The necessity of time inconsistency for intergenerational equity*

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Abstract

We show how conditions of intergenerational equity may necessitate time inconsistency of normative criteria. By formalizing the intuition that less sensitivity remains for the future if sensitivity for the interests of the present is combined with time consistency, we point out conflicts (a) between time consistency and the requirement of not letting the present be dictatorial, and (b) between time consistency and equal treatment of generations. We use the results to interpret the time inconsistency of the Chichilnisky criterion and Rank-Discounted Utilitarianism. We also illustrate how such time inconsistent criteria can be applied in the Ramsey model.

Keywords: intergenerational equity, time inconsistency

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1 Introduction

One might claim that, while time inconsistency is pervasive in individual decision-making, it has no place when formulating normative criteria. We disagree with this view for two reasons:

- Firstly, time inconsistency is not irrational; the question of rationality in this context relates to whether the decision-maker (be it is an individual person or a social planner) is conscious of his time inconsistency and, if so, how this problem is handled. A rational social planner with time inconsistent preferences must be aware that optimal plans might not be followed and instead plan in a "sophisticated" way by choosing the best plans that will actually be followed (see Pollak, 1968; Blackorby et al., 1973, for early references).

- Secondly, several normative criteria of intergenerational equity are in fact time inconsistent. Examples are Chichilnisky's (1996) 'sustainable preference' and Rank-Discounted Utilitarianism (see Zuber and Asheim, 2012). Moreover, as we investigate in this paper, one can argue that time inconsistency is unavoidable for any criteria satisfying equity axioms like anonymity or no dictatorship of the present under reasonable sensitivity, coherency, richness and robustness conditions.

There is a basic intuition for why time consistency might be in conflict with equitable treatment of generations: If sensitivity for the interests of the present is combined with time consistency, then less sensitivity remains for the future. This has two consequences:

- Impatience, thereby contradicting equal treatment in the sense of anonymity.

- No weight remains for the infinite future under sufficient continuity, leading to dictatorship of the present.

This intuition will be developed in the present paper.

Intertemporal social choice theory yields axiomatic foundation for criteria of intergenerational equity. As we explore in Section 2, such axioms can be divided into four classes:

(1) *Equity* axioms, like anonymity or non-dictatorship of the present/future.

(2) *Sensitivity* axioms, like different variants of the Pareto principle.

(3) *Separability* axioms, including stationarity.

(4) Axioms of *coherency* (transitivity), *richness* (completeness) and *robustness* (continuity).

As we show in Section 3, the conjunction of the axioms of separable future and stationarity is sufficient and necessary for the time consistency of a time invariant social welfare relation (SWR). In our paper we provide results which show how imposing a requirement of time consistency on a time invariant SWR leads to conflicts with equity axioms, under various sensitivity axioms.

We show in Section 4 that a time invariant, complete and continuous SWR that satisfies the strong Pareto principle (hence, it is representable) cannot both be time consistent and satisfy non-dictatorship of the present on the set of converging streams. Hence, the ranking of two converging streams depends only on the properties of the streams before some finite time τ, and will not change even if, beyond time τ, the less preferred stream is changed to a stream leading to bliss and the more preferred stream is changed to one leading to destitution. We show by means of two examples that the dilemma can be resolved by dropping continuity. However, we have not been able to construct an "attractive" representable SWR that combines time consistency and non-dictatorship of the present on the set of converging streams. Chichilnisky (1996) provides a class of representable and sensitive SWRs which respect both non-dictatorship of the present and non-dictatorship of the future, but they do not satisfy stationarity and hence are not time consistent.

Under the strong Pareto principle, a representable SWR cannot be anonymous, as shown by Basu and Mitra (2003). So there is an impossibility even before imposing time consistency. However, a representable and continuous SWR can satisfy even strong anonymity, in the sense of being indifferent to all permutations (including infinite ones), by weakening the strong Pareto principle to restricted dominance. Restricted dominance means that a stream is made worse by reducing the wellbeing of the present generation if the starting point is an egalitarian stream. We show in Section 5 that a time invariant SWR that satisfies restricted dominance and allows for some present-future substitution cannot both be time consistent and satisfy anonymity in its weaker finite version. We demonstrate by means of two examples how the dilemma can be resolved by weakening sensitivity even further, thereby exploring further the difficulty of constructing "attractive" representable SWRs that combine time consistency and anonymity. In Zuber and Asheim's (2012) axiomatization of Rank-Discounted Utilitarianism, they weaken the axiom of separable future to hold only on the set of non-decreasing streams. Therefore, while Rank-Discounted Utilitarianism is a representable SWR that satisfies restricted dominance and (even) strong anonymity, it need not yield a time-consistent ranking in the comparison of streams that are not non-decreasing.

In Section 6 we discuss further the properties of Chichilnisky's (1996)'s criterion and Rank-Discounted Utilitarianism. Time inconsistency turns models of economic growth into intergenerational games. We illustrate in Section 7 such game-theoretic analysis by applying Chichilnisky's (1996)'s criterion and Rank-Discounted Utilitarianism in the context of the Ramsey model. We discuss in the final Section 8 how the dilemma between time consistency and equity axioms can resolved by dropping time invariance so that the SWR becomes history dependent. The proofs that do not follow the statements of the results in the main text are contained in Appendix A.

We are not aware of previous discussions in the literature of the conflict between time consistency and the requirement of not letting the present be dictatorial. However, we are not the first to point out a conflict between time consistency and equal treatment. In the setting of a numerically representable and sensitive SWR, Koopmans (1960) shows through his Theorem 1 how impatience is implied by the axioms of separable future and stationarity (thus implied by time consistency, provided that time invariance is imposed). In fact, this is the main result of his seminal article, and the implication from stationarity to impatience is reflected by its title. More recently, in the setting of possibly incomplete SWRs, Jonsson and Voorneveld (2015, Proposition 1) show that if the alternating streams $(1, 0, 1, 0, 1, 0, \dots)$ and $(0, 1, 0, 1, 0, 1, \dots)$ are comparable under the strong Pareto principle and the axioms of separable future and stationarity, then the former stream is strictly preferred to the latter, thereby exhibiting a weak form of impatience (this result is not included in the published version: Jonsson and Voorneveld, 2018). In the present paper we show how time consistency leads to impatience also in Koopmans' (1960) stronger form even under conditions that would otherwise be consistent with the axiom of strong anonymity, in the sense of being indifferent to all permutations.

2 Framework and axioms

Denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. Let, for all $t \in \mathbb{N}$, $x_t \in \mathbb{R}$ be an indicator of the *wellbeing* of generation t, and let ${}_1\mathbf{x} = (x_1, x_2, \dots, x_t, \dots)$ be an infinite wellbeing stream. Write, for all $T \in \mathbb{N}$, ${}_1\mathbf{x}_T = (x_1, \dots, x_T)$ and ${}_{T+1}\mathbf{x} = (x_{T+1}, x_{T+2}, \dots)$; these are, respectively, the T-head and the T-tail of ${}_1\mathbf{x}$. A wellbeing stream ${}_1\mathbf{x}$ is *constant* if there is $a \in \mathbb{R}$ such that $x_t = a$ for all $t \in \mathbb{N}$, and we write ${}_1\mathbf{x} = {}_{\text{con}}a$. A wellbeing stream ${}_1\mathbf{x}$ is *non-decreasing* if $x_t \leq x_{t+1}$ for all $t \in \mathbb{N}$. A wellbeing stream ${}_1\mathbf{x}$ is *converging* if $\lim_{t\to\infty} x_t$ exists.

We normalize wellbeing to lie within the unit interval, $[0, 1]$. This normalization is common in the literature on intergenerational equity and leads to little loss of generality. Write $\mathbf{X} = [0, 1]^{|\mathbb{N}|}$ the set of possible wellbeing streams, $\mathbf{X}^+$ for the subset of non-decreasing streams and $\mathbf{X}^c$ for the subset of converging streams.

We study *social welfare relations* (SWR) $\succsim$ that rank elements of $\mathbf{X}$ (or $\mathbf{X}^c$). We interpret the SWR $\succsim$ as the social ranking made at time 1 over streams starting at time 1. We say that a *social welfare function* (SWF) $W : \mathbf{X} \to \mathbb{R}$ (or $W : \mathbf{X}^c \to \mathbb{R}$)

represents an SWR $\succsim$ if, for all ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}$ (or $\mathbf{X}^c$), ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if $W({}_1\mathbf{x}) \geq W({}_1\mathbf{y})$.

For ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}$, write ${}_1\mathbf{x} \geq {}_1\mathbf{y}$ whenever $x_t \geq y_t$ for all $t \in \mathbb{N}$; write ${}_1\mathbf{x} > {}_1\mathbf{y}$ if ${}_1\mathbf{x} \geq {}_1\mathbf{y}$ and ${}_1\mathbf{x} \neq {}_1\mathbf{y}$; and write ${}_1\mathbf{x} \gg {}_1\mathbf{y}$ whenever $x_t > y_t$ for all $t \in \mathbb{N}$.

The axioms that we will consider fall into four classes (where $\mathbf{X}^c$ substitutes for $\mathbf{X}$ for SWRs with domain $\mathbf{X}^c$). The first class consists of axioms that concerns the *coherency* (transitivity), *richness* (completeness) and *robustness* (continuity) of the SWR $\succsim$.

Axiom O (*Order*) The SWR $\succsim$ is complete, reflexive and transitive on $\mathbf{X}$.

Axiom QO (*Quasi Order*) The SWR $\succsim$ is reflexive and transitive on $\mathbf{X}$.

An SWR satisfying axiom **O** is called a social welfare order (SWO). Clearly, axiom **O** implies axiom **QO**.

Axiom C (*Continuity*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}$, if a sequence ${}_1\mathbf{x}^1, {}_1\mathbf{x}^2, \ldots, {}_1\mathbf{x}^k, \ldots$ of allocations in $\mathbf{X}$ is such that $\lim_{k\to\infty} \sup_{t\in\mathbb{N}} |x_t^k - x_t| = 0$ and, for all $k \in \mathbb{N}$, ${}_1\mathbf{x}^k \succsim {}_1\mathbf{y}$ (resp. ${}_1\mathbf{x}^k \precsim {}_1\mathbf{y}$), then ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ (resp. ${}_1\mathbf{x} \precsim {}_1\mathbf{y}$).

Axiom PFS (*Present-Future Substitution*) There exist a, b, a', $b' \in [0,1]$ with $a \leq a' \leq b' < b$ such that $(a, {}_{\text{con}}b) \sim (a', {}_{\text{con}}b')$.

Axiom REP (*Representability*) There exists an SWF W such that W represents the SWR $\succsim$.

