Characterizing sustainability in discrete time*

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Abstract

We examine the investment rule that must be satisfied by an efficient and egalitarian path in a discrete-time version of the Dasgupta-Heal-Solow model of capital accumulation and resource depletion. In the discrete-time model, competitive valuation of net investments in terms of *early* and *late* pricing differs. We redefine Hartwick's rule to require zero value of net investments at a valuation rule *intermediate* between these two. Using this definition, we show that along an efficient and egalitarian path, Hartwick's rule is followed in all time periods. We thereby establish the converse of Hartwick's result in discrete time, and we do so under weaker assumptions than those in the existing literature on how output varies as a function of capital and resource use. Our redefinition of Hartwick's rule follows naturally if discrete time is viewed as providing information at discrete points in time of an underlying continuous-time process.

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1 Introduction

Hartwick's rule for sustainability prescribes reinvesting resource rents, thus keeping the value of net investments equal to zero. Originating from Hartwick's original article (Hartwick, 1977), a series of papers (among others, Dixit, Hoel and Hammond, 1980; Withagen and Asheim, 1998; Mitra, 2002; Asheim, Buchholz and Withagen, 2003; Buchholz, Dasgupta and Mitra, 2005; Mitra et al., 2013) have contributed to our understanding of the connection between Hartwick's rule and a sustainable development with constant well-being.

In particular, there is a general relationship between implementing a path which keeps well-being constant (is *egalitarian*) and where well-being cannot be increased for some subinterval of time without being decreased at some other subinterval (is *efficient*), on the one hand, and Hartwick's rule for sustainability, on the other. This relationship can be stated through the following two results:

> *Hartwick's result.* If along an efficient path Hartwick's rule is followed forever, then an egalitarian path is implemented.
>
> *The converse of Hartwick's result.* If an efficient and egalitarian path is implemented, then Hartwick's rule is followed forever.

Asheim (2013) presents an overview of how these results obtain in three different classes of technologies: (i) the one-sector growth model, (ii) the model of capital accumulation and resource depletion used by Hartwick, and (iii) a general model with multiple capital goods.

Now, these results are established under assumptions that are rather strict: the economy is assumed to have constant technology and constant population. Furthermore, the economy is assumed to implement an efficient path in continuous time. There are ways to relax each of these assumptions and still obtain some variant of Hartwick's result and its converse. For example, if there is exogenous technological progress in the sense of a time-dependent technology, we may restore these results by including time as an additional stock.

In this paper we focus on how to relax the assumption that time is continuous. In reality, accounting for the value of net investments will take place at discrete points in time. Futhermore, Fleurbaey (2015, p. 35) suggests that the concept of sustainable development be analysed in discrete time "because discrete time makes it possible to obtain a clear distinction between what the current period does and what the next periods can do given the capital stocks they inherit". Thus, it is of interest to investigate to what extent the relationship between Hartwick's rule and a sustainable development with constant well-being survives in a discrete-time setting. This question was already raised by Dasgupta and Mitra (1983) in a discrete-time version of the model of capital accumulation and resource depletion used by Hartwick. The results were discouraging in the sense that Hartwick's rule is not exactly followed along an efficient and egalitarian path, even if technology and population are constant and rather strong assumptions are imposed on the production function.

This paper shows how the converse of Hartwick's result obtains as an exact result in the model of capital accumulation and resource depletion under weaker assumptions than those imposed by Dasgupta and Mitra (1983). In particular, we impose only regular neoclassical assumption on how net production is a function of the capital and resource use, while relaxing a requirement that resource use have a non-vanishing functional share of output as resource use approaches zero.

The proof of this result is based on the following observation made by Malinvaud (1953) in discrete time and developed in the context of efficient and egalitarian paths in continuous time by Mitra (2002): Vectors of initial stocks that are sufficient to maintain well-being at or above a given level c form a convex set $S(c)$. Furthermore, there is a relative price between capital and resource at which the vector of initial stocks that leads to an efficient and egalitarian path with well-being kept constant at c minimizes the cost of stocks evaluated over all vectors in $S(c)$. Finally, by applying a suggestion made by Dasgupta and Mitra (1983, Section 7), we show that this relative price is exactly the ratio of gross capital productivity to productivity of resource use, which is the *competitive* price ratio that compares the value of stock changes in the context of Hartwick's rule.

The fact that the competitive value of stock is minimized is sufficient to establish that the competitive valuation of net investments between time period t and time period $t+1$ is non-negative when valued at the (higher) *early* price ratio at t and non-positive when valued at the (lower) *late* price ratio at $t+1$. Hence, there exists an intermediate price ratio of capital augmentation to resource use, between the one at $t+1$ and the one at t, that leads to zero value of net investments. Thus, by redefining Hartwick's rule to require zero value of net investments at a valuation rule *intermediate* between early and late pricing, we show that Hartwick's rule is followed in all time periods along an efficient and egalitarian path.

We start out in Section 2 by presenting the assumptions that we impose on the model of capital accumulation and resource depletion. We define properties for paths in this model, derive preliminary results on the competitive price support of interior paths (with positive capital and resource use in each time period), and adapt the definition of Hartwick's rule to the discrete-time setting. We then proceed in Section 3 to prove that the converse of Hartwick's result holds even in discrete time.

In the concluding Section 4 we show how our redefinition of Hartwick's rule follows naturally if we view discrete time as providing information at discrete points in time of an underlying continuous-time process. This perspective allows us to relate our adaptation of Hartwick's rule to discrete time to the continuous-time version of the converse of Hartwick's result. Furthermore, we discuss the challenges faced when attempting to establish Hartwick's rule as a prescriptive rule in a discrete-time setting. In particular, we pose the question whether Hartwick's result can be demonstrated in discrete time, so that a policy of zero value of net investments can be used to steer the economy along an efficient and egalitarian path.

2 Framework

Consider a model with one produced good, which serves as both the capital and the consumption good, and an exhaustible resource. Labor is assumed to be constant over time. The framework described below is the standard one employed in the literature on intertemporal resource allocation in the presence of an exhaustible resource (see for example Solow, 1974; Dasgupta and Heal, 1974), modified by a discrete-time formulation of the economic dynamics, following Mitra (1978) and Dasgupta and Mitra (1983). We refer to this as the discrete-time DHS model.

Denote by k the stock of the augmentable capital good (which is assumed to be non-depreciating), and by r the flow of exhaustible resource use. Let $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ denote the production function for the capital/consumption good; the production process uses the capital stock, k, and the exhaustible resource use, r, as inputs. The net output $G(k, r)$ is used to provide consumption c—being an indicator of well-being—or to provide augmentation of k. Output $G(k, r)$ is the only source of flow of consumption or of net investment.

We will use the following standard assumptions on the production function G.

(G1) $G(k, r)$ is a function from $\mathbb{R}^2_+$ to $\mathbb{R}_+$, which is continuous, non-decreasing and concave in (k, r) on $\mathbb{R}^2_+$.

(G2) $G(k, 0) = G(0, r) = 0$ for all $(k, r) \in \mathbb{R}^2_+$.

(G3) G is strictly increasing in each argument on $\mathbb{R}^2_{++}$, and continuously differentiable in (k, r) on $\mathbb{R}^2_{++}$, with $G_k(k, r) > 0$ and $G_r(k, r) > 0$ for all $(k, r) \in \mathbb{R}^2_{++}$.

