Ordering infinite utility streams

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Abstract

We reconsider the problem of ordering infinite utility streams. As has been established in earlier contributions, if no representability condition is imposed, there exist strongly Paretoian and finitely anonymous orderings of intertemporal utility streams. We examine the possibility of adding suitably formulated versions of classical equity conditions. First, we provide a characterization of all ordering extensions of the generalized Lorenz criterion as the only strongly Paretoian and finitely anonymous rankings satisfying the strict transfer principle. Second, we offer a characterization of an infinite-horizon extension of leximin obtained by adding an equity-preference axiom to strong Pareto and finite anonymity.

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1. Introduction

Treating generations equally is one of the basic principles in the utilitarian tradition of moral philosophy. As Sidgwick [19, p. 414] observes, “the time at which a man exists cannot affect the value of his happiness from a universal point of view; and […] the interests of posterity
must concern a utilitarian as much as those of his contemporaries”. This view, which is formally expressed by the anonymity condition, is also strongly endorsed by Ramsey [16].

Following Koopmans [14], Diamond [9] establishes that anonymity is incompatible with the strong Pareto principle when ordering infinite utility streams. Moreover, he shows that if anonymity is weakened to finite anonymity—which restricts the application of the standard anonymity requirement to situations where utility streams differ in at most a finite number of components—and a continuity requirement is added, an impossibility results again. Hara et al. [12] adapt the well-known strict transfer principle due to Pigou [15] and Dalton [7] to the infinite-horizon context. They show that this principle is incompatible with strong Pareto and continuity even if the social preference is merely required to be acyclical. Basu and Mitra [5] show that strong Pareto, finite anonymity and representability by a real-valued function are incompatible.

Faced with these impossibilities, it seems to us that the most natural assumption to drop is that of continuity or representability. We view the strong Pareto principle and finite anonymity as being on much more solid ground than axioms such as continuity or representability, especially in the context of the ranking of infinite utility streams where these conditions may be considered to be overly demanding. Svensson [21] proves that strong Pareto and finite anonymity are compatible by showing that any ordering extension of an infinite-horizon variant of Suppes’ [20] grading principle satisfies the required axioms. The Suppes grading principle is a quasi-ordering that combines the Pareto quasi-ordering and finite anonymity. Given Arrow’s [1] version of Szpilrajn’s [22] extension theorem, this establishes the compatibility result. As noted by Asheim et al. [2], Svensson’s possibility result is easily converted into a characterization: ordering extensions of the Suppes grading principles are the only orderings satisfying strong Pareto and finite anonymity.

Once the possibility of satisfying these two fundamental axioms is established, another natural question to ask is what orderings satisfy additional desirable properties. Asheim and Tungodden [3] provide a characterization of an infinite-horizon version of the leximin principle by adding an equity-preference condition (the infinite-horizon version of Hammond equity; see [10]) and a preference-continuity property to strong Pareto and finite anonymity. An infinite-horizon version of utilitarianism is characterized by Basu and Mitra [6] by adding an information-invariance condition to the two fundamental axioms. Furthermore, they narrow down the class of infinite-horizon utilitarian orderings to those resulting from the overtaking criterion [23]. This is accomplished by using a consistency condition in addition to the three axioms characterizing their utilitarian orderings.

In this paper, we focus on equity properties. One of the most fundamental equity properties (if not the most fundamental) is the Pigou–Dalton transfer principle, adapted to the infinite-horizon framework by Hara et al. [12]. Our first result characterizes all orderings that satisfy strong Pareto, anonymity and the strict transfer principle. They are extensions of an infinite-horizon formulation of the well-known generalized Lorenz quasi-ordering [18].