Axioms **O** and **C** imply **REP** (provided that axiom **M** below is invoked, cf. Lemma 1(i)) and **PFS** (provided that axioms **M** and **RD** below are invoked, cf. Lemma 4(i) of Appendix B). **PFS** does not imply **C**, as illustrated by Examples 1, 2 and 3 below. Axiom **REP** implies axiom **O**, but not axioms **C** and **PFS**, as shown by Example 3 of Banerjee and Mitra (2018).

The next class provides *sensitivity* axioms. The strong Pareto principle belongs here:

Axiom SP (*Strong Pareto*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}$, if ${}_1\mathbf{x} > {}_1\mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$.

Axiom **SP** implies the following two axioms.

Axiom M (*Monotonicity*) For any ${}_1\mathbf{x}, {}_1\mathbf{y} \in \mathbf{X}$, if ${}_1\mathbf{x} > {}_1\mathbf{y}$, then $\mathbf{x} \succsim \mathbf{y}$.

Axiom RD (*Restricted Dominance*) For any $a, b \in [0,1]$, if $a < b$, then $(a, {}_{\text{con}}b) \prec {}_{\text{con}}b$.

Axiom **RD** ensures some sensitivity to the interests of the present generation. Axiom **RD** is equivalent to the strong Pareto principle on the set of non-decreasing streams when a set of additional axioms are imposed; Rank-Discounted Utilitarianism discussed in Section 6 exemplifies this. Together with axioms **QO** and **M**, axiom **RD** implies the strong Pareto principle on the diagonal: For any $a, b \in [0,1]$,

$$\text{if } a < b, \text{ then } {}_{\text{con}}a \prec {}_{\text{con}}b \tag{1}$$

by transitivity since ${}_{\text{con}}a \precsim (a, {}_{\text{con}}b) \prec {}_{\text{con}}b$ follows from **M** and **RD**.

Throughout this paper we will be concerned with SWRs that satisfy at least axioms **QO**, **PFS**, **M** and **RD**. It follows from the discussion above that this combination is strictly weaker than **O**, **C**, **M** and **RD**, which in turn is strictly weaker than **O**, **C** and **SP**. The following result is useful for our analysis.

Lemma 1 *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies **O**, **C**, and **M**. Then:*

(i) There exists a SWF $W : \mathbf{X} \to [0,1]$ *such that* W *represents* $\succsim$.

(ii) If the SWR $\succsim$ *satisfies also **RD**, then there exists a unique* SWF $W : \mathbf{X} \to [0,1]$ *representing* $\succsim$ *with the property that* $W({}_{\text{con}}b) = b$ *for all* $b \in [0,1]$.

Throughout we use the symbol W for the SWF that represents $\succsim$ and has the property that $W({}_{\text{con}}b) = b$ for all $b \in [0,1]$.

The third class consists of *ethical* axioms. The two first imposes impartiality by imposing invariance to the set of all or the set of finite permutation.

Axiom SA (*Strong Anonymity*) For any ${}_1\mathbf{x} \in \mathbf{X}$, if ${}_1\mathbf{y}$ is a permutation of ${}_1\mathbf{x}$, then ${}_1\mathbf{x} \sim {}_1\mathbf{y}$.

Axiom FA (*Finite Anonymity*) For any ${}_1\mathbf{x} \in \mathbf{X}$, if ${}_1\mathbf{y}$ is a finite permutation of ${}_1\mathbf{x}$, then ${}_1\mathbf{x} \sim {}_1\mathbf{y}$.

Since the set of finite permutations, where only a finite set of positions are reordered, is a strict subset of the set of all permutations, it follows that axiom **SA** implies axiom **FA**.

As already observed by Diamond (1965), there is conflict between sensitivity and impartiality. Lauwers (1997) shows that **SP** is incompatible with **SA**, while Basu and Mitra (2003) show that **SP** is incompatible also with **FA** if the SWR satisfies **REP**. But as we will see, if **SP** is weakened to the conjunction of axioms **M** and **RD**, then compatibility with even **SA** is restored for SWRs that satisfy **REP**.

Chichilnisky (1996) says that there is Dictatorship of the Present if, for any ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}$ with ${}_1\mathbf{x} \succ {}_1\mathbf{y}$, there exists $\tau \in \mathbb{N}$ such that for all $t \geq \tau$ and ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in \mathbf{X}$, $({}_1\mathbf{x}_t,\, {}_{t+1}\mathbf{u}) \succ ({}_1\mathbf{y}_t,\, {}_{t+1}\mathbf{v})$. Hence, the present is a dictator if only what happens before a finite time matters for the ranking of alternatives. Here, we weaken this axiom by only considering pairs of converging wellbeing streams.

Axiom DP (*Dictatorship of the Present on the set of converging streams*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}^c$ with ${}_1\mathbf{x} \succ {}_1\mathbf{y}$, there exists $\tau \in \mathbb{N}$ such that for all $t \geq \tau$ and ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in \mathbf{X}$, $({}_1\mathbf{x}_t,\, {}_{t+1}\mathbf{u}) \succ ({}_1\mathbf{y}_t,\, {}_{t+1}\mathbf{v})$.

In particular, when Axiom **DP** is satisfied, for any ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}^c$ with ${}_1\mathbf{x} \succ {}_1\mathbf{y}$, there exists $\tau \in \mathbb{N}$ such that for all $t \geq \tau$, $({}_1\mathbf{x}_t,\, {}_{\text{con}}0) \succ ({}_1\mathbf{y}_t,\, {}_{\text{con}}1)$. Hence, the ranking of two converging streams depends only on the properties of the streams before some finite time τ, and will not change even if, beyond time τ, the more preferred stream is changed to one leading to destitution and the less preferred stream is changed to one leading to bliss. Note that we do not require the arbitrary streams ${}_1\mathbf{u}$ and ${}_1\mathbf{v}$ to be converging.

The ethically commendable property is the negation of axiom **DP**

Axiom NDP (*Non-Dictatorship of the Present on the set of converging streams*) There exist ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}^c$ with ${}_1\mathbf{x} \succ {}_1\mathbf{y}$ such that, for all $\tau \in \mathbb{N}$, there exist $t \geq \tau$

and ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in \mathbf{X}$ such that $({}_1\mathbf{x}_t,\, {}_{t+1}\mathbf{u}) \precsim ({}_1\mathbf{y}_t,\, {}_{t+1}\mathbf{v})$.

The fourth and last class consists of *separability* axioms.

Axiom SEP (*Separable Present*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y}$, ${}_1\mathbf{x}'$, ${}_1\mathbf{y}' \in \mathbf{X}$ such that (i) $x_t = x'_t$ and $y_t = y'_t$ for all $t \in \{1, 2\}$ and (ii) $x_t = y_t$ and $x'_t = y'_t$ for all $t \in \mathbb{N} \setminus \{1, 2\}$, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$.

Axiom **SEP** is Postulate 3′a in Koopmans' (1960) characterization of discounted utilitarianism.

Axiom SEF (*Separable Future*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y}$, ${}_1\mathbf{x}'$, ${}_1\mathbf{y}' \in \mathbf{X}$ such that (i) $x_t = x'_t$ and $y_t = y'_t$ for all $t \in \mathbb{N} \setminus \{1\}$ and (ii) $x_1 = y_1$ and $x'_1 = y'_1$, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$.

Axiom **SEF** is Postulate 3b in Koopmans' (1960) characterization of discounted utilitarianism.

Axiom ST (*Stationarity*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y}, \in \mathbf{X}$, there exists $a \in [0, 1]$ such that ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if $(a,\, {}_2\mathbf{x}') \succsim (a,\, {}_2\mathbf{y}')$, where $x_t = x'_{t+1}$ and $y_t = y'_{t+1}$ for all $t \in \mathbb{N}$.

Axiom **ST** is Koopmans' (1960) stationarity postulate (Postulate 4). The conjunction of axioms **SEF** and **ST** is the property Independent future, as introduced (in a slightly stronger form) by Fleurbaey and Michel (2003):

Independent future. For any ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}$ with $x_1 = y_1$, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$, where $x'_t = x_{t+1}$ and $y'_t = y_{t+1}$ for all $t \in \mathbb{N}$.

Lemma 2 *The* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies Independent future if and only if* $\succsim$ *satisfies axioms* ***SEF*** *and* ***ST****.*

We end this section by noting that axiom **RD** together with axioms **QO**, **M**, **SEF** and **ST** implies sensitivity for the future: For any a, b, $c \in [0, 1]$,

$$\text{if } b < c, \text{ then } \ (a,\, {}_{\text{con}}b) \prec (a,\, {}_{\text{con}}c)\,. \tag{2}$$

This is obtained by combining (1) with **SEF** and **ST** and invoking Lemma 2.

3 Condition for time consistency

Time consistency means that the ranking of two streams with a common first element does not change after this first element has been realized. To discuss this concept we need to interpret ${}_t\mathbf{x} \succsim {}_t\mathbf{y}$ as the social ranking made by $\succsim$ at time t over streams starting at time t.

Time consistency. For any $\tau \in \mathbb{N}$ and any ${}_1\mathbf{x}, {}_1\mathbf{y} \in \mathbf{X}$ with $x_\tau = y_\tau$, ${}_\tau\mathbf{x} \succsim {}_\tau\mathbf{y}$ if and only if ${}_{\tau+1}\mathbf{x} \succsim {}_{\tau+1}\mathbf{y}$.

To relate the concept of Time consistency to the separability axioms of the last section, we must introduce the concept of Time invariance.

Time invariance. For any $\tau \in \mathbb{N}$ and any ${}_1\mathbf{x}, {}_1\mathbf{y} \in \mathbf{X}$, ${}_\tau\mathbf{x} \succsim {}_\tau\mathbf{y}$ if and only if ${}_{\tau+1}\mathbf{x}' \succsim {}_{\tau+1}\mathbf{y}'$, where $x_t = x'_{t+1}$ and $y_t = y'_{t+1}$ for all $t \in \mathbb{N}$.

With these concepts, we can state the following result.

Proposition 1 *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *is Time invariant. Then* $\succsim$ *is Time consistent if and only if* $\succsim$ *satisfies axioms* ***SEF*** *and* ***ST****.*

The terminology used above corresponds to the one applied by Halevy (2015, Section 3), who also shows that Time consistency combined with Independent future (i.e., axioms **SEF** and **ST**) implies Time invariance (Halevy, 2015, Proposition 4). See also the insightful discussion by Millner and Heal (2017).