These assumptions impose regular neoclassical properties and correspond to assumptions (A1)–(A3) of Dasgupta and Mitra (1983). However, our results do not require assumption (A4) (or (A4′)) of Dasgupta and Mitra (1983), imposing that resource use is 'important' in the sense that resource use has a non-vanishing functional share of output as resource use approaches zero. Assumptions (G1)–(G3) are always satisfied when G takes the Cobb-Douglas form (used by Solow, 1974):

$$G(k, r) = k^{\alpha} r^{\beta} \quad \text{for all } (k, r) \in \mathbb{R}^2_+ \,, \tag{1}$$

with $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$.

Denote by m the stock of the resource. The transformation of the stocks k and m from time period t (≥ 0) to the next time period $t+1$ satisfies:

$$y_t \leq G(k_t, r_t) + k_t \,,$$
$$k_{t+1} = y_t - c_t \,,$$
$$m_{t+1} = m_t - r_t \,,$$

where y_t, k_t, m_t, c_t and r_t are constrained to be non-negative for all $t \geq 0$. This dynamic transformation can be expressed through the following tranformation set:

$$\mathcal{T} = \{\big((k,m),(y,m')\big) : 0 \leq y \leq G(k, m-m') + k,\; k \geq 0,\; m \geq m' \geq 0\}\,.$$

A sequence $\{k_t, m_t, c_t\}$ is a *path* from $(k,m) \in \mathbb{R}^2_+$ if $(k_0, m_0) = (k,m)$ and

$$((k_t, m_t), (c_t + k_{t+1}, m_{t+1})) \in \mathcal{T} \;\text{ and }\; c_t \geq 0 \;\text{ for all } t \geq 0\,.$$

Denote by $\mathcal{F}(k,m)$ the set of paths from any $(k,m) \in \mathbb{R}^2_+$.

We provide a list of properties for paths: A path $\{k_t, m_t, c_t\}$ from $(k,m) \in \mathbb{R}^2_+$

- is *efficient* if there does not exist $\{k'_t, m'_t, c'_t\} \in \mathcal{F}(k,m)$ with $c'_t \geq c_t$ for all $t \geq 0$ and $c'_s > c_s$ for some $s \geq 0$,
- is *egalitarian* if $c_{t+1} = c_t$ for all $t \geq 0$,
- is *interior* if $k_t > 0$ and $m_{t+1} < m_t$ for all $t \geq 0$,
- is *non-wasteful* if $y_t = G(k_t, m_t - m_{t+1}) + k_t$ for all $t \geq 0$,
- is *competitive* if there is a non-null sequence $\{p_t, q_t\}$ of non-negative price pairs such that, for all $t \geq 0$, we have:

$$\begin{aligned} p_{t+1}y_t + q_{t+1}m_{t+1} - p_t k_t - q_t m_t \;&\geq\; p_{t+1}y + q_{t+1}m' - p_t k - q_t m \\ &\text{for all } \big((k,m),(y,m')\big) \in \mathcal{T} \end{aligned} \tag{C}$$

- satisfies *Hotelling's rule* if $\{k_t, m_t, c_t\}$ is interior and:

$$\frac{G_r(k_{t+1}, m_{t+1} - m_{t+2})}{G_r(k_t, m_t - m_{t+1})} = G_k(k_{t+1}, m_{t+1} - m_{t+2}) + 1 \quad \text{for all } t \geq 0, \tag{H}$$

– satisfies the *capital-value transversality condition* if $\{k_t, m_t, c_t\}$ is interior and:

$$\lim_{t\to\infty} \left(\frac{G_k(k_t, m_t - m_{t+1}) + 1}{G_r(k_t, m_t - m_{t+1})} \right) k_t = 0\,,$$

– satisfies *resource exhaustion* if:

$$\lim_{t\to\infty} m_t = 0\,.$$

Consider an interior path $\{k_t, m_t, c_t\}$ from $(k, m) \in \mathbb{R}^2_{++}$ and define a sequence $\{p_t, q_t\}$ of positive price pairs as follows, where the resource is numéraire:

$$p_t = \frac{G_k(k_t, m_t - m_{t+1}) + 1}{G_r(k_t, m_t - m_{t+1})} \quad \text{and} \quad q_t = 1 \quad \text{for all } t \geq 0\,. \tag{P}$$

The following result summarizes the principal connections between competitive paths and Hotelling's rule in the context of the discrete-time DHS model.

Lemma 1 *Asssume (G1)–(G3), and let $\{k_t, m_t, c_t\}$ be an interior path from $(k, m) \in \mathbb{R}^2_{++}$. Then:*

(a) If $\{k_t, m_t, c_t\}$ is non-wasteful and satisfies Hotelling's rule, then $\{k_t, m_t, c_t\}$ is competitive with associated sequence $\{p_t, q_t\}$ of price pairs given by (P).

(b) If $\{k_t, m_t, c_t\}$ is competitive, then $\{p_t, q_t\}$ given by (P) is an associated sequence of price pairs, which is unique up to a positive linear tranformation. Furthermore, $\{k_t, m_t, c_t\}$ is non-wasteful and satisfies Hotelling's rule.

Proof. (a) Assume that $\{k_t, m_t, c_t\}$ is non-wasteful and satisfies Hotelling's rule. Let $t \geq 0$ be an arbitrary time period and let $((k, m), (y, m')) \in \mathcal{T}$. Then, denoting $m - m'$ by r and $m_t - m_{t+1}$ by r_t, we have $r \geq 0$ and:

$$\begin{aligned} y - y_t \;&\leq\; \big(G(k, m - m') + k\big) - (G(k_t, m_t - m_{t+1}) + k_t) \\ &\leq\; (G_k(k_t, r_t) + 1)\,(k - k_t) + G_r(k_t, r_t)(r - r_t)\,, \end{aligned} \tag{2}$$

where we have used non-wastefulness of $\{k_t, m_t, c_t\}$ in the first line of (2), and (G1) and (G3) in the second line of (2).

Dividing through in (2) by $G_r(k_t, r_t)$ and using (H) and (P), we get:

$$
\begin{aligned}
p_{t+1}(y-y_t) \;=\; \frac{1}{G_r(k_t,r_t)}(y-y_t) \;&\le\; \left(\frac{G_k(k_t,r_t)+1}{G_r(k_t,r_t)}\right)(k-k_t)+(r-r_t) \\
&=\; p_t(k-k_t)+q_t(r-r_t)\,.
\end{aligned} \tag{3}
$$

Transposing terms in (3), we obtain that (C) is satisfied:

$$
p_{t+1}y_t + q_{t+1}m_{t+1} - p_t k_t - q_t m_t \;\ge\; p_{t+1}y + q_{t+1}m' - p_t k - q_t m\,.
$$

(b) Assume that $\{k_t, m_t, c_t\}$ is competitive with associated non-null sequence $\{p_t, q_t\}$ of non-negative price pairs.