In the presence of strong Pareto, the axiom of equity preference (the infinite-horizon version of Hammond equity) is a strengthening of the strict transfer principle. We use it to identify a subclass of the class of orderings satisfying the three axioms just mentioned. These orderings are extensions of a particular infinite-horizon incomplete version of leximin. This second result leaves a larger class of orderings than that identified by Asheim and Tungodden [3] because they employ an additional axiom. The relationship between our leximin characterization and that of Asheim and Tungodden is analogous to the relationship between Basu and Mitra’s [6] characterizations of infinite-horizon utilitarianism and of the overtaking criterion.
2. Basic definitions

The set of infinite utility streams is $X = \mathbb{R}^\mathbb{N}$, where $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{N}$ denotes the set of all natural numbers. A typical element of $X$ is an infinite-dimensional vector $x = (x_1, x_2, \ldots, x_n, \ldots)$ and, for $n \in \mathbb{N}$, we write $x^{-n} = (x_1, \ldots, x_n)$ and $x^{+n} = (x_{n+1}, x_{n+2}, \ldots)$. The standard interpretation of $x \in X$ is that of a countably infinite utility stream where $x_n$ is the utility experienced in period $n \in \mathbb{N}$. Of course, other interpretations are possible—for example, $x_n$ could be the utility of an individual in a countably infinite population.

Our notation for vector inequalities on $X$ is as follows. For all $x, y \in X$, (i) $x \geq y$ if $x_n \geq y_n$ for all $n \in \mathbb{N}$; (ii) $x > y$ if $x \geq y$ and $x \neq y$; (iii) $x \succ y$ if $x_n > y_n$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $x \in X$, $(x^{-1}_n, \ldots, x^{-n}_n)$ is a rank-ordered permutation of $x^{-n}$ such that $x^{-1}_n \leq \cdots \leq x^{-n}_n$, ties being broken arbitrarily.

A finite permutation of $\mathbb{N}$ is a bijection $\rho : \mathbb{N} \rightarrow \mathbb{N}$ such that there exists $m \in \mathbb{N}$ with $\rho(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. $x^\rho = (x_{\rho(1)}, x_{\rho(2)}, \ldots, x_{\rho(n)}, \ldots)$ is the finite permutation of $x \in X$ that results from relabelling the components of $x$ in accordance with the finite permutation $\rho$.

Two of the most fundamental axioms in this area are the strong Pareto principle and finite anonymity, defined as follows.

**Strong Pareto:** For all $x, y \in X$, if $x > y$, then $(x, y) \in P(R)$.

**Finite anonymity:** For all $x \in X$ and for all finite permutations $\rho$ of $\mathbb{N}$,

$$(x^\rho, x) \in I(R).$$

Szpiroján’s [22] fundamental result establishes that every partial order has a linear order extension. Arrow [1, p. 64] presents a variant of Szpiroján’s theorem stating that every quasi-ordering has an ordering extension; see also [11]. This implies that the sets of orderings characterized in the theorems of the following sections are non-empty.

3. Transfer-sensitive infinite-horizon orderings

Now we examine the consequences of adding the strict transfer principle to strong Pareto and finite anonymity. A Pigou–Dalton transfer is a transfer of a positive amount of utility from a better-off agent to a worse-off agent so that the relative ranking of the two agents in the post-transfer utility stream is the same as their relative ranking in the pre-transfer stream. The strict transfer principle requires that any Pigou–Dalton transfer leads to a utility stream that is strictly preferred to the pre-transfer stream.

**Strict transfer principle:** For all $x, y \in X$ and for all $m, n \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$, then $(x, y) \in P(R)$.

The strict transfer principle is the natural analogue of the corresponding condition for finite streams; see also [12]. An alternative (equivalent) formulation of the strict transfer principle involves the explicit expression of the amount transferred from $m$ to $n$ when moving from $y$ to
x (this amount is $\delta = y_m - x_m = x_n - y_n$ and is readily obtained from our statement of the axiom). Although this alternative may be more standard in the literature, we use the version introduced above because it is parallel in structure to the equity-preference axioms to be defined in the following section.