From now on, we will abuse notation slightly by writing ${}_2\mathbf{x}$ for ${}_1\mathbf{x}'$ with $x'_t = x_{t+1}$ for all $t \in N$. Hence, formally when considering ${}_2\mathbf{x}$ we will be concerned with rankings at time 1 of streams starting at time 1. However, if Time invariance is imposed, it does matter whether ${}_2\mathbf{x}$ is interpreted in an actual sense as a stream starting at time 2 and being compared to other streams at that time.

Let the SWR $\succsim$ satisfy axioms **O**, **C**, **M**, **RD**, **SEF** and **ST**. Since $\succsim$ satisfy **O**, **C**, **M** and **RD**, it follows from Lemma 1 that there exists a unique $W : \mathbf{X} \to [0, 1]$

such that W represents $\succsim$ and has the property that $W({}_{\text{con}}b) = b$ for all $b \in [0,1]$. For all $(a,b) \in [0,1]^2$, define the function $g : [0,1]^2 \to [0,1]$ by:

$$g(a,b) = W(a, {}_{\text{con}}b)\,.$$

For all ${}_1\mathbf{x} \in \mathbf{X}$, denote $W({}_2\mathbf{x})$ by b. Hence,

$$W({}_2\mathbf{x}) = b = W({}_{\text{con}}b)\,.$$

Since $\succsim$ satisfy **SEF** and **ST**, it follows from Proposition 1 that:

$$W({}_1\mathbf{x}) = W(x_1, {}_2\mathbf{x}) = W(x_1, {}_{\text{con}}b) = g(x_1, b) = g(x_1, W({}_2\mathbf{x}))\,.$$

We call g an *aggregator function* of present wellbeing and future welfare. It follows directly from the definition of g and axioms **C**, **M**, **RD**, **SEF** and **ST** that g has the following properties:

(G.1) $g(a,b) < b$ if $a < b$ and $g(a,b) = b$ if $a = b$;

(G.2) $g(a,b)$ is non-decreasing in a given b;

(G.3) $g(a,b)$ is increasing in b given a;

(G.4) $g(a,b)$ is continuous in (a,b) on $[0,1]^2$.

In particular, to establish property (G.3), let $b < c$. Then $g(b,b) \le g(b,c) < g(c,c)$ by **M** and **RD**. Therefore, by **SEF** and **ST** and invoking Proposition 1,

$$g(a,b) = g(a, g(b,b)) < g(a, g(c,c)) = g(a,c)\,.$$

If, in the set of axioms above, the combination of **M** and **RD** is strengthened to **SP**, then (G.2) is strengthened to:

(G.2′) $g(a,b)$ is increasing in a given b.

4 Time consistency leads to dictatorship of the present

The main result of the present section is to demonstrate that the combination of **O**, **C**, **SP**, **SEF** and **ST** implies **DP**, and thus contradicts **NDP**. Recall that we have weakened **DP** (and strengthened **NDP**) to be concerned with the set of converging streams. Recall also that the aggregator function $g : [0,1]^2 \to [0,1]$ satisfies (G.1), (G.2′), (G.3) and (G.4) under **O**, **C**, **SP**, **SEF** and **ST**.

To motivate the analysis, we first present a weaker version of the result, where we only consider constant streams.

Observation 1 *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies axioms* ***O****,* ***C****,* ***SP****,* ***SEF*** *and* ***ST****. Let* $\succsim$ *be represented by* $W : \mathbf{X} \to [0,1]$*. For any* a*,* $b \in [0,1]$*, if* $a < b$*, then there exists* τ *such that, for every* $t \geq \tau$*,*

$$W(\underbrace{a, \ldots, a}_{t \text{ times}}, {}_{t+1}\mathbf{u}) < W(\underbrace{b, \ldots, b}_{t \text{ times}}, {}_{t+1}\mathbf{v})$$

for all ${}_1\mathbf{u}$*,* ${}_1\mathbf{v} \in \mathbf{X}$*.*

In particular, for every $t \geq \tau$,

$$W(\underbrace{a, \ldots, a}_{t \text{ times}}, {}_{\text{con}}1) < W(\underbrace{b, \ldots, b}_{t \text{ times}}, {}_{\text{con}}0)\,.$$

So in the comparison of constant streams, only what happens before a finite τ matters. Note that τ depends on a and b.

Proof. Assume axioms **O**, **C**, **SP**, **SEF** and **ST**. Define the sequence $\{a^t\}$ by:

$$a^1 = g(a, 1) = W(a, {}_{\text{con}}1)$$
$$a^2 = g(a, a^1) = W(a, a, {}_{\text{con}}1)$$
$$\cdots$$
$$a^t = g(a, a^{t-1}) = W(\underbrace{a, \ldots, a}_{t \text{ times}}, {}_{\text{con}}1)$$

For all t, $a^t \in (a, 1)$. Furthermore, $\{a^t\}$ is decreasing and bounded below by a.

Hence $\{a^t\}$ is a converging sequence: $a^\infty := \lim_{t\to\infty} a^t \in [a, 1)$. By continuity of g, $a^\infty = g(a, a^\infty)$. If $a^\infty > a$, then

$$a^\infty = g(a, a^\infty) < g(a^\infty, a^\infty) = a^\infty$$

which is a contradiction. Therefore: $a^\infty = a$.

Define likewise the sequence $\{b^t\}$ by $b^t = g(b, b^{t-1})$ for all $t \in \mathbb{N}$ and $b^0 = 0$. The same argument implies that $b^\infty = b$.

Hence, with $a < b$, we obtain

$$\lim_{t\to\infty} W(\underbrace{a, \ldots, a}_{t \text{ times}}, {}_{\text{con}}1) = a < b = \lim_{t\to\infty} W(\underbrace{b, \ldots, b}_{t \text{ times}}, {}_{\text{con}}0)$$

Hence, by axiom **SP** there exists $\tau \in \mathbb{N}$ such that

$$\begin{aligned} W(\underbrace{a, \ldots, a}_{t \text{ times}}, {}_{t+1}\mathbf{u}) &\leq W(\underbrace{a, \ldots, a}_{t \text{ times}}, {}_{\text{con}}1) \\ &< W(\underbrace{b, \ldots, b}_{t \text{ times}}, {}_{\text{con}}0) \leq W(\underbrace{b, \ldots, b}_{t \text{ times}}, {}_{t+1}\mathbf{v}) \end{aligned}$$

for all $t \geq \tau$, and ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in \mathbf{X}$, thereby establishing the proposition. ■

Observation 1 can be generalized in two directions.

- One is to weaken the axioms invoked and show that it is still the case that only what happens before a finite τ matters in the comparison of constant streams. Such strengthening of Observation 1 is reported in Remark 1.

- Another is to keep the set of axioms of the observation and show that only what happens before a finite τ matters even in the comparisons of *converging* streams that need not be constant. This strengthening of Observation 1 is shown in Proposition 2, thereby establishing that axiom **DP** (on the set of converging streams) is implied by this set of axioms.

Remark 1 Consider the following two continuity and sensitivity axioms.

Axiom LC (*Limited Continuity*) For any a, b, a', $b' \in [0, 1]$ and sequences $\{a^k\}$,

$\{b^k\} \in [0,1]^{|\mathbb{N}|}$ with $a^k \to a$ and $b^k \to b$, if $(a^n,\, {}_{\text{con}}b^n) \succsim (a',\, {}_{\text{con}}b')$ (resp. $(a^n,\, {}_{\text{con}}b^n) \precsim (a',\, {}_{\text{con}}b')$) for all $n \in \mathbb{N}$, then $(a,\, {}_{\text{con}}b) \succsim (a',\, {}_{\text{con}}b')$ (resp. $(a,\, {}_{\text{con}}b) \precsim (a',\, {}_{\text{con}}b')$).

Axiom LD (*Limited Dominance*) For any $a, b \in [0,1]$, if $a < b$, then $(a,\, {}_{\text{con}}b) \prec {}_{\text{con}}b$ and ${}_{\text{con}}a \prec (b,\, {}_{\text{con}}a)$.

Axiom **LC** is implied by axiom **C**, and combined with axioms **O**, **M** and **RD** it implies axiom **PFS** (cf. Lemma 4). Axiom **LD** is implied by axiom **SP**, and it implies axiom **RD**. In Appendix B we show as Proposition 4 that the result of Observation 1 holds also under the weaker set of axioms **O**, **LC**, **M**, **LD**, **SEF** and **ST**.

We next turn to our main result of this section, showing that axiom **DP** (on the set of converging streams) is implied by the original set of axioms.

Proposition 2 *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies axioms* ***O****,* ***C****,* ***SP****,* ***SEF*** *and* ***ST****. Let* $\succsim$ *be represented by* $W : \mathbf{X} \to [0,1]$*. If* ${}_1\mathbf{x},\, {}_1\mathbf{y} \in \mathbf{X}^c$ *satisfy* $W({}_1\mathbf{x}) < W({}_1\mathbf{y})$*, then there exists* τ *such that, for every* $t \geq \tau$*,*

$$W({}_1\mathbf{x}_t,\, {}_{t+1}\mathbf{u}) < W({}_1\mathbf{y}_t,\, {}_{t+1}\mathbf{v})$$

for all ${}_1\mathbf{u},\, {}_1\mathbf{v} \in \mathbf{X}$*.*

We have no counterexample showing that the implication of Proposition 2 does not obtain if we permit ${}_1\mathbf{x}$ and ${}_1\mathbf{y}$ to be non-converging streams in $\mathbf{X} = [0,1]^{|\mathbb{N}|}$. Thus, it is an open question whether Proposition 2 can be strengthened by showing that that this set of axioms implies Dictatorship of the Present on the set of *all* streams.

The following two examples show how the remaining axioms can be combined with **NDP**, provided that axiom **C** is dropped.

Example 1 Let the SWF $V : \mathbf{X} \to [0,2]$ be given by:

$$V({}_1\mathbf{x}) = \begin{cases} 1 + (1-\beta)\sum_{t=1}^{\infty}\beta^{t-1}x_t & \text{if, for all } t \in \mathbb{N},\ x_t \geq \frac{1}{2}\,, \\ (1-\beta)\sum_{t=1}^{\infty}\beta^{t-1}x_t & \text{if there is } t \in \mathbb{N} \text{ such that } x_t < \frac{1}{2}\,, \end{cases}$$

where $\beta \in (0,1)$. Then V represents an SWR that satisfies axioms **REP**, **PFS**, **SP**, **SEF**, **ST** and **NDP**. Note that the SWR represented by V does not satisfy axiom **C**, as it does not even satisfy axiom **LC**. To see this, consider ${}_1\mathbf{y}$ with $a' = \frac{1}{2} - \frac{1}{2}\beta$ and $b' = 1 - \frac{1}{2}\beta$ for all $t \geq 2$ so that $V(a', {}_{\text{con}}b') = \frac{1}{2}$. Then ${}_{\text{con}}b \prec (a', {}_{\text{con}}b')$ if $b \in [0, \frac{1}{2})$ and ${}_{\text{con}}b \succ (a', {}_{\text{con}}b')$ if $b \in [\frac{1}{2}, 1]$, thus contradicting **LC**.