First, we establish that $q_{t+1} = q_t$ for all $t \ge 0$. To show this, note that

$$
\big((k,m),(y,m')\big) = ((k_t, m_t+\varepsilon),(G(k_t, m_t - m_{t+1}) + k_t, m_{t+1}+\varepsilon)) \in \mathcal{T}
$$

for all $\varepsilon \ge -m_{t+1}$ (< 0), where $t \ge 0$ is an arbitrary time period. Furthermore,

$$
\big(p_{t+1}y + q_{t+1}m' - p_t k - q_t m\big) - (p_{t+1}y_t + q_{t+1}m_{t+1} - p_t k_t - q_t m_t) \;\ge\; (q_{t+1}-q_t)\varepsilon\,,
$$

where the inequality might be strict if $\{k_t, m_t, c_t\}$ is not non-wasteful. By choosing $\varepsilon > 0$, the competitiveness of $\{k_t, m_t, c_t\}$ is contradicted if $q_{t+1} > q_t$. And likewise, by choosing $\varepsilon < 0$, the competitiveness of $\{k_t, m_t, c_t\}$ is contradicted if $q_{t+1} < q_t$. Hence, $q_t = q \ge 0$ for all $t \ge 0$.

Second, we observe that

$$
p_{t+1}y_t = p_{t+1}G(k_t, m_t - m_{t+1}) \quad \text{for all } t \ge 0, \tag{4}
$$

since $p_{t+1}y_t > p_{t+1}G(k_t, m_t - m_{t+1})$ is infeasible and $p_{t+1}y_t < p_{t+1}G(k_t, m_t - m_{t+1})$ would contract the competitiveness of $\{k_t, m_t, c_t\}$ as

$$
\big((k,m),(y,m')\big) = ((k_t, m_t),(G(k_t, m_t - m_{t+1}) + k_t, m_{t+1})) \in \mathcal{T}\,.
$$

Denoting $m - m'$ by r and $m_t - m_{t+1}$ by r_t, it follows from (4) and the competitiveness of $\{k_t, m_t, c_t\}$ that, for all $t \ge 0$:

$$
p_{t+1}G(k_t, r_t) - p_t k_t - q r_t \;\ge\; p_{t+1}G(k,r) - p_t k - q r \quad \text{for all } (k,r) \in \mathbb{R}^2_+\,. \tag{5}
$$

Since the path $\{k_t, m_t, c_t\}$ is interior, we have that $(k_t, r_t) \in \mathbb{R}^2_{++}$ for all $t \geq 0$. The first-order conditions for the maximization in (5) imply:

$$p_{t+1}\left(G_k(k_t, r_t) + 1\right) = p_t \quad \text{for all } t \geq 0\,, \tag{6}$$

$$p_{t+1} G_r(k_t, r_t) = q \quad \text{for all } t \geq 0\,. \tag{7}$$

If $q = 0$, then (7) implies that $p_{t+1} = 0$ for all $t \geq 0$ (by using (G3)), and $p_0 = 0$ by (6). However, this contradicts the fact that the sequence $\{p_t, q_t\}$ is non-null. Thus, $q > 0$, and we normalize by letting $q = 1$. Furthermore, it follows from (6) and (7) that $\{p_t, q_t\}$ is given by (P) and that this sequence is unique up to a positive linear tranformation. In particular, $p_t > 0$ for all $t \geq 0$ (using (G3)), implying that non-wastefulness follows directly from (4). Finally, (6) and (7) imply that

$$\frac{G_r(k_{t+1}, r_{t+1})}{G_r(k_t, r_t)} = \frac{p_{t+1}}{p_{t+2}} = G_k(k_{t+1}, r_{t+1}) + 1 \quad \text{for all } t \geq 0\,,$$

showing that (H) holds. ■

The following corollary is immediate.

Corollary 1 *Under (G1)–(G3), if $\{k_t, m_t, c_t\}$ is an interior and competitive path from $(k, m) \in \mathbb{R}^2_{++}$ with associated sequence $\{p_t, q_t\}$ of price pairs given by (P), then the price sequence $\{p_t\}$ is decreasing.*

Proof. We have that the price sequence $\{p_t\}$ is decreasing since

$$\frac{p_t}{p_{t+1}} = G_k(k_t, m_t - m_{t+1}) + 1 > 1 \quad \text{for all } t \geq 0\,,$$

by (6) and (G3) and the fact that $\{k_t, m_t, c_t\}$ is interior. ■

That the sequence $\{p_t\}$ of capital prices is decreasing corresponds to a positive rate of interest associated with the positive net capital productivity of G. It means that the sequence $\{p_t, q_t\}$ of positive price pairs as determined by (P) is expressed as present-value prices, where time-discounting is an integral part of the price system.

In discrete time the value of net investments at time $t \geq 0$ along a competitive path, $\{k_t, m_t, c_t\}$, can be defined in terms of *early* or *late* pricing, respectively:

$$E_t = p_t (k_{t+1} - k_t) + (m_{t+1} - m_t),$$
$$L_t = p_{t+1} (k_{t+1} - k_t) + (m_{t+1} - m_t).$$

As we will see, these measures of the value of net investments differ along an efficient and egalitarian path and cannot both be zero. Still, we will be able to show that an efficient and egalitarian path is characterized by the requirement that resource rents be reinvested in augmentable capital in all time periods, provided that we adopt the following redefinition of Hartwick's rule.

Definition 1 (Hartwick's rule in the discrete-time DHS model) *Hartwick's rule is followed in time period t along an interior and competitive path $\{k_t, m_t, c_t\}$ from $(k, m) \in \mathbb{R}^2_{++}$ with associated sequence $\{p_t, q_t\}$ of price pairs given by (P) if there exists $p \in [p_{t+1}, p_t]$ such that $p (k_{t+1} - k_t) + (m_{t+1} - m_t) = 0$.*

3 Properties of an efficient and egalitarian path

In this section, we present the main result of the paper (Theorem 1) on efficient and egalitarian paths. As prerequisites of that result, we develop basic properties associated with such paths. These can be conveniently separated into competitive conditions (presented in Section 3.1), and conditions on the value of investments (presented in Section 3.2).

Throughout, we will assume the *existence* of an efficient and egalitarian path. We do not address the independent question relating to the technological conditions which characterize this assumption. This we consider a separate research project.

Define:

$$\Omega = \{(k, m) \in \mathbb{R}^2_{++} : \text{there exists an efficient path } \{k_t, m_t, c_t\} \text{ from } (k, m) \text{ such that } c_t \text{ is a positive constant for all } t \geq 0\}.$$

Note that, in general, Ω can be empty. We will assume that:

(E) The set Ω is non-empty.

Condition (E) ensures that there is a stock pair (k, m), such that there is an efficient and egalitarian path from (k, m) with positive (constant) consumption. With a Cobb-Douglas production function, as given by (1), every pair $(k, m) \in \mathbb{R}^2_{++}$ belongs to Ω provided $\alpha > \beta$. (This can be inferred from results reported in Dasgupta and Mitra, 1983, and Cass and Mitra, 1991.)

Let $(k, m) \in \Omega$. Note that, for any efficient path $\{k_t, m_t, c_t\}$ from (k, m), such that c_t is a positive constant for all $t \geq 0$, we must have the positive constant to be unique. (For if there were two such constants, then the path with the lower constant consumption would be inefficient.) Consequently, with each $(k, m) \in \Omega$, we can associate this unique constant consumption level, and denote it by $c(k, m)$. Thus, a consequence of (E) is the following observation:

(E$'$) There are $(k, m) \in \Omega$ and an efficient and egalitarian path $\{k_t, m_t, c_t\}$ from (k, m) with $c_t = c(k, m) > 0$ for all $t \geq 0$.

