Supplement to
"Sustainable recursive social welfare functions"

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Proofs of Propositions 9 and 10, not to be included in the paper.

Proof of Proposition 9. *Part I: (1) implies (2).* Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, **WS**, and **IP**.

Say that $\succsim_0^{\mathbf{z}}$ is *sensitive* if there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $x_0 \succ_0^{\mathbf{z}} y_0$, and likewise for ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$. By **WS**, $\succsim_0^{\mathbf{z}}$ is sensitive. By **IF**, $(x_1, {}_2\mathbf{z}) \succ (y_1, {}_2\mathbf{z})$ implies $(z_0, x_1, {}_2\mathbf{z}) \succ (z_0, y_1, {}_2\mathbf{z})$. Since $\succsim_0^{\mathbf{z}}$ is sensitive, there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $x_1 \succ_1^{\mathbf{z}} y_1$, meaning that $\succsim_1^{\mathbf{z}}$ is sensitive. Since Y is not a singleton, it follows from (3) and **IF** that ${}_1\succsim^{\mathbf{z}}$ is sensitive and, by an additional application of **IF**, that ${}_2\succsim^{\mathbf{z}}$ is sensitive. By Lemma 3, $\succsim_0^{\mathbf{z}}$, ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$ are independent of ${}_0\mathbf{z}$.

By **O** and **M**, there exists a continuous function $\tilde{U} : Y \to \mathbb{R}$ satisfying $\tilde{U}(z) \geq \tilde{U}(v)$ if and only if ${}_{\text{con}}z \succsim {}_{\text{con}}v$. In view of Lemma 1, determine $\tilde{W} : \mathbf{X} \to \mathbb{R}$ by, for all ${}_0\mathbf{x} \in \mathbf{X}$, $\tilde{W}({}_0\mathbf{x}) = \tilde{U}(y)$ where ${}_{\text{con}}y \sim {}_0\mathbf{x}$. By **O**, $\tilde{W}({}_0\mathbf{x}) \geq \tilde{W}({}_0\mathbf{y})$ if and only if ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$. By construction of $\tilde{W}$, $\tilde{W}({}_{\text{con}}z) = \tilde{U}(z)$ for all $z \in Y$. By **IF**, for given

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$x_0 \in Y$, there exists an increasing transformation $\tilde{V}(\tilde{U}(x_0), \cdot) : \mathbb{R} \to \mathbb{R}$ such that, for all ${}_1\mathbf{x} \in \mathbf{X}$, $\tilde{W}(x_0, {}_1\mathbf{x}) = \tilde{V}(\tilde{U}(x_0), \tilde{W}({}_1\mathbf{x}))$. This determines $\tilde{V} : \tilde{U}(Y)^2 \to \mathbb{R}$. Since $\neg\left((x, {}_{\text{con}}z) \prec {}_{\text{con}}v\right)$ (resp. $\neg\left((x, {}_{\text{con}}z) \succ {}_{\text{con}}v\right)$) if and only if

$$\tilde{V}(\tilde{U}(x), \tilde{U}(z)) = \tilde{V}(\tilde{U}(x), \tilde{W}({}_{\text{con}}z)) = \tilde{W}(x, {}_{\text{con}}z) \geq \tilde{U}(v) \quad (\text{resp.} \leq \tilde{U}(v)),$$

RC implies that $\tilde{V}$ is continuous in (u, w) on $\tilde{U}(Y)^2$.

Hence, on the set of streams in $\mathbf{X}$ of the form $(x_0, x_1, {}_{\text{con}}v)$, $\succsim$ is represented by $\tilde{W}(x_0, x_1, {}_{\text{con}}v) = \tilde{V}(x_0, \tilde{W}(x_1, {}_{\text{con}}v)) = \tilde{V}(x_0, \tilde{V}(x_1, \tilde{U}(v)))$, which is continuous in (x_0, x_1, v) on Y^3. Since $\succsim_0^{\mathbf{z}}$, $\succsim_1^{\mathbf{z}}$, and ${}_2\succsim^{\mathbf{z}}$ are sensitive (in the case of ${}_2\succsim^{\mathbf{z}}$ also within the set of constant streams, by **O**, **M**, and **WS**), and $\succsim_0^{\mathbf{z}}$, ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$ are all independent of ${}_0\mathbf{z}$, it now follows from standard results for additively separable representations (Debreu, 1960; Gorman, 1968; Koopmans, 1986) that there exist continuous functions $U_0 : Y \to \mathbb{R}$, $U_1 : Y \to \mathbb{R}$, and $U : Y \to \mathbb{R}$, such that $W_0 : \{{}_0\mathbf{x} \in \mathbf{X} \mid x_t = v \text{ for all } t \geq 2\} \to \mathbb{R}$ defined by