To define the class of orderings satisfying the three axioms introduced thus far, we begin with a statement of Shorrocks’ [18] generalized Lorenz quasi-ordering $R^n_g$ for a society consisting of $n \in \mathbb{N}$ individuals. This quasi-ordering generalizes the standard Lorenz quasi-ordering by extending the relevant dominance criterion to comparisons involving different average (or total) utilities. For all $x, y \in X$,

$$(x^n, y^n) \in R^n_g \iff \sum_{i=1}^{k} x_{(i)}^{n} \geq \sum_{i=1}^{k} y_{(i)}^{n} \text{ for all } k \in \{1, \ldots, n\}.$$

The relation $R^n_G \subseteq X \times X$ is defined by letting, for all $x, y \in X$,

$$(x, y) \in R^n_G \iff (x^n, y^n) \in R^n_g \text{ and } x^{+n} \geq y^{+n}.$$ 

Clearly, $R^n_G$ is a quasi-ordering for all $n \in \mathbb{N}$. The infinite-horizon extension of the generalized Lorenz quasi-ordering that is of interest in this paper is defined by $R^G = \bigcup_{n \in \mathbb{N}} R^n_G$. The relation $R^G$ can be shown to be a quasi-ordering and we characterize the class of its ordering extensions in the following theorem.

**Theorem 1.** An ordering $R$ on $X$ satisfies strong Pareto, finite anonymity and the strict transfer principle if and only if $R$ is an ordering extension of $R^G$.

**Proof.** ‘If.’ Step 1: We show that the relations $R^n_G$ and their associated strict preference relations $P(R^n_g)$ are nested, that is, for all $n \in \mathbb{N}$,

$$R^n_G \subseteq R^{n+1}_G \quad \text{(1)}$$

and

$$P(R^n_G) \subseteq P(R^{n+1}_G). \quad \text{(2)}$$

To prove (1), suppose that $(x, y) \in R^n_G$. By definition, $(x^n, y^n) \in R^n_g$ and $x^{+n} \geq y^{+n}$ and, thus,

$$\sum_{i=1}^{k} x_{(i)}^{n} \geq \sum_{i=1}^{k} y_{(i)}^{n} \quad \text{for all } k \in \{1, \ldots, n\}, \quad \text{(3)}$$

$$x_{n+1} \geq y_{n+1} \quad \text{(4)}$$

and

$$x^{+(n+1)} \geq y^{+(n+1)}. \quad \text{(5)}$$

Because of (5), it is sufficient to prove that

$$\sum_{i=1}^{k} x_{(i)}^{-(n+1)} \geq \sum_{i=1}^{k} y_{(i)}^{-(n+1)} \quad \text{for all } k \in \{1, \ldots, n + 1\}. \quad \text{(6)}$$
If \( k = n + 1 \), we have
\[
\sum_{i=1}^{n+1} x_{(i)}^{-(n+1)} = \sum_{i=1}^{n} x_{(i)}^{-n} + x_{n+1}
\]
and
\[
\sum_{i=1}^{n+1} y_{(i)}^{-(n+1)} = \sum_{i=1}^{n} y_{(i)}^{-n} + y_{n+1}.
\]
Adding (3) for \( k = n \) and (4), we obtain (6) for \( k = n + 1 \).

Now let \( k \in \{1, \ldots, n\} \). We distinguish the following four cases which cover all possibilities.

**Case 1:** \( x_{n+1} \geq x_{(k)}^{-n} \) and \( y_{n+1} \geq y_{(k)}^{-n} \). This implies \( x_{(i)}^{-(n+1)} = x_{(i)}^{-n} \) and \( y_{(i)}^{-(n+1)} = y_{(i)}^{-n} \) for all \( i \in \{1, \ldots, k\} \), and (6) for this \( k \) follows immediately from (3).

**Case 2:** \( x_{n+1} \leq x_{(k)}^{-n} \) and \( y_{n+1} \leq y_{(k)}^{-n} \). This implies
\[
\sum_{i=1}^{k} x_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} x_{(i)}^{-n} + x_{n+1}
\]
and
\[
\sum_{i=1}^{k} y_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{n+1}.
\]
Adding (3) and (4), we obtain (6) for this \( k \).

**Case 3:** \( x_{n+1} < x_{(k)}^{-n} \) and \( y_{n+1} > y_{(k)}^{-n} \). This implies
\[
\sum_{i=1}^{k} x_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} x_{(i)}^{-n} + x_{n+1}
\]
and
\[
\sum_{i=1}^{k} y_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{n+1}.
\]
Combining (4) and the inequality \( y_{n+1} > y_{(k)}^{-n} \) (which is valid by definition of the present case), it follows that \( x_{n+1} \geq y_{(k)}^{-n} \). Adding this inequality and (3), we obtain (6) for this \( k \).