In what follows, we fix a stock pair $(k, m) \in \Omega$ (guaranteed by (E)), and we fix an efficient and egalitarian path (guaranteed by (E$'$)), and denote it by $\{k_t, m_t, c_t\}$; furthermore, we denote $c(k, m)$ by c.

3.1 Competitive conditions

We define:

$$S(c) = \{(k', m') \in \mathbb{R}^2_+ : \text{there exists } \{k'_t, m'_t, c'_t\} \in \mathcal{F}(k', m') \text{ with } c'_t \geq c \text{ for all } t \geq 0\}.$$

Thus, $S(c)$ is the pair of initial stocks that enable consumption to be maintained at or above $c = c(k, m) > 0$.[1] Clearly, for each $t \geq 0$, we have $(k_t, m_t) \in S(c)$.

We will note two important properties of the set $S(c)$; these are stated in Lemmas 2 and 3. These properties will be used to establish a 'cost minimization' property of an efficient and egalitarian path in Proposition 1.

[1] In the viability approach (Aubin, 1991), the set $S(c)$ is referred to as the *viability kernel*, a concept which is applied in the continuous-time DHS model by Martinet and Doyen (2007) and Doyen and Martinet (2012). Furthermore, the price pair (P_t, Q_t) established in Proposition 1 below corresponds to an element in the *normal cone* to the viability kernel $S(c)$ at (k_t, m_t).

For statement of the first of these lemmas (as well as for the proof of Proposition 1), define, for each $t \geq 0$, the set of stock vectors where neither stock exceeds (k_t, m_t):

$$R_t = \{(k', m') \in \mathbb{R}_+^2 : (k', m') \leq (k_t, m_t)\}.$$

Lemma 2 *Under (G1)–(G3) and (E), for each* $t \geq 0$, $S(c) \cap R_t = \{(k_t, m_t)\}$.

Proof. Fix an arbitrary $t \geq 0$.

Assume that there exists $\{k'_\tau, m'_\tau, c'_\tau\} \in \mathcal{F}(k', m')$ with $c'_\tau \geq c$ for all $\tau \geq 0$, where $k' < k_t$ and $m' \leq m_t$. Construct $\{k''_\tau, m''_\tau, c''_\tau\} \in \mathcal{F}(k_t, m_t)$ by letting

$$\begin{aligned}(k''_0, m''_0, c''_0) &= (k_t, m_t, G(k_t, m_t - m'_1) + k_t - k'_1), \\ (k''_\tau, m''_\tau, c''_\tau) &= (k'_\tau, m'_\tau, c'_\tau) \quad \text{for } \tau \geq 1.\end{aligned}$$

It follows from (G1) that $c''_0 = G(k_t, m_t - m'_1) + k_t - k'_1 > G(k', m' - m'_1) + k' - k'_1 \geq c'_0 \geq c = c_t$, since $k' < k_t$ and $m' \leq m_t$, while $c''_\tau = c'_\tau \geq c = c_{t+\tau}$ for all $\tau \geq 1$. This contradicts the fact that $\{k_t, m_t, c_t\}$ is efficient.

Next, assume that there exists $\{k'_\tau, m'_\tau, c'_\tau\} \in \mathcal{F}(k', m')$ with $c'_\tau \geq c$ for all $t \geq 0$, where $k' \leq k_t$ and $m' < m_t$. By (G2), $k' > 0$, since otherwise $c'_\tau = 0 < c$ for all $t \geq 0$. Construct $\{k''_\tau, m''_\tau, c''_\tau\} \in \mathcal{F}(k_t, m_t)$ by letting

$$\begin{aligned}(k''_0, m''_0, c''_0) &= (k_t, m_t, G(k_t, m_t - m'_1) + k_t - k'_1), \\ (k''_\tau, m''_\tau, c''_\tau) &= (k'_\tau, m'_\tau, c'_\tau) \quad \text{for } \tau \geq 1.\end{aligned}$$

It follows from (G3) that $c''_0 = G(k_t, m_t - m'_1) + k_t - k'_1 > G(k', m' - m'_1) + k' - k'_1 \geq c'_0 \geq c = c_t$, since $k' \leq k_t$ and $m' < m_t$, while $c''_\tau = c'_\tau \geq c = c_{t+\tau}$ for all $\tau \geq 1$. This contradicts the fact that $\{k_t, m_t, c_t\}$ is efficient. ■

Lemma 2 leads to the following corollary. This has already been established by Dasgupta and Mitra (1983, Proposition 2), but we state and prove the result here for completeness.

Corollary 2 *Under (G1)–(G3) and (E), the efficient and egalitarian path* $\{k_t, m_t, c_t\}$ *from* $(k, m) \in \Omega$ *with* $c_t = c > 0$ *for all* $t \geq 0$ *is interior and satisfies* $k_{t+1} > k_t$ *for* *all* $t \geq 0$.

Proof. Since by feasibility we have $m_{t+1} \leq m_t$, it follows from Lemma 2 that $k_{t+1} \geq k_t$, and consequently $G(k_t, m_t - m_{t+1}) \geq c > 0$. By (G2), we must have $k_t > 0$ and $m_t > m_{t+1}$, so that $\{k_t, m_t, c_t\}$ is interior. Furthermore, by Lemma 2, $k_{t+1} = k_t$ is ruled out, so we must have $k_{t+1} > k_t$. ■

Lemma 3 *Under (G1), $S(c)$ is a convex subset of $\mathbb{R}^2_+$.*

Proof. Assume that (k', m') and (k'', m'') are both in $S(c)$. Hence, there exist $\{k'_t, m'_t, c'_t\} \in \mathcal{F}(k', m')$ with $c'_t \geq c$ for all $t \geq 0$ and $\{k''_t, m''_t, c''_t\} \in \mathcal{F}(k'', m'')$ with $c''_t \geq c$ for all $t \geq 0$. Let $\lambda \in (0, 1)$, and define $\{k'''_t, m'''_t, c'''_t\}$ for all $t \geq 0$ by

$$\begin{aligned} k'''_t &= \lambda k'_t + (1-\lambda)k''_t\,, \\ m'''_t &= \lambda m'_t + (1-\lambda)m''_t\,, \\ c'''_t &= \lambda c'_t + (1-\lambda)c''_t\,. \end{aligned}$$

Since $c'''_t = \lambda c'_t + (1-\lambda)c''_t \geq \lambda c + (1-\lambda)c = c$ for all $t \geq 0$, it suffices to show that $c'''_t + k'''_{t+1} \leq G(k'''_t, m'''_t - m'''_{t+1}) + k'''_t$ for all $t \geq 0$. This holds since:

$$\begin{aligned} c'''_t + k'''_{t+1} &= \lambda(c'_t + k'_{t+1}) + (1-\lambda)(c''_t + k''_{t+1}) \\ &\leq \lambda\left(G(k'_t, m'_t - m'_{t+1}) + k'_t\right) + (1-\lambda)\left(G(k''_t, m''_t - m''_{t+1}) + k''_t\right) \\ &\leq G(\lambda k'_t + (1-\lambda)k''_t, \lambda(m'_t - m'_{t+1}) + (1-\lambda)(m''_t - m''_{t+1})) + \lambda k'_t + (1-\lambda)k''_t \\ &= G(k'''_t, m'''_t - m'''_{t+1}) + k'''_t\,, \end{aligned}$$

where the second inequality follows since, by (G1), G is concave. ■

Proposition 1 *Under (G1)–(G3) and (E), for each $t \geq 0$, there exists a non-null and non-negative price pair (P_t, Q_t) such that (k_t, m_t) minimizes $P_t k' + Q_t m'$ over all $(k', m') \in S(c)$.*

