^c(t)/k^c(t) \leq F(k^c(t), r^c(t))/k^c(t) \leq F(1,1/k_0)/k_0 \leq \lambda\,. \tag{A2}$$

In case (i)(b),

$$\dot{k}^c(t)/k^c(t) \leq F(k^c(t), r^c(t))/k^c(t) \leq F(1, r^c(t)/k^c(t)) \leq F(1,1/k_0) \leq \lambda\,. \tag{A3}$$

In case (ii), we have that $r^c(t) > k^c(t)/k_0 \geq 1$. So

$$\dot{k}^c(t) \leq F(k^c(t), r^c(t)) \leq r^c(t)F(k^c(t)/r^c(t),1) \leq r^c(t)F(k_0,1)$$

and

$$\dot{k}^c(t)/k^c(t) \leq r^c(t)F(k_0,1)/k_0 \leq \lambda r^c(t)\,. \tag{A4}$$

Let $\Lambda = \{t \in [0,T) : r^c(t) > k^c(t)/k_0\}$. Then, by (A2)–(A4), we have:

$$\int_{[0,T)} \left(\dot{k}^c(t)/k^c(t)\right) dt \leq \int_\Lambda \lambda r^c(t)dt + \int_{[0,T)\setminus\Lambda} \lambda dt \leq \lambda(S+T)\,.$$

Thus, for every $t \in (0,T)$, $\ln k^c(t) \leq \ln k_0 + \lambda(S+T)$, showing that $k^c(T) < \infty$. ■

[3]See Apostol [1] Theorem 7.36, p. 164.

Proof of Lemma 4. By the definition of $C(k_0, m_0)$, there is a feasible path $(c'(t), k'(t), r'(t))$ from (k_0, m_0) with $c'(t) \geq c'$ for $t \geq 0$. Construct $(c(t), k(t), r(t))$ as follows:

$$\begin{aligned} c(t) &= F(k(t), r(t)) - \dot{k}(t) && \text{for all } t \geq 0 \\ k(t) &= (1-\lambda)k_0 + \lambda k'(t) && \text{for all } t \geq 0 \\ r(t) &= \lambda[r'(t) + (\epsilon/e^t)] && \text{for all } t \geq 0\,, \end{aligned} \tag{A5}$$

where $\lambda = c/c' < 1$ and $\epsilon = (1-\lambda)m_0/\lambda > 0$. We must show that $(c(t), k(t), r(t))$ is a feasible interior path from (k_0, m_0) with $c(t) > c$ for $t \geq 0$.

Clearly, $k(t)$ is a differentiable function of t, with $\dot{k}(t) = \lambda \dot{k}'(t)$ for $t \geq 0$, and $(c(t), r(t))$ are continuous functions of t. Using (A5), for $t \geq 0$,

$$\begin{aligned} c(t) = F(k(t), r(t)) - \dot{k}(t) &> F(\lambda k'(t), \lambda r'(t)) - \lambda \dot{k}'(t) \\ &\geq \lambda[F(k'(t), r'(t)) - \dot{k}'(t)] = \lambda c'(t) \geq \lambda c' = c\,. \end{aligned}$$

Also, $k(t) \geq (1-\lambda)k_0 > 0$ for $t \geq 0$, and (1) is satisfied since $k(0) = (1-\lambda)k_0 + \lambda k_0 = k_0$. Again using (A5), $r(t) > 0$ for $t \geq 0$. So the path $(c(t), k(t), r(t))$ is interior. Moreover,

$$\int_0^t r(\tau)d\tau \leq \lambda \int_0^t r'(\tau)d\tau + \lambda\epsilon < \lambda m_0 + (1-\lambda)m_0 = m_0\,.$$

Thus, the path $(c(t), k(t), r(t))$ is feasible. ∎

Proof of Lemma 5. Suppose on the contrary that there is $\bar{k} \in (0, \infty)$ such that $k(t) \leq \bar{k}$ for all $t \geq 0$. We have that $F(\bar{k}, 0) = 0$ and $F(\bar{k}, \cdot)$ is continuous, concave and increasing on $\mathbb{R}_+$. Using Jensen's inequality, we have for all $T > 0$:

$$\frac{1}{T}\int_0^T F\left(\bar{k}, r(t)\right) dt \leq F\left(\bar{k},\, \frac{1}{T}\int_0^T r(t)dt\right) \leq F\left(\bar{k}, \frac{m_0}{T}\right)\,.$$