**Case 4:** \( x_{n+1} > x_{(k)}^{-n} \) and \( y_{n+1} < y_{(k)}^{-n} \). This implies
\[
\sum_{i=1}^{k} x_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} x_{(i)}^{-n} + x_{(k)}^{-n}
\]
and
\[
\sum_{i=1}^{k} y_{(i)}^{-(n+1)} = \sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{n+1}.
\]
The inequality $y_{n+1} < y_k$ (which is satisfied by definition of the present case) implies

$$\sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{(k)}^{-n} \geq \sum_{i=1}^{k-1} y_{(i)}^{-n} + y_{n+1}.$$ 

Combining this inequality with (3) yields (6) for this $k$.

To establish (2), suppose that $(x, y) \in P(R_G^n)$. By definition, at least one of the following two statements is true:

$$(x^n, y^n) \in P(R_G^n) \quad \text{and} \quad x^n \geq y^n,$$  

$$(x^n, y^n) \in R_G^n \quad \text{and} \quad x^n > y^n.$$  

If (7) is true, it follows that the inequalities in (3) are satisfied and at least one of them is strict. Now $(x, y) \in P(R_G^{n+1})$ follows from noting that, in all cases distinguished in the proof of (1), the presence of a strict inequality in (3) yields (6) with at least one strict inequality.

If (8) is true, it follows as in the proof of (1) that the inequalities in (6) are satisfied. If $x_{n+1} > y_{n+1}$, it follows immediately that one of these inequalities must be strict and, together with $x^{(n+1)} \geq y^{(n+1)}$, we obtain $(x, y) \in P(R_G^{n+1})$. If $x_{n+1} = y_{n+1}$, we must have $x^{(n+1)} > y^{(n+1)}$ which, together with (6), establishes that $(x, y) \in P(R_G^{n+1})$.

Step 2: We now show that, for all $x, y \in X$,

$$(x, y) \in P(R_G) \iff \exists n \in \mathbb{N} \text{ such that } (x, y) \in P(R_G^n).$$

Suppose first that $(x, y) \in P(R_G)$. By definition, there exists $n \in \mathbb{N}$ such that $(x, y) \in R_G^n$. Moreover, $(y, x) \notin R_G^n$ because otherwise we obtain $(y, x) \in R_G$ by definition and thus a contradiction to our hypothesis that $(x, y) \in P(R_G)$. Hence $(x, y) \in P(R_G^n)$.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $(x, y) \in P(R_G^n)$. Suppose there exists $m \in \mathbb{N}$ such that $(x, y) \in R_G^m$. Clearly, $m \neq n$; otherwise we immediately obtain a contradiction. If $m > n$, $(x, y) \in P(R_G^m)$ and (repeated if necessary) application of (2) together imply $(x, y) \in P(R_G^n)$, contradicting the assumption $(y, x) \notin R_G^n$. If $m < n$, $(x, y) \notin R_G^n$ and (repeated if necessary) application of (1) together imply $(y, x) \in R_G^m$, contradicting the hypothesis $(x, y) \in P(R_G^n)$. We conclude that $(x, y) \in R_G^n$ and $(x, y) \notin R_G^m$ for all $m \in \mathbb{N}$. By definition, this implies $(x, y) \in P(R_G)$.

Step 3: Next, we prove that $R_G$ is a quasi-ordering. Reflexivity is immediate because, for all $x \in X$, $(x, x) \in R_G^n$ for all $n \in \mathbb{N}$ and hence $(x, x) \in R_G$. To prove that $R_G$ is transitive, suppose that $(x, y), (y, z) \in R_G$. By definition, there exist $m, n \in \mathbb{N}$ such that $(x, y) \in R_G^m$ and $(y, z) \in R_G^n$. Let $k = \max\{m, n\}$. By (repeated if necessary) application of (1), $(x, y), (y, z) \in R_G^k$ and by the transitivity of $R_G^k$, $(x, z) \in R_G^k$ which, in turn, implies $(x, z) \in R_G$.