Proof. Fix an arbitrary $t \geq 0$ and define:

$$D = \{(k'' - k', m'' - m') : (k'', m'') \in R_t \text{ and } (k', m') \in S(c)\}\,.$$

Clearly, R_t is a convex subset of $\mathbb{R}^2$, and since $S(c)$ is a convex subset of $\mathbb{R}^2$ by Lemma 3, so is D. Suppose $D \cap \mathbb{R}^2_{++} \neq \emptyset$. Then there would exist $(k'', m'') \in R_t$ and $(k', m') \in S(c)$ such that $(k'', m'') \gg (k', m')$. Since $(k'', m'') \leq (k_t, m_t)$, this means there would exist $(k', m') \in S(c)$ with $(k', m') \ll (k_t, m_t)$, which is impossible by Lemma 2. Thus, we can apply the separation theorem 3.5 in Nikaido (1968, p. 35) to assert that there exists $(P_t, Q_t) \in \mathbb{R}^2_+$, with (P_t, Q_t) non-null, such that:

$$P_t \tilde{k} + Q_t \tilde{m} \leq 0 \quad \text{for all } (\tilde{k}, \tilde{m}) \in D\,.$$

This means that for all $(k'', m'') \in R_t$ and $(k', m') \in S(c)$, we must have:

$$P_t(k'' - k') + Q_t(m'' - m') \leq 0\,.$$

Since $(k_t, m_t) \in R(t)$, we must have:

$$P_t k_t + Q_t m_t \leq P_t k' + Q_t m' \quad \text{for all } (k', m') \in S(c)\,.$$

This establishes the proposition. ■

The following result shows that the efficient and egalitarian path $\{k_t, m_t, c_t\}$ from $(k, m) \in \Omega$ with $c_t = c > 0$ for all $t \geq 0$ satisfies Hotelling's rule, among other properties. Note that the proof of this result only uses the efficiency and interiority of the path; the fact that $\{k_t, m_t, c_t\}$ is egalitarian is not directly invoked.

Proposition 2 *Under (G1)–(G3) and (E), the efficient and egalitarian path* $\{k_t, m_t, c_t\}$ *from* $(k, m) \in \Omega$ *with* $c_t = c > 0$ *for all* $t \geq 0$ *is non-wasteful and satisfies Hotelling's rule and resource exhaustion.*

Proof. (i) If $c_s + k_{s+1} < G(k_s, m_s - m_{s+1}) + k_s$ for some $s \geq 0$, then consumption can be increased at time s without changing consumption at any other time $t \geq 0$. This contradicts the fact that $\{k_t, m_t, c_t\}$ is efficient. Hence, $c_s + k_{t+1} = G(k_t, m_t - m_{t+1}) + k_t$ for all $t \geq 0$, showing that $\{k_t, m_t, c_t\}$ is non-wasteful.

(ii) Since m_t is non-increasing in t, and non-negative, the sequence $\{m_t\}$ has a non-negative limit: $\lim_{t\to\infty} m_t \geq 0$. Suppose $\lim_{t\to\infty} m_t > 0$. Then resource use can be increased at some time $s \geq 0$ without changing resource use at any other time $t \geq 0$. By (G3) and the fact that $\{k_t, m_t, c_t\}$ is interior, production and therefore consumption can be increased at time s, without changing consumption at any other time $t \geq 0$; this contradicts the fact that $\{k_t, m_t, c_t\}$ is efficient. Hence, we must have $\lim_{t\to\infty} m_t = 0$, showing that $\{k_t, m_t, c_t\}$ satisfies resource exhaustion.

(iii) By Corollary 2, $\{k_t, m_t, c_t\}$ is interior, so it only remains to show that (H) holds. Fix an arbitrary $t \geq 0$. Denote $m_t - m_{t+1}$ by r_t and $m_{t+1} - m_{t+2}$ by r_{t+1}. The triple (k_{t+1}, r_t, r_{t+1}) clearly minimizes the sum $r'_t + r'_{t+1}$ of resource use in time periods t and $t+1$ over all triples $\left(k'_{t+1}, r'_t, r'_{t+1}\right) \gg 0$ that satisfy the set of constraints:

$$\left.\begin{aligned} c + k'_{t+1} &\leq G(k_t, r'_t) + k_t \,, \\ c + k_{t+2} &\leq G(k'_{t+1}, r'_{t+1}) + k'_{t+1} \,. \end{aligned}\right\}$$

For if there is a triple $(k'_{t+1}, r'_t, r'_{t+1}) \gg 0$ which satisfies this set of constraints, and $\varepsilon := (r_t + r_{t+1}) - (r'_t + r'_{t+1}) > 0$, then $(k, m - \varepsilon) \in S(c)$, and this has been shown to be impossible by Lemma 2.

Clearly $(k_{t+1}, r_t, r_{t+1}) \gg 0$ satisfies each of the constraints in this set with equality (since $\{k_t, m_t, c_t\}$ is non-wasteful). Therefore $(k'_{t+1}, r'_t, r'_{t+1}) := (k_{t+1}, r_t + \theta, r_{t+1} + \theta)$ satisfies each of the constraints with strict inequality (by (G3)), so that Slater's Condition is satisfied. Since G is concave (by (G1)), we can invoke the version of the Kuhn-Tucker theorem due to Arrow, Hurwicz and Uzawa (1961) to assert that there exist $(p_{t+1}, p_{t+2}) \in \mathbb{R}^2_+$ such that the following first-order conditions hold:

$$p_{t+2}\left(G_k(k_{t+1}, r_{t+1}) + 1\right) = p_{t+1} \,,$$

$$p_{t+1} G_r(k_t, r_t) = 1 \,,$$

$$p_{t+2} G_r(k_{t+1}, r_{t+1}) = 1 \,.$$

Since $\{k_t, m_t, c_t\}$ is interior, we can use (G3) and the second and third equations above to infer that $(p_{t+1}, p_{t+2}) \in \mathbb{R}^2_{++}$. Now, the above set of equations yields:

$$\frac{G_r(k_{t+1}, r_{t+1})}{G_r(k_t, r_t)} = \frac{p_{t+1}}{p_{t+2}} = G_k(k_{t+1}, r_{t+1}) + 1 \,.$$

Since $t \geq 0$ is arbitrary, this establishes (H). ■

Corollary 3 *Under (G1)–(G3) and (E), the efficient and egalitarian path* $\{k_t, m_t, c_t\}$ *from* $(k, m) \in \Omega$ *with* $c_t = c > 0$ *for all* $t \geq 0$ *is competitive with associated sequence* $\{p_t, q_t\}$ *of price pairs given by (P).*