$$W_0(x_0, x_1, {}_{\text{con}}v) = U_0(x_0) + U_1(x_1) + U(v) \tag{1}$$

is an SWF. By repeated applications of **IF**, it follows from Lemma 1 that W_0 can be extended to all ${}_0\mathbf{x} \in \mathbf{X}$:

$$W_0({}_0\mathbf{x}) = U_0(x_0) + U_1(x_1) + U(W^*({}_2\mathbf{x}))\,,$$

where $W^* : \mathbf{X} \to Y$ maps any ${}_0\mathbf{y} \in \mathbf{X}$ into *some* $z \in Y$ satisfying ${}_{\text{con}}z \sim {}_0\mathbf{y}$. It follows from **IF** that $W_1 : \mathbf{X} \to \mathbb{R}$ defined by

$$W_1({}_0\mathbf{x}) = U_1(x_0) + U(W^*({}_1\mathbf{x}))\,,$$

is also an SWF. The additively separable structure between time 0 and times $1, 2, \dots$ means that, for all ${}_0\mathbf{x} \in \mathbf{X}$, $W_1({}_0\mathbf{x}) = \delta W_0({}_0\mathbf{x}) + \epsilon$, $U_1(x_0) = \delta U_0(x_0) + \epsilon$, and

$$U(W^*({}_1\mathbf{x})) = \delta\big(U_1(x_1) + U(W^*({}_2\mathbf{x}))\big) + \epsilon\,. \tag{2}$$

Furthermore, by inserting ${}_{\text{con}}z$ in (2) and keeping in mind that $U(W^*({}_{\text{con}}z)) = U(z)$, we obtain $U(z) = \delta\big(U_1(z) + U(z)\big) + \epsilon$, or equivalently,

$$U_1(z) = \tfrac{1-\delta}{\delta}U(z) - \tfrac{\epsilon}{\delta} \tag{3}$$

for all $z \in Y$. By defining $W : \mathbf{X} \to \mathbb{R}$ by, for all ${}_0\mathbf{x} \in \mathbf{X}$, $W({}_0\mathbf{x}) = U(W^*({}_0\mathbf{x}))$, it follows from (2) and (3) that the SWF W satisfies $W({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W({}_1\mathbf{x})$ for all ${}_0\mathbf{x} \in \mathbf{X}$, where $\delta \in (0, 1)$ since both $\succsim_0^{\mathbf{z}}$ and ${}_1\succsim^{\mathbf{z}}$ are sensitive. By **M**, W is monotone and U is non-decreasing. By **WS**, $U(Y)$ is not a singleton; hence, $U \in \mathcal{U}$.

If **WS** is strengthened to **RD**, then it follows from (3), (1), and repeated applications of **IF** that $U(Y)$ is increasing; hence, $U \in \mathcal{U}_I$.

Part II: (2) implies (1). Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ is an SWF and satisfies, for some $U \in \mathcal{U}$ and $\delta \in (0,1)$, $W({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W({}_1\mathbf{x})$ for all ${}_0\mathbf{x} \in \mathbf{X}$. Note that, for each $U \in \mathcal{U}$ and each $\delta \in (0,1)$, $V : U(Y)^2 \to \mathbb{R}$ defined by $V(u,w) = (1-\delta)u + \delta w$ is an element of $\mathcal{V}(U)$; hence,

$$\{V : U(Y)^2 \to \mathbb{R} \mid V(u,w) = (1-\delta)u + \delta w \text{ for some } \delta \in (0,1)\} \;\subseteq\; \mathcal{V}(U)\,.$$

Also, $W({}_{\text{con}}z) = (1-\delta)U(z) + \delta W({}_{\text{con}}z)$ implies $W({}_{\text{con}}z) = U(z)$. Hence, by Proposition 2, if $U \in \mathcal{U}_I$, it remains to be shown that the SWR $\succsim$, represented by the SWF W, satisfies **IP**. The following argument shows that $\succsim$ satisfies **IP**.