Step 4: Now let $R$ be an ordering extension of $R_G$. We complete the proof of the ‘if’ part by showing that $R$ satisfies the required axioms.

To establish that strong Pareto is satisfied, suppose that $x, y \in X$ are such that $x > y$. Let $n = \min\{m \in \mathbb{N} \mid x_m > y_m\}$. By definition, $(x, y) \in P(R_G^n)$. By (9), $(x, y) \in P(R_G)$, and because $R$ is an ordering extension of $R_G$, we obtain $(x, y) \in P(R)$.

Next, we show that finite anonymity is satisfied. Let $x \in X$ and let $\rho$ be a finite permutation of $\mathbb{N}$. By definition, there exists $m \in \mathbb{N}$ such that $\rho(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. By definition of $R_G^m$, $(x^\rho, x) \in I(R_G^m)$. By definition of $R_G$, this implies $(x^\rho, x) \in I(R_G)$. Because $R$ is an ordering extension of $R_G$, we obtain $(x^\rho, x) \in I(R)$.
Finally, we show that the strict transfer principle is satisfied. Consider \( x, y \in X \) and \( m, n \in \mathbb{N} \) such that \( x_k = y_k \) for all \( k \in \mathbb{N} \setminus \{m, n\} \), \( y_m > x_m \geq x_n > y_n \) and \( x_n + x_m = y_n + y_m \). Let \( j = \max\{m, n\} \). By definition of \( R^f_G \), we obtain \( (x, y) \in R^f_G \). By (9), \( (x, y) \in R_G \) and, because \( R \) is an ordering extension of \( R_G \), \( (x, y) \in R \).

‘Only if.’ Suppose \( R \) is an ordering on \( X \) satisfying the three axioms of the theorem statement. To prove that \( R \) is an ordering extension of \( R_G \), we have to establish the set inclusions \( R_G \subseteq R \) and \( P(R_G) \subseteq P(R) \).

Suppose \( x, y \in X \) are such that \( (x, y) \in R_G \). By definition, there exists \( n \in \mathbb{N} \) such that

\[
\sum_{i=1}^{k} x_{(i)}^{-n} \geq \sum_{i=1}^{k} y_{(i)}^{-n} \quad \text{for all } k \in \{1, \ldots, n\}
\]

and \( x^{+n} \succeq y^{+n} \). By anonymity, we can without loss of generality assume that \( x_{(i)}^{-n} = x_i \) and \( y_{(i)}^{-n} = y_i \) for all \( i \in \{1, \ldots, n\} \). Employing an argument analogous to that used by Shorrocks [18, Theorem 2], we let \( w \in X \) be such that \( w_j = y_j \) for all \( j \in \{1, \ldots, n-1\} \), \( w_n = y_n + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i \) and \( w^{+n} = x^{+n} \). We have \( w \succeq y \) and thus \( (w, y) \in R \) by reflexivity (if \( w = y \)) or by strong Pareto (if \( w > y \)). Furthermore, \( (x^{-n}, w^{-n}) \in R^w_G \) and \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \). If \( x^{-n} = w^{-n} \), \( (x, w) \in R \) follows from reflexivity (note that \( x^{+n} = w^{+n} \) by definition). If \( x^{-n} \neq w^{-n} \), it follows that \( x^{+n} \) can be reached from \( w^{-n} \) through a finite sequence of Pigou–Dalton transfers (see [13]). Thus, by (repeated if necessary) application of the strict transfer principle (and transitivity if necessary), we obtain \( (x, w) \in R \) (note again that \( x^{+n} = w^{+n} \)). Transitivity now implies \( (x, y) \in R \).