Then we get:

$$k(T) - k_0 = \int_0^T \dot{k}(t)dt \leq \int_0^T F(\bar{k}, r(t))dt - Tc \leq T\left[F\left(\bar{k}, \frac{m_0}{T}\right) - c\right]\,.$$

Since $\lim_{T\to\infty} F\left(\bar{k}, m_0/T\right) = 0$, this implies $k(T) < 0$ for large T, contradicting that $k(t) \geq 0$ for $t \geq 0$. ∎

Proof of Lemma 6. Assume that $(c(t), k(t), r(t))$ is a feasible and interior path from (k_0, m_0).

Part 1: If $(c(t), k(t), r(t))$ satisfies (HoR), then it is competitive. Since F is twice continuously differentiable and $k(t)$ and $r(t)$ are differentiable, we may define $p(t)$ by (P) and set $q_1(t) = p(t)$ and $q_2(t) = 1$ for all $t \geq 0$. Note that, for each $t \geq 0$, $(c(t), k(t), m(t), \dot{k}(t), \dot{m}(t)) \in Y$. Furthermore, for all $(c', k', m', \dot{k}', \dot{m}') \in Y$, we have by **A2** and **A3**:

$$\begin{aligned} (c' + \dot{k}') - (c(t) + \dot{k}(t)) &\leq F(k', -\dot{m}') - F(k(t), -\dot{m}(t)) \\ &\leq F_1(k(t), r(t))\,(k' - k(t)) + F_2(k(t), r(t))\,(-\dot{m}' + \dot{m}(t))\,. \end{aligned} \tag{A6}$$

Multiplying through (A6) by $p(t) > 0$, and using (HoR), (P) and the definitions of $q_1(\cdot)$ and $q_2(\cdot)$, yields:

$$p(t)\,(c' - c(t)) + q_1(t)\big(\dot{k}' - \dot{k}^c(t)\big) \le -\dot{q}_1(t)\,(k' - k(t)) + q_2(t)\,(-\dot{m}' + \dot{m}(t))\ . \tag{A7}$$

Transposing terms in (A7) and noting that $\dot{q}_2(t) = 0$ for $t \ge 0$, we obtain

$$\begin{aligned} & p(t)c(t) + q_1(t)\dot{k}(t) + q_2(t)\dot{m}(t) + \dot{q}_1(t)k(t) + \dot{q}_2(t)m(t) \\ \ge\ & p(t)c' + q_1(t)\dot{k}' + q_2(t)\dot{m}' + \dot{q}_1(t)k' + \dot{q}_2(t)m' \end{aligned}$$

for all $(c', k', m', \dot{k}', \dot{m}') \in Y$ and all $t \ge 0$.

Part 2: If $(c(t), k(t), r(t))$ is competitive, then it satisfies (HoR). By the premise, there are functions $p(\cdot) : [0, \infty) \to \mathbb{R}_{++}$ and $(q_1(\cdot), q_2(\cdot)) : [0, \infty) \to \mathbb{R}^2$, where $p(t)$ is continuous and $(q_1(t), q_2(t))$ are differentiable, such that, for all $t \ge 0$, $(c(t), k(t), m(t), \dot{k}(t), \dot{m}(t))$ maximizes instantaneous profits $p(t)c' + q_1(t)\dot{k}' + q_2(t)\dot{m}' + \dot{q}_1(t)k' + \dot{q}_2(t)m'$ over all quintuples $(c', k', m', \dot{k}', \dot{m}') \in Y$.

Since $(c(t), k(t), r(t))$ is interior, $m(t) > 0$ for all $t \ge 0$, implying that $\dot{q}_2(t) = 0$ for all $t \ge 0$, since otherwise there is some $\tau \ge 0$ such that instantaneous profit could be increased at time τ by $m' \ne m(\tau)$. Furthermore, $q_1(t) \ge p(t) > 0$ for all $t \ge 0$, since otherwise there is some $\tau \ge 0$ such that instantaneous profit could be increased by $\dot{k}' < k(\tau)$ and $c' > c(\tau)$ with $c' + \dot{k}' = c(\tau) + k(\tau)$. Finally, $q_1(t)F_1(k(t), r(t)) = -\dot{q}_1(t)$ and $q_1(t)F_2(k(t), r(t)) = q_2(t)$ for all $t \ge 0$, since otherwise there is some $\tau \ge 0$ such that instantaneous profit could be increased at time τ by $k' \ne k(\tau)$ or $\dot{m}' \ne \dot{m}(\tau)$.