Proof. This is an immediate consequence of Proposition 2 and Lemma 1. ■

We can now state an important property of an efficient and egalitarian path, namely that it satisfies the 'cost minimization' property in terms of competitive prices. In the proof of this result we utilize the observation that, for each $t \geq 0$, $S_t(c) \subseteq S(c)$, where $S_t(c)$ denotes the set of initial pairs of stocks that enable consumption to be maintained at or above c, provided that $\{k'_\tau, m'_\tau, c'_\tau\}$ with $(k'_\tau, m'_\tau, c'_\tau) = (k_{t+1+\tau}, m_{t+1+\tau}, c_{t+1+\tau})$ for all $\tau \geq 0$ is followed from (k_{t+1}, m_{t+1}):

$$S_t(c) = \{(k', m') : \big((k', m'), (c + k_{t+1}, m_{t+1})\big) \in \mathcal{T}\} . \tag{8}$$

Hence, by Proposition 1 and the fact that $(k_t, m_t) \in S_t(c)$, we have that, for each $t \geq 0$, (k_t, m_t) minimizes $P_t k' + Q_t m'$ over all $(k', m') \in S_t(c)$, which, by the differentiability of G, leads to the result that the price ratio P_t/Q_t equals the competitive price ratio p_t as given by (P).

Proposition 3 *Assume (G1)–(G3) and (E), and let the price sequence* $\{p_t\}$ *be determined by (P) from the efficient and egalitarian path* $\{k_t, m_t, c_t\}$ *from* $(k, m) \in \Omega$ *with* $c_t = c > 0$ *for all* $t \geq 0$*. Then, for each* $t \geq 0$*, we have that* (k_t, m_t) *minimizes* $p_t k' + m'$ *over all* $(k', m') \in S(c)$*.*

Proof. Fix an arbitrary $t \geq 0$. Define $X = \{(k', m') \in \mathbb{R}^2 : k' > 0, m' > m_{t+1}\}$, and let $x : X \to \mathbb{R}$ determine the excess of consumption in time period t relative to c, if the stock vector (k', m') at time t is transformed to the stock vector (k_{t+1}, m_{t+1}) at time $t + 1$ in a non-wasteful manner:

$$x(k', m') \;=\; \big(G(k', m' - m_{t+1}) + k'\big) - k_{t+1} - c\,.$$

Then, X is an open set in $\mathbb{R}^2$ and x is continuously differentiable on X (by (G3)). Furthermore, since $\{k_t, m_t, c_t\}$ is non-wasteful by Proposition 2, we have $x(k_t, m_t) = 0$. Note that by interiority of $\{k_t, m_t, c_t\}$ and (G3),

$$x_m(k_t, m_t) \;=\; G_r(k_t, m_t - m_{t+1}) \;\neq\; 0\,.$$

Hence, we can invoke the Implicit Function Theorem (see Rudin, 1976, Theorem 9.28, pp. 224–5) to assert that there is an open interval A containing k_t, and a unique function $m : A \to \mathbb{R}$ which, for each $k' \in A$, determines the smallest resource stock $m(k') = m'$ enabling consumption to equal c at time t when (k', m') at time t is transformed to (k_{t+1}, m_{t+1}) at time $t + 1$. In effect, the function m maps the frontier of the set $S_t(c)$, as defined by (8), in a neighborhood of (k_t, m_t). We have:

(i) $(k', m(k')) \in X$ and $x(k', m(k')) = 0$ for all $k' \in A$,

(ii) $m(k_t) = m_t$,

(iii) m is continuously differentiable on A.

Using (i) and (iii), we have: $x_k(k_t, m(k_t)) + x_m(k_t, m(k_t)) \cdot m'(k_t) = 0$. And, using (ii) and the definition of x, we obtain:

$$(G_k(k_t, m_t - m_{t+1}) + 1) \;+\; G_r(k_t, m_t - m_{t+1}) \cdot m'(k_t) \;=\; 0\,. \tag{9}$$

We obtain $(k', m(k')) \in S(c)$ for all $k' \in A$ by combining (a) the property that, for each $k' \in A$, $m(k')$ enables consumption to equal c at time t when $(k', m(k'))$ at time t is transformed to (k_{t+1}, m_{t+1}) at time $t+1$ with (b) the fact that $(k_{t+1}, m_{t+1}) \in S(c)$. Using Proposition 1, we can therefore write:

$$P_t k_t + Q_t m_t \;\leq\; P_t k' + Q_t m(k') \quad \text{for all } k' \in A\,.$$

Since A is an open set, the following first-order condition must hold:

$$P_t + Q_t m'(k_t) = 0\,,$$

which, using (9), yields:

$$P_t \,=\, Q_t \cdot \frac{G_k(k_t, m_t - m_{t+1}) + 1}{G_r(k_t, m_t - m_{t+1})}\,.$$

Thus, since (P_t, Q_t) is non-null and non-negative, we have $(P_t, Q_t) \in \mathbb{R}^2_{++}$ and:

$$\frac{P_t}{Q_t} \,=\, \frac{G_k(k_t, m_t - m_{t+1}) + 1}{G_r(k_t, m_t - m_{t+1})} \,=\, p_t$$

by (P), thereby establishing the result. ■

3.2 Conditions on the value of net investments

We are now in a position to state the main result of the paper.

Theorem 1 *Under (G1)–(G3), if the path $\{k_t, m_t, c_t\}$ from $(k, m) \in \Omega$ with $c_t = c > 0$ for all $t \geq 0$ is efficient and egalitarian, then Harwick's rule as redefined in Definition 1 is satisfied in all time periods $t \geq 0$. Furthermore, for all $t \geq 0$, the value of net investments, E_t, measured in early pricing is non-negative and non-increasing, and the value of net investments, L_t, measured in late pricing is non-positive and non-decreasing. The two measures differ for all $t \geq 0$, but both converge to 0 as $t \to \infty$.*

Proof. Assume that the path $\{k_t, r_t, c_t\}$ from $(k, m) \in \mathbb{R}^2_{++}$ is efficient and egalitarian. Hence, by Corollary 3, the path is competitive with associated sequence $\{p_t, q_t\}$ of price pairs given by (P). Furthermore, by Proposition 3, for all $t \geq 0$, $p_{t+1}k_{t+1} + m_{t+1} \leq p_{t+1}k_t + m_t$, or equivalently:

$$L_t \,=\, p_{t+1}\,(k_{t+1} - k_t) + (m_{t+1} - m_t) \leq 0\,, \tag{10}$$

since $(k_t, m_t) \in S(c)$ for all $t \geq 0$. Likewise, for all $t \geq 0$, $p_t k_t + m_t \leq p_t k_{t+1} + m_{t+1}$, or equivalently:

$$E_t \,=\, p_t\,(k_{t+1} - k_t) + (m_{t+1} - m_t) \geq 0\,. \tag{11}$$

Hence, by (10) and (11), for all $t \geq 0$, there exists $p \in [p_{t+1}, p_t]$ such that

$$p\,(k_{t+1} - k_t) + (m_{t+1} - m_t) = 0\,,$$

since, by Corollaries 1 and 2, the price sequence $\{p_t\}$ is decreasing and the capital sequence $\{k_t\}$ is increasing. This shows that Hartwick's rule as redefined in Definition 1 is satisfied in all time periods $t \geq 0$ along an efficient and egalitarian path.