Now let \( x, y \in X \) be such that \( (x, y) \in P(R_G) \). Because \( P(R_G) \subseteq R_G \) by definition and \( R_G \subseteq R \) as just established, it follows that \( (x, y) \in R \). If \( (y, x) \in R \), there exists \( m \in \mathbb{N} \) such that \( (y, x) \in R^m_G \). By (9), there exists \( n \in \mathbb{N} \) such that \( (x, y) \in P(R^m_G) \). We now obtain a contradiction using the same argument as in the proof of (9) and, thus, the hypothesis \( (x, y) \in R \) must be false. Together with \( (x, y) \in R \), it follows that \( (x, y) \in P(R) \). \( \square \)

4. Infinite-horizon leximin

An equity property that has received a considerable amount of attention in finite settings is Hammond equity and some of its variations. The infinite-horizon version we use is defined as follows.

**Equity preference:** For all \( x, y \in X \) and for all \( m, n \in \mathbb{N} \), if \( x_k = y_k \) for all \( k \in \mathbb{N} \setminus \{m, n\} \) and \( y_m > x_m \geq x_n > y_n \), then \( (x, y) \in R \).

Equity preference is the extension of Hammond’s [10] equity axiom to the infinite-horizon environment. The axiom is used in Asheim and Tungodden [3]; see also Asheim et al. [4] for an alternative version which they call Hammond equity for the future. A condition which is stronger than Hammond’s equity axiom is used by d’Aspremont and Gevers [8] who require \( (x, y) \in P(R) \) rather than merely \( (x, y) \in R \) in the conclusion of the axiom. In the presence of strong Pareto, the two axioms are equivalent. Moreover, strong Pareto and equity preference together imply the following property which, in turn, obviously implies the strict transfer principle.

**Strict equity preference:** For all \( x, y \in X \) and for all \( m, n \in \mathbb{N} \), if \( x_k = y_k \) for all \( k \in \mathbb{N} \setminus \{m, n\} \) and \( y_m > x_m \geq x_n > y_n \), then \( (x, y) \in P(R) \).

To see that strict equity preference is implied by strong Pareto and equity preference, suppose that \( R \) satisfies the first two axioms, and let \( x, y \in X \) and \( m, n \in \mathbb{N} \) be such that \( x_k = y_k \) for all
Again, let $\mathbf{1} \in \mathbb{N} \setminus \{m, n\}$ and $y_m > x_m \geq x_n > y_n$. Let $z \in X$ be such that $z_k = x_k = y_k$ for all $k \in \mathbb{N} \setminus \{m, n\}$ and $x_n > z_m > z_n > y_n$. By strong Pareto, $(x, z) \in P(R)$ and by equity preference, $(z, y) \in R$. Thus, transitivity implies $(x, y) \in P(R)$ and strict equity preference is satisfied.

If the strict transfer principle is replaced by equity preference (which, in the presence of strong Pareto, is a strengthening), the only remaining orderings are infinite-horizon versions of the leximin criterion. For each $n \in \mathbb{N}$, we denote the usual leximin ordering on $\mathbb{R}^n$ by $R^n_L$, that is, for all $x, y \in X$,

$$(x^n, y^n) \in R^n_L \iff x^n = y^n \text{ is a permutation of } y^n \text{ or there exists } m \in \{1, \ldots, n\} \text{ such that } x_{(k)}^n = y_{(k)}^n \text{ for all } k \in \{1, \ldots, n\} \setminus \{m, \ldots, n\} \text{ and } x_{(m)}^n > y_{(m)}^n.$$  

Again, let $n \in \mathbb{N}$ and define a relation $R^n_L \subseteq X \times X$ by letting, for all $x, y \in X$,

$$(x, y) \in R^n_L \iff (x^n, y^n) \in R^n_L \text{ and } x^{+n} \geq y^{+n}.$$  

This relation can be shown to be a quasi-ordering for all $n \in \mathbb{N}$. Finally, let $R^n_L = \bigcup_{n \in \mathbb{N}} R^n_L$. This relation is a quasi-ordering but it is not complete—some infinite utility streams are not ranked by $R^n_L$. Our next result characterizes all ordering extensions of $R^n_L$.

**Theorem 2.** An ordering $R$ on $X$ satisfies strong Pareto, finite anonymity and equity preference if and only if $R$ is an ordering extension of $R^n_L$.