Differentiating $q_1(t)F_2(k(t), r(t)) = q_2(t)$ w.r.t. t and applying $\dot{q}_2(t) = 0$ yields

$$\dot{q}_1(t)F_2(k(t), r(t)) + q_1(t)\dot{F}_2(k(t), r(t)) = 0\,,$$

which combined with $q_1(t)F_1(k(t), r(t)) = -\dot{q}_1(t)$ implies

$$\frac{\dot{F}_2(k(t), r(t))}{F_2(k(t), r(t))} = -\frac{\dot{q}_1(t)}{q_1(t)} = F_1(k(t), r(t))\,,$$

thereby establishing that $(c(t), k(t), r(t))$ satisfies (HoR). ■

Proof of Lemma 7. Assume that $(c(t), k(t), r(t))$ is an interior and competitive path from (k_0, m_0) that satisfies (CVT) and is resource exhausting.

Suppose that $(c'(t), k'(t), r'(t))$ is another feasible path from (k_0, m_0). By the proof of Lemma 6, $(c(t), k(t), r(t))$ is competitive at prices defined by (P) and $q_1(t) = p(t)$ and $q_2(t) = 1$ for all $t \ge 0$. We therefore obtain:

$$\begin{aligned} \int_0^T p(t)\,(c'(t) - c(t))\,dt \ &\le \int_0^T \frac{d}{dt}(p(t)\,(k(t) - k'(t)))\,dt + \int_0^T (\dot{m}(t) - \dot{m}'(t))\,dt \\ &= p(T)\,(k(T) - k'(T)) + \int_0^T (r'(t) - r(t))\,dt \\ &\le p(T)k(T) + m_0 - \int_0^\infty r(t)dt\,, \end{aligned}$$

since $p(T)k'(T) \geq 0$ and $\int_0^T r'(t)dt \leq m_0$ for all $T \geq 0$. From the premise that $(c(t), k(t), r(t))$ satisfies (CVT) and is resource exhausting, it now follows that

$$\limsup_{T\to\infty} \int_0^T p(t)\left(c'(t) - c(t)\right) dt \leq 0\,.$$

Since $p(t) > 0$ for all $t \geq 0$, this establishes that $(c(t), k(t), r(t))$ is efficient. ■

Online appendix B: An example without maximin existence

Consider the case where F is given by $F(k, r) = k + r$ and $(k_0, m_0) = (1, 1)$.

Sustainability. We first claim that $C(k_0, m_0)$ is non-empty. To establish this, simply define:

$$k(t) = 1, \quad r(t) = 0, \quad c(t) = 1 \quad \text{for all } t \geq 0\,,$$

and note that $\dot{k}(t) = 0$ for $t \geq 0$. Thus, $(c(t), k(t), r(t))$ is a path from $(k_0, m_0) = (1, 1)$, and $c(t) = 1 > 0$ for all $t \geq 0$. This establishes our claim.

An upper bound on sustainable consumption. We now claim that that there is no path $(c(t), k(t), r(t))$ satisfying:

$$c(t) \geq 2 \quad \text{for all } t \geq 0\,. \tag{B1}$$

Suppose, there were such a path. We then establish the following steps.

Step 1: We must have $k(t) < 2$ for all $t \geq 0$. For if $k(t) \geq 2$ for some $t \geq 0$, then we can define $T = \inf\{t \geq 0 : k(t) \geq 2\}$. By continuity of $k(t)$, we must have $k(T) = 2$. Since $k(0) = 1$, we know that $T > 0$. Furthermore,

$$k(t) < 2 \quad \text{for all } t \in [0, T)\,. \tag{B2}$$

Denote $(2 - k(t))$ by $\alpha(t)$ for all $t \in [0, T]$, and let $\sigma \equiv \int_0^{T/2} \alpha(t)dt$. Then, by (B2), $\sigma > 0$, and using the fact that for all $t \in [0, T)$,

$$\dot{k}(t) = k(t) + r(t) - c(t) \leq k(t) + r(t) - 2 = r(t) - \alpha(t)\,,$$

we obtain for all $\tau \in [T/2, T)$,

$$k(\tau) - k(0) = \int_0^\tau \dot{k}(t)dt = \int_0^\tau r(t)dt - \int_0^\tau \alpha(t)dt \leq \int_0^\tau r(t)dt - \int_0^{T/2} \alpha(t)dt \leq m(0) - \sigma = 1 - \sigma.$$