Example 2 Define the following subsets of $\mathbf{X}$ (keeping in mind that $\mathbf{X} = [0,1]^{|\mathbb{N}|}$): $\mathbf{N} := \{{}_1\mathbf{x} \in \mathbf{X} : \sum_{t=1}^{\infty} x_t < \infty\}$ and $\mathbf{I} := \mathbf{X} \backslash \mathbf{N}$. Let the SWF $V : \mathbf{X} \to [0,2]$ be given by:

$$V({}_1\mathbf{x}) = \begin{cases} 1 + (1-\beta)\sum_{t=1}^{\infty} \beta^{t-1} x_t & \text{if } {}_1\mathbf{x} \in \mathbf{I}\,, \\ \frac{\sum_{t=1}^{\infty} x_t}{1+\sum_{t=1}^{\infty} x_t} & \text{if } {}_1\mathbf{x} \in \mathbf{N}\,, \end{cases}$$

where $\beta \in (0,1)$. Then V represents an SWR that satisfies axioms **REP**, **PFS**, **SP**, **SEF**, **ST** and **NDP**. To see that V does not satisfy axiom **LC** (and thus not axiom **C**), consider a sequence $\{b^k\} \in (0,1]^{|\mathbb{N}|}$ such that $\lim_{k\to\infty} b^k = 0$. Then ${}_{\text{con}}b^k \succ (1, {}_{\text{con}}0)$ for all $k \in \mathbb{N}$, while ${}_{\text{con}}0 \prec (1, {}_{\text{con}}0)$, thus contradicting **LC**.

The SWF of Example 2 represents discounted utilitarianism on the set **I** of streams with unbounded sum of wellbeing. If subsistence requires wellbeing bounded away from zero, then **I** is the relevant set of streams in the context of sustainability. The non-continuity between streams in **I** and streams in **N** is an artifact to ensure that the SWR represented by the SWF of Example 2 satisfies **NDP**. A similar comment can be made in the case of Example 1 with $\frac{1}{2}$ as subsistence level. Chichilnisky (1996) defines 'sustainable preference' by the axioms **REP**, **SP** and **NDP** (noting that her axiom Non-Dictatorship of the Future follows from **SP**). Examples 1 and 2 show that there exist 'sustainable preferences' that are both Time consistent and Time invariant. However, they illustrate also that axioms discussed in the literature actually fail to put adequate restrictions on SWRs, and the commonly accepted representation results obtained from these axioms are driven by the imposition of continuity. In particular, a 'sustainable preference' without the additional imposition of continuity might not have commendable properties. Thus, the dilemma posed by a choice between time consistency and intergenerational equity remains.

5 Time consistency leads to impatience

In this section we show how combinations of axioms including **SEF** and **ST** imply impatience. We start by reproducing a result due to Koopmans (1960). Recall again that the aggregator function $g : [0,1]^2 \to [0,1]$ satisfies (G.1), (G.2′), (G.3) and (G.4) under the axioms **O**, **C**, **SP**, **SEF** and **ST**.

Observation 2 (Koopmans, 1960) *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies axioms **O**, **C**, **SP**, **SEF** and **ST**. Let* $\succsim$ *be represented by* $W : \mathbf{X} \to [0,1]$*. For any* a*,* $b \in [0,1]$*, if* $a < b$*, then*

$$W(a,\, b,\, {}_{\mathrm{con}}b) < W(b,\, a,\, {}_{\mathrm{con}}b)\,. \tag{3}$$

If $\succsim$ *satisfies also axiom **SEP**, then*

$$W(a,\, b,\, {}_3\mathbf{x}) < W(b,\, a,\, {}_3\mathbf{x}) \tag{4}$$

for all ${}_3\mathbf{x} \in \mathbf{X}$.

Proof. Assume axioms **O**, **C**, **SP**, **SEF** and **ST**, and let $a < b$. Define the sequence $\{c^t\}$ by:

$$\begin{aligned}
c^0 &= g(a,b) = W(a,\, {}_{\mathrm{con}}b) \\
c^1 &= g(b,c^0) = W(b,\, a,\, {}_{\mathrm{con}}b) \\
&\quad\cdots \\
c^t &= g(b,c^{t-1}) = W(\underbrace{b,\ldots,b}_{t \text{ times}},\, a,\, {}_{\mathrm{con}}b)
\end{aligned}$$

Suppose $c^0 \geq c^1$. By induction, $c^{t-1} \geq c^t$ for all t, implying that $g(a,b) = c^0 \geq c^t$ for all t. However, by the proof of Observation 1,

$$c^t \;\geq\; W(\underbrace{b,\ldots,b}_{t \text{ times}},\, {}_{\mathrm{con}}0) \;=:\; b^t \;\to\; b \;=\; g(b,b) \;>\; g(a,b)\,,$$

which is a contradiction. Therefore:

$$W(a,\, b,\, {}_{\text{con}}b) \;=\; c^0 \;<\; c^1 \;=\; W(b,\, a,\, {}_{\text{con}}b)\,,$$

which establishes (3). Now, (4) follows directly if axiom **SEP** is added. ■

Observation 2 demonstrates that the combination of **O**, **C**, **SP**, **SEF** and **ST** contradicts axiom **FA** and thus precludes equal treatment of generations. In particular, the observation shows that the streams $(a,\, b,\, {}_{\text{con}}b)$ and $(b,\, a,\, {}_{\text{con}}b)$ with $a < b$ meets condition (40) of impatience in Koopmans (1960). The result actually follows from part (a) of Koopmans's (1960) Theorem 1.

However, as shown by Diamond (1965), axioms **O**, **C** and **SP** alone contradict axiom **FA**. In fact, as demonstrated by Basu and Mitra (2003), even if axioms **O** and **C** are replaced by **REP**, it is impossible to satisfy both axioms **SP** and **FA**. Hence, it is not the imposition of Time consistency (in the setting of Time invariance) through axioms **SEF** and **ST** that rules out equal treatment.

The following main result of this section shows that axiom **FA** is contradicted by axioms **SEF** and **ST** even if axioms **O**, **C** and **SP** are weakened to axioms **QO**, **PFS** and **RD**. The significance of this result is that axioms **QO**, **PFS** and **RD** alone do not contradict **FA**. In particular, the SWR represented by $W({}_1\mathbf{x}) = \inf_{t\in\mathbb{N}} x_t$ satisfies axioms **O**, **C**, **M**, **RD** and **SA**, and thus, axioms **QO**, **PFS**, **RD** and **FA** since **O**, **C**, **M** and **RD** imply **QO** and **PFS**, and **SA** strengthens **FA**. Rank-Discounted Utilitarianism discussed in Section 6 is another example. Hence, under this set of axioms it is the requirement of Time consistency that causes the conflict with equal treatment as a criterion of intergenerational equity.

Proposition 3 *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies axioms* ***QO****,* ***PFS****,* ***RD****,* ***SEF*** *and* ***ST****. Then axiom* ***FA*** *does not hold.*

Proof. Assume that the SWR $\succsim$ satisfies axioms **QO**, **RD**, **SEF** and **ST**. It suffices to show that axiom **PFS** contradicts axiom **FA**. We prove this by contradiction by supposing that **FA** holds even though there exist a, b, a', $b' \in [0,1]$ with

$a \leq a' \leq b' < b$ such that $(a, {}_{\text{con}}b) \sim (a', {}_{\text{con}}b')$. So suppose that (i) $a, b, a', b' \in [0,1]$ satisfy that $a \leq a' \leq b' < b$ and $(a, {}_{\text{con}}b) \sim (a', {}_{\text{con}}b')$ *and* (ii) **FA** holds. We obtain:

$$(a, b', {}_{\text{con}}b) \prec (a, {}_{\text{con}}b) \sim (a', {}_{\text{con}}b') \sim (b', a', {}_{\text{con}}b') \sim (b', a, {}_{\text{con}}b)\,,$$

where the strict preference follows from **RD**, **SEF** and **ST** since $b' < b$, the first indifference follows since $(a, {}_{\text{con}}b) \sim (a', {}_{\text{con}}b')$, the second indifference follows from **FA**, and the third indifference follows from combining $(a, {}_{\text{con}}b) \sim (a', {}_{\text{con}}b')$ with **SEF** and **ST**. But then, by transitivity (**QO**), we have that **FA** fails, leading to a contradiction. Hence, **FA** cannot hold under the assumptions of the proposition. ■

We proceed by providing two examples showing how **REP** (thus implying axioms **O** and **QO**), **PFS** (or even **C**), **M**, **SEF** and **ST** can be combined with even **SA** (implying invariance to set of *all* permutation) provided that the sensitivity entailed by **RD** is relaxed. The first one is a variant of Example 2, retaining sensitivity only for summable streams, while the second one has only asymptotic sensitivity.

Example 3 As in Example 2, let $\mathbf{N} := \{{}_1\mathbf{x} \in \mathbf{X} : \sum_{t=1}^{\infty} x_t < \infty\}$ and $\mathbf{I} := \mathbf{X} \backslash \mathbf{N}$. Let the SWF $V : \mathbf{X} \to [0,1]$ be given by:

$$V({}_1\mathbf{x}) = \begin{cases} 1 & \text{if } {}_1\mathbf{x} \in \mathbf{I}\,, \\ \frac{\sum_{t=1}^{\infty} x_t}{1+\sum_{t=1}^{\infty} x_t} & \text{if } {}_1\mathbf{x} \in \mathbf{N}\,. \end{cases}$$

Then V represents an SWR that satisfies axioms **REP**, **PFS**, **M**, **SEF**, **ST** and **SA**. Note that the SWR does not satisfy axiom **RD** as $V(a, {}_{\text{con}}b) = 1 = V({}_{\text{con}}b)$ if $a < b$, while satisfying the strong Pareto principle on the subset of summable streams. As in Example 2 the SWR represented by V does not satisfy **LC**. To see this, consider a sequence $\{b^k\} \in (0,1]^{|\mathbb{N}|}$ such that $\lim_{k\to\infty} b^k = 0$. Then ${}_{\text{con}}b^k \succ (1, {}_{\text{con}}0)$ for all $k \in \mathbb{N}$, while ${}_{\text{con}}0 \prec (1, {}_{\text{con}}0)$, thus contradicting **LC**.