Let ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$, and let $(x_0,\, x_1)\ {}_0\!\succsim_1^{\mathbf{z}} (y_0,\, y_1)$, or equivalently, $W(x_0,\, x_1,\, {}_2\mathbf{z}) \geq W(y_0,\, y_1,\, {}_2\mathbf{z})$. We have to show that $(x_0,\, x_1)\ {}_0\!\succsim_1^{\mathbf{v}} (y_0,\, y_1)$, or equivalently, $W(x_0,\, x_1,\, {}_2\mathbf{v}) \geq W(y_0,\, y_1,\, {}_2\mathbf{v})$. By the properties of W,

$$\begin{aligned} W(x_0,\, x_1,\, {}_2\mathbf{z}) - W(y_0,\, y_1,\, {}_2\mathbf{z}) &= (1-\delta)\big[\big(U(x_0) - U(y_0)\big) + \delta\big(U(x_1) - U(y_1)\big)\big] \\ &= W(x_0,\, x_1,\, {}_2\mathbf{v}) - W(y_0,\, y_1,\, {}_2\mathbf{v})\,, \end{aligned}$$

since $W({}_0\mathbf{x}') = (1-\delta)\big(U(x_0') + \delta U(x_1')\big) + \delta^2 W({}_2\mathbf{x}')$ for all ${}_0\mathbf{x}' \in \mathbf{X}$.

If $U \in \mathcal{U} \backslash \mathcal{U}_I$, then above analysis goes through, except that it does not follow that the SWR $\succsim$ satisfies **RD**. Instead, the property that $U(Y)$ is not a singleton implies that SWR $\succsim$ satisfies **WS**. ∎

Proof of Proposition 10. Fix $U \in \mathcal{U}$ and $\delta \in (0,1)$, and let ${}_0\mathbf{x} \in \mathbf{X}$, implying that there exist $\underline{y}$, $\bar{y} \in Y$ such that, for all $t \in \mathbb{Z}_+$, $\underline{y} \leq x_t \leq \bar{y}$.

Part I: Existence. For each $T \in \mathbb{Z}_+$, consider the following finite sequence:

$$\begin{aligned} &w(T,T) = U(\bar{y}) \\ &w(T-1,T) = (1-\delta)U(x_{T-1}) + \delta w(T,T) = (1-\delta)U(x_{T-1}) + \delta U(\bar{y}) \\ &\ldots \\ &w(0,T) = (1-\delta)U(x_0) + \delta w(1,T) = (1-\delta)\sum\nolimits_{t=0}^{T-1} \delta^t U(x_t) + \delta^T U(\bar{y}) \end{aligned}$$

Since $w(t,T)$ is non-increasing in T for given $t \leq T$ and bounded below by $U(\underline{y})$, $\lim_{T\to\infty} w(t,T)$ exists for all $t \in \mathbb{Z}_+$. Define the monotone mapping $W_\delta : \mathbf{X} \to \mathbb{R}$ by

$$W_\delta({}_0\mathbf{x}) := \lim\nolimits_{T\to\infty} w(0,T) = (1-\delta)\sum\nolimits_{t=0}^{\infty} \delta^t U(x_t)\,.$$

As $w(0,T) = (1-\delta)U(x_0) + \delta w(1,T)$, we have that $W_\delta({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W_\delta({}_1\mathbf{x})$.

Part II: Uniqueness. Suppose there exists a monotone mapping $W : \mathbf{X} \to \mathbb{R}$ satisfying $W({}_0\mathbf{y}) = (1-\delta)U(y_0) + \delta W({}_1\mathbf{y}))$ for all ${}_0\mathbf{y} \in \mathbf{X}$ such that $W({}_0\mathbf{x}) \neq W_\delta({}_0\mathbf{x})$. Since $W({}_t\mathbf{x}) - W_\delta({}_t\mathbf{x}) = \delta\big(W({}_{t+1}\mathbf{x}) - W_\delta({}_{t+1}\mathbf{x})\big)$ for all $t \in \mathbb{Z}_+$,

$$|W({}_T\mathbf{x}) - W_\delta({}_T\mathbf{x})| = \tfrac{1}{\delta^T}|W({}_0\mathbf{x}) - W_\delta({}_0\mathbf{x})| > U(\bar{y}) - U(\underline{y})$$

for some $T \in \mathbb{Z}_+$. However this contradicts that, for all $T \in \mathbb{Z}_+$,

$$U(\underline{y}) = W({}_{\text{con}}\underline{y}) \leq W({}_T\mathbf{x}) \leq W({}_{\text{con}}\bar{y}) = U(\bar{y})$$

(and likewise for $W_\delta({}_T\mathbf{x})$) by the facts that W is monotone and $W({}_{\text{con}}z) = (1-\delta)U(z) + \delta W({}_{\text{con}}z)$ implies $W({}_{\text{con}}z) = U(z)$. ■

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