Thus, for all $\tau \in [T/2, T)$, we get $k(\tau) \leq k(0) + (1 - \sigma) = (2 - \sigma)$. So, by continuity of $k(t)$, we obtain $k(T) \leq 2 - \sigma$, and this contradicts the fact that $k(T) = 2$. This completes Step 1.

Define $\alpha(t) = 2 - k(t)$ for all $t \geq 0$. Then, $\alpha(t) > 0$ for all $t \geq 0$ by Step 1, and therefore:

$$\beta \equiv \int_0^1 \alpha(t)dt > 0\,.$$

Step 2: We must have $k(t) \leq 2 - \beta$ for all $t \geq 1$. To see this, note that for all $t \geq 0$,

$$\dot{k}(t) = k(t) + r(t) - c(t) \leq k(t) + r(t) - 2 = r(t) - \alpha(t)\,.$$

so that for all $T \geq 1$,

$$k(T) - k(0) = \int_0^T \dot{k}(t)dt = \int_0^T r(t)dt - \int_0^T \alpha(t)dt \leq \int_0^T r(t)dt - \int_0^1 \alpha(t)dt \leq m(0) - \beta = 1 - \beta,$$

and consequently, $k(T) \leq k(0) + (1-\beta) = 2 - \beta$ for all $T \geq 1$. This completes Step 2.

Step 3: $k(t) < 0$ for all $t > (2+\beta)/\beta$. For all $t \geq 0$, we have $\dot{k}(t) = k(t) + r(t) - c(t)$, and we can write for all $T > 1$,

$$\begin{aligned} k(T) - k(0) &= \int_0^T \dot{k}(t)dt = \int_0^T k(t)dt + \int_0^T r(t)dt - \int_0^T c(t)dt \\ &\leq \int_0^1 k(t)dt + \int_1^T k(t)dt + \int_0^T r(t)dt - 2T \\ &< 2 + (2-\beta)(T-1) + 1 - 2T = 1 - \beta(T-1)\,, \end{aligned} \tag{B3}$$

the third line of (B3) following from Steps 1 and 2. Thus, $k(T) \leq k(0) + 1 - \beta(T-1) = 2 - \beta(T-1)$ for all $T > 1$. For $T > (2+\beta)/\beta$, we have $(T-1) > 2/\beta$, and so $\beta(T-1) > 2$. Thus, for $T > (2+\beta)/\beta$, $k(T) \leq 2 - \beta(T-1) < 0$ and this establishes Step 3.

By Step 3, the hypothesis that there is a path $(c(t), k(t), r(t))$ satisfying (B1) must be false, and this establishes our claim.

We have now demonstrated that an upper bound of $C(k_0, m_0)$ is 2. We will show in the next section that this is also its least upper bound.

The supremum of $C(k_0, m_0)$. We now show that, given any $\varepsilon \in (0,1)$, there is a path $(c(t), k(t), r(t))$ satisfying:

$$c(t) \geq 2 - \varepsilon \quad \text{for all } t \geq 0\,.$$

Given the ε, define:

$$n = (1/\varepsilon) \quad \text{and} \quad T = 2/(n+1)^3\,, \tag{B4}$$

and determine the path of resource depletion by:

$$r(t) = \begin{cases} \dfrac{2}{T} - \dfrac{2t}{T^2} & \text{for } t \in [0,T]\,, \\ 0 & \text{for } t > T\,. \end{cases}$$

Then, $r(t) \geq 0$ for $t \in [0,T]$, with $r(T) = 0$, and $r(t) \to 0$ as $t \to T$. Thus $r(t)$ is continuous for $t \geq 0$. Furthermore,

$$\int_0^\infty r(t)dt = \int_0^T r(t)dt = \int_0^T \left\{\frac{2}{T} - \frac{2t}{T^2}\right\} dt = 2 - \frac{2}{T^2}\left[\frac{t^2}{2}\right]_0^T = 2 - 1 = 1 = m_0\,.$$