**Proof.** ‘If.’ As in the proof of Theorem 1, we begin by showing that the relations $R^n_L$ and their associated strict preference relations $P(R^n_L)$ are nested, that is, for all $n \in \mathbb{N}$,

$$R^n_L \subseteq R^{n+1}_L,$$  

and

$$P(R^n_L) \subseteq P(R^{n+1}_L).$$  

To prove (10), suppose that $(x, y) \in R^n_L$. By definition, $(x^n, y^n) \in R^n_L$ and $x^{+n} \geq y^{+n}$. Then, either $x^n$ is a permutation of $y^n$ and $x^{+n} \geq y^{+n}$, or there exists $j \in \{1, \ldots, n\}$ such that $x_{(k)}^n = y_{(k)}^n$ for all $k \in \{1, \ldots, n\} \setminus \{j, \ldots, n\}$, $x_{(j)}^n > y_{(j)}^n$ and $x^{+n} \geq y^{+n}$. In both cases, $(x^{-(j+1)}, y^{-(j+1)}) \in R^{n+1}_L$ and $x^{+(n+1)} \geq y^{+(n+1)}$, that is, $(x, y) \in R^{n+1}_L$.

To establish (11), suppose that $(x, y) \in P(R^n_L)$. By definition, at least one of the following two statements is true:

$$(x^n, y^n) \in P(R^n_L) \text{ and } x^{+n} \geq y^{+n},$$  

$$(x^n, y^n) \in R^n_L \text{ and } x^{+n} > y^{+n}.$$  

By (10), it follows that $(x, y) \in R^{n+1}_L$. To prove that $(x, y) \in P(R^{n+1}_L)$, suppose, by way of contradiction, that $(y, x) \in R^{n+1}_L$. Then, by definition,

$$(x^n, y^n) \in I(R^n_L) \text{ and } x^{+n} = y^{+n},$$  

contradicting (12) and (13).

Using the same arguments as in the proof of (9) in Theorem 1 (replacing $R_G$ and $R^n_G$ with $R_L$ and $R^n_L$), it follows that, for all $x, y \in X$,

$$(x, y) \in P(R_L) \iff \exists n \in \mathbb{N} \text{ such that } (x, y) \in P(R^n_L)$$  

(14)
and, furthermore, that \( R_L \) is a quasi-ordering and that any ordering extension of \( R_L \) satisfies strong Pareto and finite anonymity.

We complete the proof of the ‘if’ part by showing that any ordering extension \( R \) of \( R_L \) satisfies equity preference. Consider \( x, y \in X \) and \( m, n \in \mathbb{N} \) such that \( x_k = y_k \) for all \( k \in \mathbb{N} \setminus \{m, n\} \) and \( y_m > x_m > x_n > y_n \). Let \( j = \max\{m, n\} \). By definition of \( R_j \), we obtain \((x, y) \in R_j^j \). By (14), \((x, y) \in R_\mathbb{N} \) and, because \( R \) is an ordering extension of \( R_\mathbb{N} \), \((x, y) \in R \).

‘Only if.’ Suppose \( R \) is an ordering on \( X \) satisfying the three axioms of the theorem statement. Fix \( n \in \mathbb{N} \) and \( z \in X \) and define the relation \( Q^n(z) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) as follows. For all \( x, y \in X \),

\[(x^{-n}, y^{-n}) \in Q^n(z) \iff ((x^{-n}, z^{+n}), (y^{-n}, z^{+n})) \in R.\]

\( Q^n(z) \) is an ordering because \( R \) is. Furthermore, it is clear that

\[(x^{-n}, y^{-n}) \in P(Q^n(z)) \iff ((x^{-n}, z^{+n}), (y^{-n}, z^{+n})) \in P(R) \quad (15)\]

for all \( x, y \in X \). The three axioms imply that \( Q^n(z) \) must satisfy the \( n \)-person versions of the axioms and, using Hammond’s [10, Theorem 7.2] characterization of \( n \)-person leximin (see also [8, Theorem 5]), it follows that

\[Q^n(z) = R^n_L.\]  

(16)

Because \( n \) and \( z \) were chosen arbitrarily, (16) is true for all \( n \in \mathbb{N} \) and for any \( z \in X \).