Example 4 Let the SWF $W : \mathbf{X} \to [0,1]$ be given by:

$$W({}_1\mathbf{x}) \;=\; \alpha \liminf_{t\to\infty} x_t \;+\; (1-\alpha)\limsup_{t\to\infty} x_t\,,$$

where $\alpha \in (0,1)$. Then W represents an SWR that satisfies axioms **O**, **C**, **M**, **SEF**, **ST** and **SA**. Even though the SWR does not satisfy axiom **RD** (as required in Lemma 1), W has the property that $W({}_{\text{con}}b) = b$ for all $b \in [0,1]$. In particular, the SWR satisfies a "uniform" Pareto principle in the sense that for all ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}$ and $\varepsilon > 0$, if $x_t \geq y_t + \varepsilon$ for all $t \in \mathbb{N}$, then ${}_1\mathbf{x}$ is strictly preferred to ${}_1\mathbf{y}$. For more on the axiomatic basis for related SWRs, see Chambers (2009) and Sakai (2016).

These two examples illustrate that one cannot have both sensitivity for the present generation and asymptotic sensitivity in an SWR that treats generations equally and is time consistent. In fact, axiom **RD** is a minimalist way of capturing both types of sensitivity (when combined with the incontrovertible **M**). And as soon as one imposes axiom **RD** (in combination with **QO** and **PFS**), we know that axioms **SEF** and **ST** lead to impatience, so equal treatment as captured by **FA** is destroyed.

We end this section by noting that there is another route out of the dilemma discussed here, namely to drop the minimal continuity requirement that axiom **PFS** represents. Undiscounted utilitarianism and leximin as defined in the setting of infinite wellbeing streams by Basu and Mitra (2007) and Bossert et al. (2007), respectively, satisfy axiom **FA** as well as axioms **QO**, **SP**, **SEF** and **ST**, and thus all the remaining axioms of Proposition 3, as axiom **SP** implies axiom **RD**. However, they fail axiom **PFS**. Moreover, if we impose completeness by strengthening **QO** to **O**, then these criteria are necessarily non-constructive (see Zame, 2007; Lauwers, 2010), and they cannot satisfy axiom **REP**.

6 Time inconsistent criteria

In Section 3 we have shown that Time consistency is equivalent to axioms **SEF** and **ST** under Time invariance.

In Section 4 we have shown that axioms **O**, **C**, **SEF** and **ST** in combination with axiom **SP** contradict **NDP** (on the set of converging streams). The criterion that Chichilnisky (1996) proposes in her Theorems 1 and 2 satisfies **O**, **C**, **SP** and **NDP**. Thus, either **SEF** or **ST** must fail. In the first part of this section we take a closer look at Chichilnisky's criterion (henceforth referred to as the *C-criterion*) and show that it satisfies **SEF**, but fails **ST**.

In Section 5 we have shown that axioms **QO**, **PFS**, **SEF** and **ST** in combination with axiom **RD** contradict **FA**. The criterion of *Rank-Discounted Utilitarianism* (RDU) suggested by Zuber and Asheim (2012) satisfies **O**, **C**, **M**, **RD** and **SA**. Thus, since axioms **O**, **C**, **M** and **RD** imply axioms **QO** and **PFS** and **SA** strengthens **FA**, this leads to the conclusion that RDU must fail either **SEF** or **ST**. In the second part of this section we investigate the RDU criterion and show that it satisfies **ST**, but fails **SEF**.

As preliminaries to studying both criteria, let $u : [0,1] \to \mathbb{R}$ be an increasing and continuous function that maps positive wellbeing into *transformed wellbeing* (or *generalized utility*), and define the *Time-Discounted Utilitarian* (TDU) welfare function $W^T_\beta : \mathbf{X} \to [0,1]$ for the discount factor $\beta \in (0,1)$ as follows:

$$W^T_\beta({}_1\mathbf{x}) = u^{-1}\left[(1-\beta)\sum\nolimits_{t=1}^{\infty}\beta^{t-1}u(x_t)\right] .$$

By multiplying with $(1-\beta)$, one obtains average discounted generalized utility and, by using the inverse u^{-1} of the utility function u, average discounted generalized utility is mapped into $[0,1]$, leading to an SWF in the class considered by Lemma 1.

The C-criterion (see Chichilnisky, 1996, for a presentation and analysis of this criterion, including an axiomatization) evaluates streams according to a convex combination of TDU welfare and the limit of wellbeing. It is represented by the welfare function $W^C({}_1\mathbf{x}) : \mathbf{X}^c \to [0,1]$ defined as follows:

$$W^C({}_1\mathbf{x}) \;=\; (1-\gamma)\, W^T_\beta({}_1\mathbf{x}) \;+\; \gamma \lim_{t\to\infty} x_t \,,$$

where $\gamma \in (0, 1)$. This is a particular version within the class of representations considered in Chichilnisky's (1996) Theorems 1 and 2.

The C-criterion satisfies axioms **O**, **C**, **SP**, **SEP**, **SEF** and **NDP** on the set $\mathbf{X}^c$ of converging streams. However, it does not satisfy **ST**. The reason is that, as when time is advanced, the weight on the stream in TDU part increases, while the weight on the limit is not affected. Therefore, it is not the case that for all ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}^c$, there exists $a \in [0, 1]$ such that ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if $(a, {}_2\mathbf{x}') \succsim (a, {}_2\mathbf{y}')$, where $x_t = x'_{t+1}$ and $y_t = y'_{t+1}$ for all $t \in \mathbb{N}$. Rather, there exist ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}^c$ with

$$W^T_\beta({}_1\mathbf{x}) > W^T_\beta({}_1\mathbf{y}) \quad \text{and} \quad \lim_{t\to\infty} x_t < \lim_{t\to\infty} y_t$$

such that, for all $a \in [0, 1]$, ${}_1\mathbf{x} \succ {}_1\mathbf{y}$, but $(a, {}_2\mathbf{x}') \prec (a, {}_2\mathbf{y}')$. In particular, if ${}_1\mathbf{x} = (1, 1, {}_{\text{con}}0)$ and ${}_1\mathbf{y} = (0, 0, {}_{\text{con}}1)$ with $\beta = 1/2$, $\gamma = 1/4$ and $u(a) = a$, then $W^T_\beta({}_1\mathbf{x}) = 3/4$ and $W^T_\beta({}_1\mathbf{y}) = 1/4$ so that $W^C({}_1\mathbf{x}) - W^C({}_1\mathbf{y}) = 1/8$ and $W^C(a, {}_2\mathbf{x}') - W^C(a, {}_2\mathbf{y}') = -1/16$ for all $a \in [0, 1]$.

One gets another perspective on the time inconsistency of the C-criterion by noting that, on the set $\mathbf{X}^c$ of converging streams, the limit of wellbeing equals the limit of TDU welfare as the discount factor goes to 1. Hence, the following welfare function is an alternative representation of the C-criterion:

$$W^C({}_1\mathbf{x}) \,=\, (1-\gamma)\, W^T_\beta({}_1\mathbf{x}) \,+\, \gamma \lim_{\delta\to 1} W^T_\delta({}_1\mathbf{x})\,.$$

This means that the C-criterion might be looked at as a Pareto-efficient and non-dictatorial aggregation of heterogeneous preferences, with $(1-\gamma)$ weight on TDU and γ weight on undiscounted utilitarianism. Thus, by Jackson and Yariv (2015, Proposition 1), the C-criterion is time inconsistent.

Under RDU (see Zuber and Asheim, 2012, for a presentation and analysis of this criterion, including an axiomatization), a wellbeing stream is first reordered into a non-decreasing stream, so that discounting is according to rank, not according to time. The definition takes into account the fact that a stream like $(1, 0, 0, 0, \dots)$, with elements of infinite rank, cannot be reordered into a non-decreasing stream. Therefore, let $\lambda({}_1\mathbf{x})$ denote lim inf of ${}_1\mathbf{x}$, and let $\Lambda({}_1\mathbf{x}) := \{t \in \mathbb{N} \mid x_t < \lambda({}_1\mathbf{x})\}$. If $|\Lambda({}_1\mathbf{x})| = \infty$, then let ${}_{[1]}\mathbf{x} = (x_{[1]}, x_{[2]}, \dots)$ be a non-decreasing reordering of all elements x_t with $t \in \Lambda({}_1\mathbf{x})$ (so that $x_{[r]} \leq x_{[r+1]}$ for all ranks $r \in \mathbb{N}$). If $|\Lambda({}_1\mathbf{x})| < \infty$, then let $(x_{[1]}, x_{[2]}, \dots, x_{[|\Lambda({}_1x)|]})$ be a non-decreasing reordering of all elements x_t with $t \in \Lambda({}_1\mathbf{x})$ (so that $x_{[r]} \leq x_{[r+1]}$ for all ranks $r \in \{1, \dots, |\Lambda({}_1\mathbf{x})| - 1\}$), and set $x_{[r]} = \lambda({}_1\mathbf{x})$ for all $r > |\Lambda({}_1\mathbf{x})|$. Define the RDU welfare function $W^R : \mathbf{X} \to [0,1]$ for $\beta \in (0,1)$ as follows:

$$W^R({}_1\mathbf{x}) = W^T_\beta({}_{[1]}\mathbf{x})\,.$$

RDU is characterized by axioms **O**, **C**, **M**, **RD**, **RSEP**, **RSEF**, **RST** and **SA**, where axioms **RSEP**, **RSEF** and **RST** are axioms **SEP**, **SEF** and **ST** restricted to the set $\mathbf{X}^+$ of non-decreasing streams:

Axiom RSEP (*Restricted Separable Present*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y}$, ${}_1\mathbf{x}'$, ${}_1\mathbf{y}' \in \mathbf{X}^+$ such that (i) $x_t = x'_t$ and $y_t = y'_t$ for all $t \in \{1,2\}$ and (ii) $x_t = y_t$ and $x'_t = y'_t$ for all $t \in \mathbb{N} \setminus \{1,2\}$, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$.

Axiom RSEF (*Restricted Separable Future*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y}$, ${}_1\mathbf{x}'$, ${}_1\mathbf{y}' \in \mathbf{X}^+$ such that (i) $x_t = x'_t$ and $y_t = y'_t$ for all $t \in \mathbb{N} \setminus \{1\}$ and (ii) $x_1 = y_1$ and $x'_1 = y'_1$, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$.

Axiom RST (*Restricted Stationarity*) For any ${}_1\mathbf{x}$, ${}_1\mathbf{y}, \in \mathbf{X}^+$, there exists $a \in [0,1]$ with $a \leq \min\{x_1, y_1\}$ such that ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if $(a, {}_2\mathbf{x}') \succsim (a, {}_2\mathbf{y}')$, where $x_t = x'_{t+1}$ and $y_t = y'_{t+1}$ for all $t \in \mathbb{N}$.