Denote $(1-\varepsilon^2)$ by λ, and determine the capital path by:

$$k(t) = \begin{cases} 1 + \lambda\left[\dfrac{2t}{T} - \dfrac{t^2}{T^2}\right] & \text{for } t \in [0,T]\,, \\ 1 + \lambda & \text{for } t > T\,. \end{cases} \tag{B5}$$

Note that $k(0) = 1 = k_0$ and $k(t) \geq 1$ for all $t \geq 0$. Furthermore,

$$\dot{k}(t) = \lambda \left[\frac{2}{T} - \frac{2t}{T^2}\right] = \lambda r(t) \geq 0 \quad \text{for } t \in [0, T)$$

with $\dot{k}(T-) = 0 = \dot{k}(T+)$. Thus, $k(t)$ is a C^1 function on $\mathbb{R}_+$, and $\dot{k}(t) = \lambda r(t)$ for all $t \geq 0$.

If we now determine the consumption path by:

$$c(t) = (1 - \lambda) r(t) + k(t) \quad \text{for all } t \geq 0\,, \tag{B6}$$

then clearly $c(t) \geq 0$ for $t \geq 0$, and $c(t) = F(k(t), r(t)) - \lambda r(t) = F(k(t), r(t)) - \dot{k}(t)$ for $t \geq 0$. So $\{c(t), k(t), r(t)\}$ is a path from (k_0, m_0).

It remains to show that $c(t) \geq 2 - \varepsilon$ for all $t \geq 0$. For $t > T$, we have $r(t) = 0$ and $k(t) = 1 + \lambda$, so by (B5) and (B6),

$$c(t) = k(t) = 1 + \lambda = 2 - \varepsilon^2 > 2 - \varepsilon \quad \text{for } t > T\,. \tag{B7}$$

So, we now concentrate on $t \in [0, T]$.

Define $N = nT/(n + 1)$ Then,

$$\begin{aligned} k(N) &= 1 + \lambda \left[\frac{2N}{T} - \frac{N^2}{T^2}\right] = 1 + \lambda \left[\frac{2n}{(n+1)} - \frac{n^2}{(n+1)^2}\right] \\ &= 1 + \frac{\lambda n}{(n+1)} \left[2 - \frac{n}{(n+1)}\right] = 1 + \frac{\lambda n}{(n+1)} \frac{(n+2)}{(n+1)} = 1 + \lambda \left[1 - \frac{1}{(n+1)^2}\right]. \end{aligned} \tag{B8}$$

Now, by choice of n in (B4), we have:

$$(n+1)^2 \geq (n+1) = \frac{1}{\varepsilon} + 1 = \frac{1+\varepsilon}{\varepsilon}$$

and so:

$$\frac{1}{(n+1)^2} \leq \frac{\varepsilon}{1+\varepsilon}\,.$$

Thus,

$$\left[1 - \frac{1}{(n+1)^2}\right] \geq \left[1 - \frac{\varepsilon}{1+\varepsilon}\right] = \frac{1}{1+\varepsilon}\,. \tag{B9}$$

Using (B9) in (B8), we get:

$$k(N) \geq 1 + \frac{\lambda}{1+\varepsilon} = 1 + \frac{1-\varepsilon^2}{1+\varepsilon} = 2 - \varepsilon\,.$$

Since $\dot{k}(t) \geq 0$ for all $t \geq 0$, we have $k(t) \geq 2 - \varepsilon$ for all $t \in [N, T]$. Consequently, using (B6),

$$c(t) = (1 - \lambda) r(t) + k(t) \geq k(t) \geq 2 - \varepsilon \quad \text{for all } t \in [N, T]\,. \tag{B10}$$

Finally, we turn to $t \in [0, N)$. Here, we have by (B6),

$$\begin{aligned} c(t) &= (1-\lambda) r(t) + k(t) \geq (1-\lambda) r(t) + 1 = \varepsilon^2 r(t) + 1 = \varepsilon^2 \left[\frac{2}{T} - \frac{2t}{T^2}\right] + 1 \\ &\geq \varepsilon^2 \left[\frac{2}{T} - \frac{2N}{T^2}\right] + 1 = \varepsilon^2 \left[\frac{2}{T} - \frac{2nT}{(n+1)T^2}\right] + 1 \\ &= \frac{2\varepsilon^2}{T} \left[1 - \frac{n}{(n+1)}\right] + 1 = \frac{2\varepsilon^2}{(n+1)T} + 1 = \varepsilon^2 (n+1)^2 + 1 > \varepsilon^2 n^2 + 1 = 2\,, \end{aligned} \tag{B11}$$

the second line of (B11) following from the definition of N, and the last line of (B11) following from the definitions of T and n in (B4).