To prove that \( R \) is an ordering extension of \( R_L \), we first establish the set inclusion \( R_L \subseteq R \). Suppose that \( x, y \in X \) are such that \((x, y) \in R_L \). By definition of \( R_L \), there exists \( n \in \mathbb{N} \) such that \((x, y) \in R^n_L \) that is,

\[(x^{-n}, y^{-n}) \in R^n_L \quad \text{and} \quad x^{+n} \geq y^{+n}.\]

Hence, by (16),

\[(x^{-n}, y^{-n}) \in Q^n(z) \text{ and } x^{+n} \geq y^{+n}\]

for all \( z \in X \). Choosing \( z = y \) and using the definition of \( Q^n(z) \), it follows that \((x^{-n}, y^{+n}), (y^{-n}, y^{+n}) \in R \). Because \( x^{+n} \geq y^{+n} \), reflexivity (if \( x^{+n} = y^{+n} \)) or the conjunction of strong Pareto and transitivity (if \( x^{+n} > y^{+n} \)) implies \((x^{-n}, x^{+n}), (y^{-n}, y^{+n}) \) or \((x, y) \in R \).

We complete the proof by establishing the set inclusion \( P(R_L) \subseteq P(R) \). Let \( x, y \in X \) be such that \((x, y) \in P(R_L) \). By (14), there exists \( n \in \mathbb{N} \) such that \((x, y) \in P(R^n_L) \). Thus, (12) or (13) is true.

If (12) holds, (16) implies

\[(x^{-n}, y^{-n}) \in P(Q^n(z)) \text{ and } x^{+n} \geq y^{+n}\]

for all \( z \in X \). Setting \( z = y \) and using (15), we obtain \((x^{-n}, y^{+n}), (y^{-n}, y^{+n}) \) or \((x^{-n}, x^{+n}), (y^{-n}, y^{+n}) \) or \((x, y) \in P(R) \) and, using reflexivity or strong Pareto and transitivity as in the proof of the set inclusion \( R_L \subseteq R \), we obtain \((x, y) \in P(R) \).

If (13) holds, (16) implies

\[(x^{-n}, y^{-n}) \in Q^n(z) \text{ and } x^{+n} > y^{+n}\]

for all \( z \in X \). Setting \( z = y \), it follows that \((x^{-n}, y^{+n}), (y^{-n}, y^{+n}) \in R \) as a consequence of the definition of \( Q^n(z) \) and, by strong Pareto and transitivity, \((x^{-n}, x^{+n}), (y^{-n}, y^{+n}) \) or \((x, y) \in P(R) \). □
5. Concluding remarks

The results of this paper reinforce the findings of earlier contributions regarding the existence of orderings of infinite utility streams with attractive properties. In particular, we provide characterizations of two classes of such orderings. Given the existential nature of the proofs, we do not provide explicit constructions of these orderings. However, this feature is by no means unique to our approach. Extending quasi-orderings to orderings often requires non-constructive techniques; see, for example, Richter’s [17] use of Szpilrajn’s [22] extension theorem in the context of rational choice.

A plausible conclusion to be drawn is that impossibility results such as those of Diamond [9], Basu and Mitra [5] and Hara et al. [12] can be avoided if continuity or representability assumptions are dispensed with. Because continuity and representability can be considered rather demanding in infinite-horizon settings, this confirms, in our view, that the state of affairs in this area is not as disappointing and negative as has been suggested by the impossibility results of many earlier contributions.

The techniques employed to characterize infinite-horizon versions of the generalized-Lorenz criterion and of lexicomin appear to be very powerful and applicable to the extension of other finite-population social-choice rules; see also the characterization of infinite-horizon utilitarianism by Basu and Mitra [6]. We hope that our approach will stimulate further research in the area of intergenerational social choice by identifying alternative sets of attractive axioms and characterizing the social orderings that satisfy them.

The classes of orderings characterized in this paper are relatively large: there are many comparisons of utility streams that are not determined by the axioms employed. An issue to be addressed in future work is to examine to what extent the ranking of more pairs of streams can be determined by employing plausible additional axioms.

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