Since the path $\{k_t, m_t, c_t\}$ is competitive and, by Proposition 2, non-wasteful, it follows from (C) that, for all $t \geq 1$:

$$p_{t+1}(c_t + k_{t+1}) + m_{t+1} - p_t k_t - m_t \;\geq\; p_{t+1}(c_{t-1} + k_t) + m_t - p_t k_{t-1} - m_{t-1}\,,$$
$$p_t(c_{t-1} + k_t) + m_t - p_{t-1}k_{t-1} - m_{t-1} \;\geq\; p_t(c_t + k_{t+1}) + m_{t+1} - p_{t-1}k_t - m_t\,.$$

By re-arranging and using the fact that $\{k_t, m_t, c_t\}$ is egalitarian (so that $c_t = c_{t-1}$ for all $t \geq 1$), we have that:

$$L_t = p_{t+1}(k_{t+1} - k_t) + (m_{t+1} - m_t) \;\geq\; p_t(k_t - k_{t-1}) + (m_t - m_{t-1}) = L_{t-1}\,, \quad (12)$$
$$E_t = p_t(k_{t+1} - k_t) + (m_{t+1} - m_t) \;\leq\; p_{t-1}(k_t - k_{t-1}) + (m_t - m_{t-1}) = E_{t-1} \quad (13)$$

hold for all $t \geq 1$. By (10) and (12), the value of net investments, L_t, measured in late pricing is non-positive and non-decreasing. Likewise, by (11) and (13), the value of net investments, E_t, measured in early pricing is non-negative and non-increasing.

By Corollaries 1 and 2, the price sequence $\{p_t\}$ is decreasing and the capital sequence $\{k_t\}$ is increasing, implying that, at any $t \geq 0$,

$$E_t - L_t = (p_t - p_{t+1})(k_{t+1} - k_t) > 0\,, \quad (14)$$

so that the two measures differ for all $t \geq 0$.

Since, for all $t \geq 0$, L_t is non-positive and non-decreasing and E_t is non-negative and non-increasing, it follows that the limits, $L = \lim_{t\to\infty} L_t$ and $E = \lim_{t\to\infty} E_t$, exist, and furthermore, $L \leq 0 \leq E$. Hence, to show that $L = 0 = E$, it is sufficient to show that $E \leq L$, or equivalently, by (14):

$$\lim_{t\to\infty} p_t(k_{t+1} - k_t) \;\leq\; \lim_{t\to\infty} p_{t+1}(k_{t+1} - k_t)\;. \quad (15)$$

Using Corollary 2 and (P) and writing $r_t = m_t - m_{t+1}$, we have:

$$p_t\,(k_{t+1}-k_t) \;=\; p_{t+1}\,(G_k(k_t,r_t)+1)\,(k_{t+1}-k_t) \;\le\; p_{t+1}\left(\frac{G(k_t,r_t)}{k_t}+1\right)(k_{t+1}-k_t)$$

for all $t \ge 0$, by (G1) and (G2). Hence, by (15), $E \le L$ is established by showing:

$$\lim_{t\to\infty} \frac{G(k_t,r_t)}{k_t} \;=\; 0\,. \tag{16}$$

To show this, the following claim is helpful.

Claim: $\lim_{t\to\infty} k_t = \infty$. Suppose the claim is not true. Then, by Corollary 2, there exists $\bar{k} > 0$ such that $k_t \le \bar{k}$ for all $t \ge 0$ and, by (G1):

$$c_t \le G(k_t, m_t - m_{t+1}) + k_t - k_{t+1} < G(k_t, m_t - m_{t+1}) \le G(\bar{k}, m_t)\,. \tag{17}$$

We have $\lim_{t\to\infty} G(\bar{k}, m_t) = 0$ by (G1) and (G2), as $\lim_{t\to\infty} m_t = 0$ by Proposition 2. By (17), this contradicts that $c_t = c > 0$ for all $t \ge 0$.

By the claim we may choose $s \ge 0$ such that $k_t \ge 1$ for all $t \ge s$, implying that

$$\frac{G(k_t,r_t)}{k_t} \;\le\; G(1, r_t/k_t) \;\le\; G(1,r_t) \;=\; G(1, m_t - m_{t+1}) \;\le\; G(1, m_t)$$

for all $t \ge s$, by (G1). We have $\lim_{t\to\infty} G(1, m_t) = 0$ by (G1) and (G2), as $\lim_{t\to\infty} m_t = 0$ by Proposition 2. This shows (16). ■

For the sake of completeness, we state the following result, using the style of proof in Mitra et al. (2013, Proposition 5).

Proposition 4 *Under (G1)–(G3) and (E), the efficient and egalitarian path* $\{k_t, m_t, c_t\}$ *from* $(k, m) \in \Omega$ *with* $c_t = c > 0$ *for all* $t \ge 0$ *satisfies the capital-value transversality condition.*

It is noteworthy that the capital-value transversality condition obtains without imposing an assumption that resource use be 'important' (in the sense of (A4) or (A4′) of Dasgupta and Mitra, 1983).

Proof. By Corollary 2, $\{k_t, m_t, c_t\}$ is interior. Hence, by (P), it remains to show that $\lim_{t\to\infty} p_t k_t = 0$. We show this by first establishing an intermediate claim.

Claim: $\lim_{t\to\infty} p_t = 0$. Let $t \ge 1$. Then:

$$\begin{aligned} p_t(k_t - k_0) &= p_t \sum\nolimits_{\tau=0}^{t-1} (k_{\tau+1} - k_\tau) = \sum\nolimits_{\tau=0}^{t-1} p_t (k_{\tau+1} - k_\tau) \\ &\le \sum\nolimits_{\tau=0}^{t-1} p_{\tau+1} (k_{\tau+1} - k_\tau) \le \sum\nolimits_{\tau=0}^{t-1} (m_\tau - m_{\tau+1}) \le m_0 \,, \end{aligned}$$

where the first inequality follows since, by Corollaries 1 and 2, the price sequence $\{p_t\}$ is decreasing and the capital sequence $\{k_t\}$ is increasing, and the second inequality follows from (10). Now, $\lim_{t\to\infty} p_t = 0$ follows since, by the proof of Theorem 1, $\lim_{t\to\infty} k_t = \infty$.