Combining (B7), (B10) and (B11), we have $c(t) \geq 2 - \varepsilon$ for all $t \geq 0$, and so:

$$\inf_{t \geq 0} \; c(t) \geq 2 - \varepsilon \, .$$

Combining the result of this part with the previous one, we conclude that the supremum of $C(k_0, m_0)$ is equal to 2. However, as shown in the previous part, there is no path in $C(k_0, m_0)$ which attains this supremum. Thus, there is no maximin path in this model.

It is of interest to note that in discussing basically the same example of the production function, Dasgupta and Heal [3, p. 18] claim that there exists an optimal path in this case. In fact they make the claim that the Dirac delta function is the optimal strategy for optimal depletion of the resource. However, given that the integral of resource depletion needs to be well-defined, at least as a Lebesgue integral, it follows that we cannot admit the Dirac delta-function as a feasible path of resource depletion.

Online appendix C: Steepness and speed in (k, m) space

Introduction. Consider a path (i) satisfying Hotelling's rule (HoR) and (ii) with growing consumption over time, starting from initial stocks $(k_0, m_0) \in \mathbb{R}^2_{++}$. A path which is optimal according to the undiscounted utilitarian criterion will satisfy both these properties. Let us denote this path as $(c_u(t), k_u(t), r_u(t))$ with a subscript u, even though it is not necessarily the utilitarian optimal path; its initial consumption is $c_u(0)$.

Consider, next, the Hartwick path $(c_h(t), k_h(t), r_h(t))$ where consumption is kept constant at $c_u(0)$; that is:

$$c_h(t) = c_u(0) \;\text{ for all } t \geq 0$$

Such a path is well-defined, since we know that $\dot{c}_u(t) > 0$ for all $t \geq 0$. The path $(c_h(t), k_h(t), r_h(t))$ also satisfies Hotelling's rule [2, Proposition 1]. It does not exhaust the resource stock m_0 (and therefore it is "long-run inefficient").

We are interested in establishing two properties regarding the resource use at time $t = 0$. First, we want to show that the initial resource use on the u path is not the same as that on the h path:

$$r_u(0) \neq r_h(0) \tag{C1}$$

Second, we want to show that, in fact, the initial resource use on the u path exceeds the resource use on the h path:

$$r_u(0) > r_h(0) \tag{C2}$$

The reason for separating the results (C1) and (C2) is that (C1) essentially reveals a property about the *steepness* of the trajectories followed by the two paths (evaluated at the same initial stocks) in (k, m) space, while (C2) reveals a property about the *speed* of accumulating the capital stock and depleting the resource stock, which cannot directly be described in (k, m) space.

For the analysis of this appendix, we introduce the following notation:

$$p(t) = \frac{1}{F_2(k_u(t), r_u(t))} \quad \text{for all } t \geq 0\,, \tag{C3}$$

and $V(r) \equiv F(k_0, r) - c_u(0) - F_2(k_0, r)r$.

The value of net investments $p(t)\dot{k}_u(t) - r_u(t)$ is positive. By the identity of [2, Proposition 1] and the fact that $(c_u(t), k_u(t), r_u(t))$ obeys Hotelling's rule (HoR), we obtain

$$p(t)\dot{c}_u(t) + \frac{d}{dt}\Big[p(t)\dot{k}_u(t) - r_u(t)\Big] = 0 \ \text{ for all } t \geq 0 \tag{C4}$$

by applying (C3). Since $p(t) > 0$ and $\dot{c}_u(t) > 0$ for all $t \geq 0$, (C4) implies that:

$$\frac{d}{dt}\Big[p(t)\dot{k}_u(t) - r_u(t)\Big] < 0 \ \text{ for all } t \geq 0\,. \tag{C5}$$