However, RDU satisfies even axiom **ST** since, for all ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}$, it holds that ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if $(0, {}_2\mathbf{x}') \succsim (0, {}_2\mathbf{y}')$, where $x_t = x'_{t+1}$ and $y_t = y'_{t+1}$ for all $t \in \mathbb{N}$. The key is that, by setting $a = 0$, the internal ranking of the elements of ${}_1\mathbf{x}$ and ${}_1\mathbf{y}$ remains unchanged when the wellbeing level $a = 0$ is imposed at time 1 and the elements of ${}_1\mathbf{x}$ and ${}_1\mathbf{y}$ are moved one period forward.

Thus, it follows from Proposition 2 that RDU cannot satisfy **SEF**. In view of the fact that RDU satisfies **RSEF**, an example that illustrates the failure of **SEF** must involve streams that are not non-decreasing. In particular, if

$$
{}_1\mathbf{x} = (\tfrac{2}{5}, \tfrac{2}{5}, {}_{\text{con}}\tfrac{2}{5}) \qquad {}_1\mathbf{y} = (\tfrac{2}{5}, 0, {}_{\text{con}}1)
$$

$$
{}_1\mathbf{x}' = (0, \tfrac{2}{5}, {}_{\text{con}}\tfrac{2}{5}) \qquad {}_1\mathbf{y}' = (0, 0, {}_{\text{con}}1)
$$

with $\beta = 1/2$ and $u(a) = a$, then $W^R({}_1\mathbf{x}) = 8/20 > 7/20 = W^R({}_1\mathbf{y})$, while $W^R({}_1\mathbf{x}') = 4/20 < 5/20 = W^R({}_1\mathbf{y}')$, thereby contradicting **SEF**.

RDU fails also **SEP**. It follows from the fact that RDU satisfies **RSEP** that an example that illustrates the failure of **SEP** must also involve streams that are not non-decreasing. In particular, if

$$
{}_1\mathbf{x} = (\tfrac{1}{5}, 1, {}_{\text{con}}1) \qquad {}_1\mathbf{y} = (\tfrac{2}{5}, \tfrac{2}{5}, {}_{\text{con}}1)
$$

$$
{}_1\mathbf{x}' = (\tfrac{1}{5}, 1, {}_{\text{con}}\tfrac{2}{5}) \qquad {}_1\mathbf{y}' = (\tfrac{2}{5}, \tfrac{2}{5}, {}_{\text{con}}\tfrac{2}{5})
$$

with $\beta = 1/2$ and $u(a) = a$, then $W^R({}_1\mathbf{x}) = 12/20 > 11/20 = W^R({}_1\mathbf{y})$, while $W^R({}_1\mathbf{x}') = 3/10 < 4/10 = W^R({}_1\mathbf{y}')$, thereby contradicting **SEP**. The intuition for this failure is that we might want to treat the conflict between the worst-off and the second worst-off generation presented by the first comparison differently from the conflict between the worst-off and the best-off generation put forward by the second comparison.

RDU satisfies the strong Pareto principle on the set of non-decreasing streams. Hence, this criterion illustrates how axiom **RD** can be equivalent to the strong Pareto principle on the set of non-decreasing streams when a set of additional axioms are imposed.

7 Consequences of time inconsistency

In the present section we investigate the consequences of the time inconsistency of the C-criterion and RDU in the *Ramsey model* (Ramsey, 1928). In this model, at each $t \in \mathbb{N}$ net production, $f(k_{t-1})$, depends on the stock of capital, k_{t-1}, and is split between wellbeing, x_t, and net accumulation of capital, $k_t - k_{t-1}$:

$$x_t + (k_t - k_{t-1}) = f(k_{t-1}),$$

with $k_0 = k$ as initial condition. Capital is assumed to be non-negative, and the net production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be increasing, strictly concave and continuously differentiable, with $f(0) = 0$, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$. Assume also that the utility function $u : \mathbb{R}_+ \to \mathbb{R}$, in addition to being an increasing and continuous function, is strictly concave, continuously differentiable, and satisfies $\lim_{x\to 0} u'(x) = \infty$. Note that we, for the analysis of this section, allow wellbeing to take on values in $\mathbb{R}_+$, so the restriction to $[0, 1]$ is relaxed and the set of possible wellbeing streams $\mathbf{X}$ equals $\mathbb{R}_+^{|\mathbb{N}|}$.

A *Markov strategy* σ maps from the capital stock to a feasible flow of wellbeing:

$$\sigma : \ k \in \mathbb{R}_+ \ \mapsto \ \sigma(k) \in [0, k + f(k)]$$

For given initial condition k, a Markov strategy σ implements a capital stream:

$$\mathbf{k}(\sigma, k) = {}_0\mathbf{k} = (k_0, k_1, \ldots, k_t, \ldots)$$
$$\text{where } k_0 = k \text{ and, for } t \in \mathbb{N},\ k_t = k_{t-1} + f(k_{t-1}) - \sigma(k_{t-1}),$$

and a wellbeing stream:

$$\mathbf{x}(\sigma, k) = {}_1\mathbf{x} = (x_1, x_2, \ldots, x_t, \ldots) \text{ where for } t \in \mathbb{N},\ x_t = \sigma(k_{t-1}).$$

A Markov strategy σ is a *Markov-perfect equilibrium* under an SWR represented by the SWF $W : \mathbf{X} \to \mathbb{R}$ if:

$$\text{For all } k \in \mathbb{R}_+,\ W(\mathbf{x}(\sigma, k)) = \max_{x\in[0,k+f(k)]} W(x, \mathbf{x}(\sigma, k + f(k) - x)).$$

A Markov-perfect equilibrium is a way of formalizing *sophisticated planning* if the

SWR is time inconsistent.

It follows from Beals and Koopmans (1969) that there is a unique optimum under TDU, with both capital and consumption streams being strictly monotone, with capital converging to $k_\infty(\beta)$, and with wellbeing converging to $f(k_\infty(\beta))$, where $k_\infty(\beta)$ is defined by $\beta(1 + f'(k_\infty(\beta)) = 1$. Since the capital stream is strictly monotone, there exists a TDU Markov strategy, σ^T, which implements the TDU optimum. Clearly, σ^T is a Markov-perfect equilibrium under the SWR represented by W_β^T, because otherwise it would have been possible to improve upon the TDU optimum.

Under the C-criterion there is no optimal stream in the Ramsey model. The reason is that it is profitable to delay the concern for the infinite future, while infinite delay is worse. Furthermore, it is straightforward to argue that σ^T is a Markov-perfect equilibrium under the C-criterion. The reason is that there is no possibility for any one generation to influence the infinite future, since under σ^T wellbeing converges to $f(k_\infty(\beta))$ independently of the initial capital stock. Hence, it is a best reply for each generation to maximize the TDU criterion, which is what is achieved by following the TDU Markov strategy, σ^T.

There are other and better non-Markovian equilibria under the C-criterion. To see this, let $\sigma^T_{[k^*,\infty)}$ be the Markov strategy that implements the TDU optimum given that the stock of capital is constrained to remain at least k^*, where $k^* > k_\infty(\beta)$. Let a non-Markovian strategy $(\Sigma_1, \Sigma_2, \dots, \Sigma_t, \dots)$ be defined by, for all $t \in \mathbb{N}$:

$$\Sigma_t({}_0\mathbf{k}_{t-1}) = \begin{cases} \sigma^T(k_{t-1}) & \text{if } \exists \tau \in \{0, \dots, t-1\} \text{ such that } k_\tau < k^* \\ \sigma^T_{[k^*,\infty)}(k_{t-1}), & \text{otherwise.} \end{cases}$$

Hence, if the initial capital stock k is at least as large as $k^* > k_\infty(\beta)$, following $(\Sigma_1, \Sigma_2, \dots, \Sigma_t, \dots)$ implies that k_t converges to k^*. For an interval of k^* values exceeding $k_\infty(\beta)$ this is an equilibrium strategy, as the loss in terms of TDU welfare is more than compensated for by a higher limit of wellbeing. Asheim and Ekeland (2016) show how such equilibria can be Markovian in continuous time.

Under RDU, there exists a unique optimum for any $k \in \mathbb{R}_+$ (Zuber and Asheim, 2012, Proposition 10). This optimum can be implemented by the following Markov strategy whereby the TDU optimum is implemented with a small initial capital stock (so that initial capital productivity is high) and the maximin path is implemented with a large initial capital stock (so that initial capital productivity is low):

$$\sigma^R(k) \;=\; \begin{cases} \sigma^T(k_{t-1}) & \text{if } k_t < k_\infty(\beta)\,, \\ f(k)\,, & \text{if } k_t \geq k_\infty(\beta)\,. \end{cases}$$

Furthermore, σ^R is a Markov-perfect equilibrium under RDU. Hence, for any $k \in \mathbb{R}_+$, the RDU optimum is time consistent.

However, there are also other and worse Markov-perfect equilibria under RDU. The following is one example of a Markov-perfect equilibrium where capital is depleted asymptotically, so that the limit of wellbeing as time goes to infinity equals zero:

$$\sigma(k) \;=\; f(k) \,+\, \delta k \;\text{ for all } k\,, \quad \delta \in (0,1]\,.$$

This is a Markov-perfect equilibrium as any one generation cannot prevent that that the limit of wellbeing as time goes to infinity equals zero, so that all present wellbeing choices are equally bad. The existence of this Markov-perfect equilibrium is caused by the RDU criterion not being continuous at infinity.

The existence of such unattractive Markov-perfect equilibria calls for refinements that picks out the unique RDU optimum, as this is the obvious consequence of sophisticated planning whereby the best stream that will actually be followed under RDU is chosen. A refinement that works in the case is *Revision-proof equilibrium* (Asheim, 1997).

Our application of the C-criterion and RDU in the Ramsey model shows how time inconsistent criteria of intergenerational equity can be applied in a growth model. However, there is a multiplicity of equilibria, suggesting that the principles for selection must be carefully considered.

8 Concluding remarks

In this paper we have shown how conditions of intergenerational equity may necessitate time inconsistency of normative criteria. Furthermore, we have illustrated how such time inconsistency leads to interesting game theoretic challenges.

More precisely, we have shown conditions under which criteria of intergenerational equity that cannot both be time consistent and time invariant. Throughout, we have insisted that the criteria be time invariant, which corresponds to consequentialist social decision-making. In particular, we have pointed to sophisticated planning as a rational manner to tackle the problem of time inconsistency, as illustrated in Section 7.