To establish $\lim_{t\to\infty} p_t k_t = 0$, it is sufficient to show that, for all $\epsilon > 0$, there exists $s' \ge 0$ such that $p_t k_t \le \varepsilon$ for all $t \ge s'$. As $\lim_{t\to\infty} m_t = 0$ by Proposition 2, there exists $s \ge 0$ such that $m_s \le \varepsilon/2$. By the Claim, there exists $s' > s$ such that $p_t k_s \le \varepsilon/2$ for all $t \ge s'$. Since, as in the Claim,

$$\begin{aligned} p_t(k_t - k_s) &= p_t \sum\nolimits_{\tau=s}^{t-1} (k_{\tau+1} - k_\tau) = \sum\nolimits_{\tau=s}^{t-1} p_t (k_{\tau+1} - k_\tau) \\ &\le \sum\nolimits_{\tau=s}^{t-1} p_{\tau+1} (k_{\tau+1} - k_\tau) \le \sum\nolimits_{\tau=s}^{t-1} (m_\tau - m_{\tau+1}) \le m_s \end{aligned}$$

for all $t > s$, it follows from the choice of s and s' that

$$p_t k_t \le p_t k_s + m_s \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $t \ge s'$, thereby establishing the capital-value transversality condition. ■

Say that an interior and competitive path $\{k'_t, m'_t, c'_t\}$ from $(k', m') \in \mathbb{R}^2_+$ has *finite consumption value* if the associated price sequence $\{p_t\}$ determined by (P) satisfies $\sum_{t=0}^{\infty} p_{t+1} c'_t < \infty$. This is equivalent to $\sum_{t=0}^{\infty} p_{t+1} < \infty$ if $c'_t = c > 0$ for all $t \ge 0$. As defined by Burmeister and Hammond (1977) and Dixit, Hoel and Hammond (1980) in continuous time, a path is a *regular maximin path* if it

(a) is egalitarian,

(b) has finite consumption value, and

(c) satisfies the capital-value transversality condition and resource exhaustion.

By Propositions 2 and 4 we have established that, under (G1)–(G3) and (E), the efficient and egalitarian path $\{k_t, m_t, c_t\}$ from $(k, m) \in \Omega$ with $c_t = c > 0$ for all

$t \geq 0$ satisfies (a) and (c). However, we cannot show, based on these assumptions, that $\{k_t, m_t, c_t\}$ has finite consumption value. The failure of (b) might arise if an increase of the initial resource stock beyond m cannot be turned into a uniform increase in consumption; see Mitra et al. (2013, Theorem 2) for an investigation of this possibility in the continuous-time DHS model.[2]

4 Concluding remarks

We have shown how the *converse of* Hartwick's result obtains in a discrete-time version of the Dasgupta-Heal-Solow model of capital accumulation and resource depletion under weak assumptions on the technology. We have established this result under the proviso that the valuation of net investments are made using a valuation rule that is intermediate between early and late pricing.

In the continuous-time DHS model, early and late pricing coincide and, as shown by Mitra (2002) in a general continuous-time model, the converse of Hartwick's rule holds as an exact result: If there exists an efficient and egalitarian path $(k(\tau), m(\tau), c(\tau))$ from $(k, m) \in \mathbb{R}^2_{++}$, then the path is interior and competitive at associated prices $(p(\tau), q(\tau))$. Furthermore, $q(\tau)$ is a positive constant which can be normalized to 1, $p(\tau) > 0$ decreases as τ increases, $\dot{k}(\tau) > 0$ for $\tau \geq 0$, and:

$$p(\tau)\dot{k}(\tau) + \dot{m}(\tau) = 0 \text{ for all } \tau \geq 0 \,. \tag{18}$$

Let $t \in \{0, 1, 2, ...\}$. Since $\dot{k}(\tau) > 0$ for all $\tau \geq 0$, it is possible to define p as the weighted mean of the competitive capital prices in the interval $[t, \, t+1]$, where the weights are given by the rates of capital investment:

$$p = \frac{\int_t^{t+1} p(\tau)\dot{k}(\tau)d\tau}{\int_t^{t+1} \dot{k}(\tau)d\tau} \,. \tag{19}$$

Then, by using (18) and noting that $\int_t^{t+1} \dot{k}(\tau)d\tau = k(t+1) - k(t)$ and $\int_t^{t+1} \dot{m}(\tau)d\tau =$

[2] Mitra (1978) showed that an interior and efficient path has finite consumption value if resource use has a functional share of output that is bounded away from zero. Hence, under this additional assumption, an efficient and egalitarian path with $c_t = c > 0$ for all $t \geq 0$ would satisfy (b).

$m(t+1) - m(t)$, we obtain:

$$p\,(k(t+1) - k(t)) + (m(t+1) - m(t)) \;=\; \int_t^{t+1} \Big(p(\tau)\dot{k}(\tau) + \dot{m}(\tau)\Big)\, d\tau \;=\; 0\,. \quad (20)$$

Thus, our discrete-time converse of Hartwick's result holds when p in (20) is precisely the weighted mean of the competitive prices in the interval $[t,\, t+1]$, as given by (19). Furthermore, since $p(\tau)$ is decreasing in τ, we will have $p \in [p(t+1),\, p(t)]$. Finally, the Mean Value Theorem of Integral Calculus will ensure (under continuity of the functions $p(\tau)$ and $\dot{k}(\tau)$) that there is in fact a point of time $s \in [t,\, t+1]$ such that the competitive price at s, $p(s)$, equals p as given by (19).

The question remains whether some variant of Hartwick's result can established in the discrete-time setting: Is it the case that, if along an efficient path Hartwick's rule is followed forever, then an egalitarian path will be implemented?

If the efficient path is interior (so that capital and resource use are positive in each time period), then it follows by the proof of Proposition 2 and Corollary 3 that the path is competitive. Therefore, we obtain the following analogs to (12) and (13) for a path that is not necessarily egalitarian:

$$p_{t+1}(c_t - c_{t-1}) \,+\, (L_t - L_{t-1}) \,\geq\, 0\,, \quad (21)$$

$$p_t(c_t - c_{t-1}) \,+\, (E_t - E_{t-1}) \,\leq\, 0\,. \quad (22)$$

Thus, keeping the value of net investments constant in the present-value prices determined by (P) leads to non-increasing consumption if early pricing is applied, and non-decreasing consumption if late pricing is applied. The latter result can be rephrased as saying that keeping the value of net investments constant in terms of late pricing ensures 'sustained development', as defined by Pezzey (1997). Keeping the value of net investments in present-value prices *constant* (but not necessarily equal to zero) corresponds to the so-called Dixit-Hammond-Hoel rule (Dixit, Hoel and Hammond, 1980).

One can now proceed to show that keeping the value of net investments in early pricing constant and negative forever contradicts feasibility of the path, and investigate the assumptions under which keeping the value of net investments in late pricing constant and positive forever contradicts efficiency of the path. However, as we have seen from the main result of this paper, even though an efficient and egalitarian path is characterized by a non-negative competitive value of net investments in early pricing and a non-positive competitive value of net investments in late pricing, these measures of net investments will not remain constant. Rather, a mixture of these measures, determined by the efficient and egalitarian path and being specific to each time period, remains constant and equal to zero in each time period.

Thus, neither the value of net investments in early pricing nor the value of net investments in late pricing value can be used to steer the economy along a path that is exactly egalitarian. In addition, there is the problem that appears also in the continuous-time setting of determining the initial rate of resource use, so that the egalitarian path exhausts the resource precisely in the limit when time is approaching infinity. Clearly, exhaustion in finite time would lead to infeasibility, while letting part of the resource stock remain unutilized would lead to inefficiency.

Hence, the problems that arise also in continuous time—limiting the applicability of Hartwick's rule as a prescriptive rule for sustainability—remains also in the discrete-time setting. In addition, with discrete time, as we have seen from the main result of this paper, the exact relative price between capital and resource to be applied in each time period is derived from the efficient and egalitarian path. Thus, in discrete time it seems challenging to use Hartwick's rule to steer the economy along an efficient and egalitarian path.

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