We now claim that:

$$p(t)\dot{k}_u(t) - r_u(t) > 0 \ \text{ for all } t \geq 0\,. \tag{C6}$$

Suppose contrary to (C6), there is some $\tau \geq 0$ such that $p(\tau)\dot{k}_u(\tau) - r_u(\tau) \leq 0$. Then, by (C5), for $T \equiv \tau + 1$, we have that $\theta \equiv p(T)\dot{k}_u(T) - r_u(T) < 0$ and:

$$p(t)\dot{k}_u(t) - r_u(t) \leq \theta \ \text{ for all } t \geq T\,. \tag{C7}$$

Furthermore, since $\dot{p}(t) = -p(t)F_1(k_u(t), r_u(t)) < 0$ (by (HoR)) and $k_u(t) > 0$ for all $t \geq 0$,

$$\frac{d}{dt}\left[p(t)k_u(t)\right] - r_u(t) = \dot{p}(t)k_u(t) + p(t)\dot{k}_u(t) - r_u(t) < p(t)\dot{k}_u(t) - r_u(t) \leq \theta$$

for all $t \geq T$ by (C7). Therefore, for all $S > T$,

$$p(S)k_u(S) < p(T)k_u(T) + \int_T^S r_u(t)dt + \theta(S - T) \leq \left[p(T)k_u(T) + m_0 - \theta T\right] + \theta S\,. \tag{C8}$$

However, the right hand side of (C8) goes to $-\infty$ as $S \to \infty$ (since $\theta < 0$), while the right hand side of (C8) is non-negative for all $S > T$. This contradiction establishes (C6).

Steepness is lower and speed is higher. The Hartwick path $(c_h(t), k_h(t), r_h(t))$ which keeps consumption constant at $c_u(0)$ satisfies Hartwick's rule (HaR), also at $t = 0$:

$$V(r_h(0)) = F(k_0, r_h(0)) - c_u(0) - F_2(k_0, r_h(0))r_h(0) = 0\,.$$

By (C6), the path $(c_u(t), k_u(t), r_u(t))$ with strictly increasing consumption has positive value of net investments, also at $t = 0$: $p(0)\dot{k}_u(0) - r_u(0) > 0$, or equivalently, using (C3):

$$V(r_u(0)) = F(k_0, r_u(0)) - c_u(0) - F_2(k_0, r_u(0))r_u(0) > 0\,.$$

Since $V'(r) = -F_{22}(k_0, r) > 0$, we have established (C2) and thus (C1).

By Lemma 1, $r_h(0) = \mathbf{r}(c_u(0), k_0)$ is the unique solution to the problem of minimizing resource input per unit of capital accumulation, keeping consumption fixed at $c_u(0)$. Hence, the steepness of the trajectory in (k, m) space is maximized if and only if resource input equals $r_h(0)$. By (C1), it follows that the path $(c_u(t), k_u(t), r_u(t))$ with strictly increasing consumption corresponds to a flatter trajectory in (k, m) space than the trajectory corresponding to the Hartwick path $(c_h(t), k_h(t), r_h(t))$ path, at the initial point (k_0, m_0):

$$\frac{\dot{k}_u(0)}{-\dot{m}_u(0)} = \frac{\dot{k}_u(0)}{r_u(0)} < \frac{\dot{k}_h(0)}{r_h(0)} = \frac{\dot{k}_h(0)}{-\dot{m}_h(0)} \,. \tag{C9}$$

We have, by (C2),

$$\dot{k}_u(0) = F(k_0, r_u(0)) - c_u(0) > F(k_0, r_h(0)) - c_u(0) = F(k_0, r_h(0)) - c_h(0) = \dot{k}_h(0) \,.$$

Thus, capital accumulation occurs at a faster rate on the u path compared to the h path, and resource stock depletion occurs at a faster rate on the u path compared to the h path (at least initially). Since (C9) holds, the higher resource stock depletion effect outweighs the higher capital accumulation effect on the u path relative to the h path.

Remark. Solow [5, p. 39] compares the undiscounted utilitarian optimal path with the maximin path from the same initial stocks (in a framework in which the production function is of the Cobb-Douglas form, and the utility function is iso-elastic) and says that the undiscounted utilitarian optimal path "will use up the pool of natural resources more slowly" than the maximin path. It should be noted carefully that our exercise above is not concerned with this comparison. In Solow's comparison, the maximin path is a Hartwick path which is egalitarian (and so satisfies Hotelling's rule), *and it is also efficient.* The h path in our exercise is inefficient, as it is constrained to consume forever the low initial consumption on the u path.

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