An alternative route is to insist that the criteria be time consistent, and relax the requirement of time invariance. In the case of the C-criterion this means that the weight on the TDU part of the criterion vanishes as time goes to infinity, so that asymptotically only the limit of wellbeing matters. In the case of RDU, insistence of time consistency and relaxation of time invariance makes the criterion history dependent, as past wellbeing levels must be allowed to enter into the ranking of future wellbeing levels. This alternative approach has merit and is worthy of exploration.

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A Proofs

Proof of Lemma 1. (i) Consider any ${}_1\mathbf{x} \in \mathbf{X}$. By axioms **O**, **C** and **M**, there exists $a \in [0,1]$ such that ${}_{\text{con}}a \sim {}_1\mathbf{x}$. By letting $W({}_1\mathbf{y}) = a$ for all ${}_1\mathbf{y} \in \mathbf{X}$ with ${}_1\mathbf{y} \sim {}_1\mathbf{x}$, we obtain a SWF $W : \mathbf{X} \to [0,1]$ that represents $\succsim$.

(ii) Let ${}_{\text{con}}a \sim {}_1\mathbf{x}$. Suppose there exists $b \in [0,1]$ such that $a \neq b$ and ${}_{\text{con}}b \sim {}_1\mathbf{x}$. Without loss of generality, suppose $a < b$. Then by **M** and **RD**,

$$ {}_1\mathbf{x} \sim {}_{\text{con}}a \precsim (a,\, {}_{\text{con}}b) \prec {}_{\text{con}}b \sim {}_1\mathbf{x}\,, $$

which contradicts axiom **O**. Hence, there exists a *unique* $a \in [0,1]$ such that ${}_{\text{con}}a \sim {}_1\mathbf{x}$. Thus, $W : \mathbf{X} \to [0,1]$ defined by, for all ${}_1\mathbf{x} \in \mathbf{X}$, $W({}_1\mathbf{x}) = a$ where ${}_{\text{con}}a \sim {}_1\mathbf{x}$, is a SWF that represents $\succsim$ and has the property that $W({}_{\text{con}}b) = b$ for all $b \in [0,1]$. ■

Proof of Lemma 2. *Only if.* Assume that $\succsim$ satisfies Independent future.

Let ${}_1\mathbf{x}, {}_1\mathbf{y}, {}_1\mathbf{x}', {}_1\mathbf{y}' \in \mathbf{X}$ satisfy (i) $x_t = x'_t$ and $y_t = y'_t$ for all $t \in \mathbb{N}\setminus\{1\}$ and (ii) $x_1 = y_1$ and $x'_1 = y'_1$. Let $x''_t = x_{t+1} = x'_{t+1}$ and $y''_t = y_{t+1} = y'_{t+1}$ for all $t \in \mathbb{N}$. By Independent future, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}'' \succsim {}_1\mathbf{y}''$ and ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$ if and only if ${}_1\mathbf{x}'' \succsim {}_1\mathbf{y}''$. Hence, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$, thereby showing that $\succsim$ satisfies **SEF**.

It follows directly from Independent future that, for any ${}_1\mathbf{x}, {}_1\mathbf{y}, \in \mathbf{X}$, and *for any* $a \in [0,1]$, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if $(a,\, {}_2\mathbf{x}') \succsim (a,\, {}_2\mathbf{y}')$, where $x_t = x'_{t+1}$ and $y_t = y'_{t+1}$ for all $t \in \mathbb{N}$, thereby showing that $\succsim$ satisfies **ST**.

If. Assume that $\succsim$ satisfies axioms **SEF** and **ST**. Let ${}_1\mathbf{x}, {}_1\mathbf{y} \in \mathbf{X}$ satisfy $x_1 = y_1$. By **ST**, there exists $a \in [0,1]$ such that $(a,\, {}_2\mathbf{x}) \succsim (a,\, {}_2\mathbf{y})$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$, where $x'_t = x_{t+1}$ and $y'_t = y_{t+1}$ for all $t \in \mathbb{N}$. By **SEF** and the fact that $x_1 = y_1$, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if $(a,\, {}_2\mathbf{x}) \succsim (a,\, {}_2\mathbf{y})$. Hence, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$, thereby showing that $\succsim$ satisfies Independent future. ■

Proof of Proposition 1. Let the SWR $\succsim$ be Time invariant.

Only if. Assume that $\succsim$ is Time consistent. Let ${}_1\mathbf{x}, {}_1\mathbf{y} \in \mathbf{X}$ satisfy $x_1 = y_1$. Let $x'_t = x_{t+1}$ and $y'_t = y_{t+1}$ for all $t \in \mathbb{N}$. By Time consistency, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_2\mathbf{x} \succsim {}_2\mathbf{y}$. By Time invariance, ${}_2\mathbf{x} \succsim {}_2\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$. Hence, ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$, thereby showing that $\succsim$ satisfies Independent future. By Lemma 2, $\succsim$ satisfies axioms **SEF** and **ST**.

If. Assume that $\succsim$ satisfies axioms **SEF** and **ST**. By Lemma 2, $\succsim$ satisfies Independent future. Let ${}_1\mathbf{x}$, ${}_1\mathbf{y}, \in \mathbf{X}$ satisfy $x_\tau = y_\tau$. Let $x'_t = x_{t+\tau-1}$, $y'_t = y_{t+\tau-1}$, $x''_t = x_{t+\tau}$ and $y''_t = y_{t+\tau}$ for all $t \in \mathbb{N}$. By repeated use of Time invariance, ${}_\tau\mathbf{x} \succsim {}_\tau\mathbf{y}$ if and only if ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$ and ${}_{\tau+1}\mathbf{x} \succsim {}_{\tau+1}\mathbf{y}$ if and only if ${}_1\mathbf{x}'' \succsim {}_1\mathbf{y}''$. By Independent future, ${}_1\mathbf{x}' \succsim {}_1\mathbf{y}'$ if and only if ${}_1\mathbf{x}'' \succsim {}_1\mathbf{y}''$. Hence, ${}_\tau\mathbf{x} \succsim {}_\tau\mathbf{y}$ if and only if ${}_{\tau+1}\mathbf{x} \succsim {}_{\tau+1}\mathbf{y}$, thereby showing that $\succsim$ satisfies Time consistency. ■

Lemma 3 below is the key to proving Proposition 2. For the statement of this lemma, define for all ${}_1\mathbf{x} \in \mathbf{X}$ the sequences $\{\ell_t({}_1\mathbf{x})\}$ and $\{h_t({}_1\mathbf{x})\}$ as follows:

$$\text{For all } t \geq 1, \quad \ell_t({}_1\mathbf{x}) = W({}_1\mathbf{x}_t, {}_{\text{con}}0) \quad \text{and} \quad h_t({}_1\mathbf{x}) = W({}_1\mathbf{x}_t, {}_{\text{con}}1)\,.$$

Clearly, $\{\ell_t({}_1\mathbf{x})\}$ is a non-decreasing sequence, bounded above by $W({}_1\mathbf{x})$, by axiom **SP**. So it converges to a limit; denote this limit by $L({}_1\mathbf{x})$. Similarly (by axiom **SP**) we have that $\{h_t({}_1\mathbf{x})\}$ is a non-increasing sequence, bounded below by $W({}_1\mathbf{x})$. So it converges to a limit; denote this limit by $H({}_1\mathbf{x})$. Hence, we have:

$$L({}_1\mathbf{x}) \;\leq\; W({}_1\mathbf{x}) \;\leq\; H({}_1\mathbf{x})\,. \tag{5}$$

Lemma 3 *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies* ***O****,* ***C****,* ***SP****,* ***SEF*** *and* ***ST****. Let* $\succsim$ *be represented by* $W : \mathbf{X} \to [0,1]$*. Then*

$$L({}_1\mathbf{x}) \;=\; W({}_1\mathbf{x}) \;=\; H({}_1\mathbf{x})$$

for all ${}_1\mathbf{x} \in \mathbf{X}^c$*.*

Proof. Assume axioms **O**, **C**, **SP**, **SEF** and **ST**. Let ${}_1\mathbf{x} \in \mathbf{X}^c$ be given, and let a denote $\lim_{t\to\infty} x_t$. We will show that $W({}_1\mathbf{x}) = H({}_1\mathbf{x})$. Given (5), this is established if, for every $\varepsilon > 0$,

$$\text{there is } \tau \geq 1 \text{ such that} \quad W({}_1\mathbf{x}_t, {}_{\text{con}}1) \;\leq\; W({}_1\mathbf{x}) + \varepsilon \quad \text{for all } t \geq \tau\,.$$

Let $\varepsilon > 0$ be given. Using axiom **C**, we can find $\delta > 0$ such that whenever ${}_1\mathbf{y} \in \mathbf{X}$ satisfies $\sup_{t\in\mathbb{N}} |y_t - x_t| \leq \delta$, we have $|W({}_1\mathbf{y}) \,-\, W({}_1\mathbf{x})| \leq \varepsilon$.

Case 1: $a \in [0,1)$. Since $\lim_{t\to\infty} x_t = a$, there exists $\tau' \geq 1$ such that $|x_t - a| \leq \delta/3$ for all $t \geq \tau'$. Let $\bar{a} = a + \delta/3$. Define the sequence $\{\bar{a}_t\}$ inductively by $\bar{a}_1 = g(\bar{a}, 1)$ and, for all $t \geq 1$, $\bar{a}_{t+1} = g(\bar{a}, \bar{a}_t)$. By the proof of Observation 1, $\{\bar{a}_t\}$ is decreasing and converges to: $\lim_{t\to\infty} \bar{a}_t = \bar{a}$. Hence, there exists $\tau'' \geq 1$ such that $\bar{a}_{\tau''} \leq \bar{a} + \delta/3$. By construction, $x_t \leq a + \delta/3 = \bar{a} \leq \bar{a}_{\tau''}$ and $x_t \geq a - \delta/3 \geq \bar{a}_{\tau''} - 2\delta/3 - \delta/3 = \bar{a}_{\tau''} - \delta$ for all $t \geq \tau'$. Let

$\tau := \tau' + \tau''$. Then, for all $t \geq \tau$:

$$W({}_1\mathbf{x}_t,\, {}_{\text{con}}1) \,\leq\, W({}_1\mathbf{x}_\tau,\, {}_{\text{con}}1) \,\leq\, W({}_1\mathbf{x}_{\tau'}, \underbrace{\bar{a}, \ldots, \bar{a}}_{\tau''\text{ times}}, {}_{\text{con}}1) \qquad \text{by axiom } \mathbf{SP},$$

$$= W({}_1\mathbf{x}_{\tau'},\, {}_{\text{con}}\bar{a}_{\tau''}) \qquad \text{by using (a) the fact that, by the definition of } \bar{a}_{\tau''},$$

$$W({}_{\text{con}}\bar{a}_{\tau''}) = \bar{a}_{\tau''} = g(\bar{a}, \bar{a}_{\tau''-1}) = g(\bar{a}, g(\bar{a}, \bar{a}_{\tau''-2})) = \cdots = W(\underbrace{\bar{a}, \ldots, \bar{a}}_{\tau''\text{ times}}, {}_{\text{con}}1)\,,$$

and (b) the Independent future property (i.e., axioms **SEF** and **ST**) repeatedly,

$$\leq W({}_1\mathbf{x}) + \varepsilon \qquad \text{since } |x_t - \bar{a}_{\tau''}| \leq \delta \text{ for all } t \geq \tau'\,,$$

thereby completing Case 1.

Case 2: $a = 1$. Since $\lim_{t\to\infty} x_t = 1$, there exists $\tau \geq 1$ such that $x_t \geq 1 - \delta$ for all $t \geq \tau$. Hence, for all $t \geq \tau$:

$$W({}_1\mathbf{x}_t,\, {}_{\text{con}}1) \,\leq\, W({}_1\mathbf{x}_\tau,\, {}_{\text{con}}1) \,\leq\, W({}_1\mathbf{x}) + \varepsilon\,,$$

thereby completing Case 2.

The result that $W({}_1\mathbf{x}) = L({}_1\mathbf{x})$ can be shown in an analogous manner. ■

Proof of Proposition 2. Assume axioms **O**, **C**, **SP**, **SEF** and **ST**. Let ${}_1\mathbf{x}$, ${}_1\mathbf{y} \in \mathbf{X}^c$ satisfy $W({}_1\mathbf{x}) < W({}_1\mathbf{y})$. Then, $\varepsilon := W({}_1\mathbf{y}) - W({}_1\mathbf{x}) > 0$. By Lemma 3 we can choose τ large enough so that for all $t \geq \tau$,

$$W({}_1\mathbf{x}_t,\, {}_{\text{con}}1) < W({}_1\mathbf{x}) + \varepsilon/2 \quad \text{and} \quad W({}_1\mathbf{y}_t,\, {}_{\text{con}}0) > W({}_1\mathbf{y}) - \varepsilon/2\,. \tag{6}$$

Thus, for all $t \geq \tau$, using (6) and the definition of ε,

$$\begin{aligned} W({}_1\mathbf{x}_t,\, {}_{\text{con}}1) &< W({}_1\mathbf{x}) + \varepsilon/2 \\ &= W({}_1\mathbf{y}) - \varepsilon + \varepsilon/2 = W({}_1\mathbf{y}) - \varepsilon/2 < W({}_1\mathbf{y}_t,\, {}_{\text{con}}0)\,. \end{aligned} \tag{7}$$

Let ${}_1\mathbf{u}$, ${}_1\mathbf{v}$ be arbitrary elements of $\mathbf{X}$. Then, by (7) and axiom **SP**,

$$W({}_1\mathbf{x}_t,\, {}_{t+1}\mathbf{u}) \,\leq\, W({}_1\mathbf{x}_t,\, {}_{\text{con}}1) \,<\, W({}_1\mathbf{y}_t,\, {}_{\text{con}}0) \,\leq\, W({}_1\mathbf{y}_t,\, {}_{t+1}\mathbf{v})$$

for all $t \geq \tau$, thereby establishing the proposition. ■

B Proposition 4

Proposition 4 *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies axioms* ***O****,* ***LC****,* ***M****,* ***LD****,* ***SEF*** *and* ***ST****. For any* a*,* $b \in [0,1]$*, if* $a < b$*, then there exists* τ *such that, for every* $t \geq \tau$*,*

$$(\underbrace{a, \ldots, a}_{t \text{ times}},\ {}_{t+1}\mathbf{u}) \prec (\underbrace{b, \ldots, b}_{t \text{ times}},\ {}_{t+1}\mathbf{v})$$

for all ${}_1\mathbf{u}$*,* ${}_1\mathbf{v} \in \mathbf{X}$*.*

The following lemma is useful for the proof of Proposition 4.

Lemma 4 *Assume that the* SWR $\succsim$ *on* $\mathbf{X}$ *satisfies axioms* ***O****,* ***LC****,* ***M****, and let* a*,* $b \in [0,1]$ *satisfy* $a < b$*. Then:*

(i) There exists $a' \in [a,b]$ *such that* ${}_{con}a' \sim (a,\ {}_{\mathrm{con}}b)$*. If the* SWR $\succsim$ *satisfies also axiom* ***RD****, then* $a' < b$*. If the* SWR $\succsim$ *satisfies also axioms* ***RD****,* ***SEF*** *and* ***ST****, then* $a < a'$*.*

(ii) There exists $b' \in [a,b]$ *such that* ${}_{con}b' \sim (b,\ {}_{\mathrm{con}}a)$*. If the* SWR $\succsim$ *satisfies also axiom* ***LD****, then* $a < b'$*. If the* SWR $\succsim$ *satisfies also axioms* ***LD****,* ***SEF*** *and* ***ST****, then* $b' < b$*.*

Proof. (i) By axioms **O**, **LC**, **M**, there exists $a' \in [a,b]$ such that ${}_{\mathrm{con}}a' \sim (a,\ {}_{con}b)$. To show $a' < b$, note by the choice of a' and axiom **RD**, ${}_{\mathrm{con}}a' \sim (a,\ {}_{\mathrm{con}}b) \prec (b,\ {}_{\mathrm{con}}b) = {}_{\mathrm{con}}b$, so ${}_{\mathrm{con}}a' \prec {}_{\mathrm{con}}b$ by transitivity. Then, since $a' > b$ would imply ${}_{\mathrm{con}}a' \succ {}_{\mathrm{con}}b$ (by (1)), while $a' = b$ would imply ${}_{\mathrm{con}}a' \sim {}_{\mathrm{con}}b$ (by reflexivity), we have: $a' < b$. To show $a' > a$, suppose $a' \leq a$. Then, by the choice of a', axiom **M**, the result that $b > a'$, and the fact that (2) holds under **O**, **M**, **RD**, **SEF** and **ST**:

$${}_{\mathrm{con}}a' \sim (a,\ {}_{\mathrm{con}}b) \succsim (a',\ {}_{\mathrm{con}}b) \succ (a',\ {}_{\mathrm{con}}a') = {}_{\mathrm{con}}a',$$

so that ${}_{\mathrm{con}}a' \succ {}_{\mathrm{con}}a'$ by transitivity. Since this contradicts reflexivity, we have: $a < a'$.

(ii) By axioms **O**, **LC**, **M**, there exists $b' \in [a,b]$ such that ${}_{\mathrm{con}}b' \sim (b,\ {}_{con}a)$. To show $b' > a$, note by the choice of b' and axiom **LD**, ${}_{\mathrm{con}}b' \sim (b,\ {}_{\mathrm{con}}a) \succ (a,\ {}_{\mathrm{con}}a) = {}_{\mathrm{con}}a$, so ${}_{\mathrm{con}}b' \succ {}_{\mathrm{con}}a$ by transitivity. Then, since $b' < a$ would imply ${}_{\mathrm{con}}b' \prec {}_{\mathrm{con}}a$ (by (1)), while $b' = a$ would imply ${}_{\mathrm{con}}b' \sim {}_{\mathrm{con}}a$ (by reflexivity), we have: $b' > a$. To show $b' < b$, suppose $b' \geq b$. Then, by the choice of b', axiom **M**, the result that $a < b'$, and the fact that (2) holds under **O**, **M**, **LD**, **SEF** and **ST**:

$${}_{\mathrm{con}}b' \sim (b,\ {}_{\mathrm{con}}a) \precsim (b',\ {}_{\mathrm{con}}a) \prec (b',\ {}_{\mathrm{con}}b') = {}_{\mathrm{con}}b',$$

so that ${}_{\text{con}}b' \prec {}_{\text{con}}b'$ by transitivity. Since this contradicts reflexivity, we have: $b' < b$. ■

Proof of Proposition 4. Assume axioms **O**, **LC**, **M**, **LD**, **SEF** and **ST**. Use Lemma 4(i) to define the sequence $\{a^t\}$ by:

$$
\begin{aligned}
{}_{\text{con}}a^1 &\sim (a, {}_{\text{con}}1) \\
{}_{\text{con}}a^2 &\sim (a, {}_{\text{con}}a^1) \sim (a, a, {}_{\text{con}}1) \\
&\cdots \\
{}_{\text{con}}a^t &\sim (a, {}_{\text{con}}a^{t-1}) \sim (\underbrace{a, \dots, a}_{t \text{ times}}, {}_{\text{con}}1)
\end{aligned}
$$

For all t, $a^t \in (a, 1)$. Furthermore, it follows from Lemma 4(i) that $\{a^t\}$ is decreasing and bounded below by a. Hence $\{a^t\}$ is a converging sequence: $a^\infty := \lim_{t\to\infty} a^t \in [a, 1)$. By **LC** and **M**, ${}_{\text{con}}a^\infty \sim (a, {}_{\text{con}}a^\infty)$. If $a^\infty > a$, then

$$
a^\infty = (a, {}_{\text{con}}a^\infty) \prec (a^\infty, {}_{\text{con}}a^\infty) = a^\infty
$$

which is a contradiction. Therefore: $a^\infty = a$.

Use likewise Lemma 4(ii) to define the sequence $\{b^t\}$ by $b^t \sim (b, {}_{\text{con}}b^{t-1})$ for all $t \in \mathbb{N}$ and $b^0 = 0$. The same argument implies that $b^\infty = b$. Since Lemma 4(ii) relies on **LD**, the strengthening of **RD** to **LD** is needed for this part of the proof.

Hence, with $a < b$, it follows from **M** and (1) that there exists $\tau \in \mathbb{N}$ such that

$$
\begin{aligned}
(\underbrace{a, \dots, a}_{t \text{ times}}, {}_{t+1}\mathbf{u}) &\precsim (\underbrace{a, \dots, a}_{t \text{ times}}, {}_{\text{con}}1) \sim {}_{\text{con}}a^t \\
&\prec {}_{\text{con}}b^t \sim (\underbrace{b, \dots, b}_{t \text{ times}}, {}_{\text{con}}0) \precsim (\underbrace{b, \dots, b}_{t \text{ times}}, {}_{t+1}\mathbf{v})
\end{aligned}
$$

for all $t \geq \tau$, and ${}_1\mathbf{u}$, ${}_1\mathbf{v} \in \mathbf{X}$. Thus, the proposition is established by transitivity. ■