Econometrica, Vol. 33, No. 1 (January, 1965)

THE EVALUATION OF INFINITE UTILITY STREAMS[1]

BY PETER A. DIAMOND

The impatience implications of several axiom sets assumed for preferences over an infinite future are explored. Sufficient conditions for the existence of a continuous utility function are presented. This analysis is done both for the product topology and the metric function equating the distance between two infinite streams to the maximal one-period difference between them.

1. INTRODUCTION

CONCERN WITH optimal economic growth (and other problems which are similar in having no natural termination date) has led to an examination of the problem of evaluating a stream, of consumption for example, which extends over an infinite future. One approach to this problem is the selection of a specific functional for the evaluation. Thus, Ramsey[2] chose to maximize the integral of undiscounted utility as the criterion for optimal growth. This approach, while permitting explicit choice of timing preference, lacks generality in the nature of the evaluation and fails to define a sensitive ordering in parts of the program space.[3]

This leads naturally to the approach of assuming the existence of a preference ordering over the alternative streams, and examining the implications of various axiom sets imposed on these preferences. This, then, was the approach of Koopmans[4] and is the one followed in this paper. The implications derived are of two types. For some axiom sets a preference is shown for present utility over that to be enjoyed in the distant future. For other sets the impossibility of treating all time periods the same is shown. Before presenting these results, sufficient conditions on preferences are derived for the existence of a utility function.

2. NOTATION

Throughout the paper, time will be considered in discrete units. A utility stream, U, for the entire future can then be denoted as $U=(u_1, u_2, \ldots, u_t, \ldots)$ where u_t is the one period utility level associated with consumption in period t. This can be interpreted in two ways. The simpler interpretation is that there is a single consumption good and u_t is the quantity of that good consumed in period t.

[1] The original version of this paper was part of a Ph. D. thesis written at the Massachusetts Institute of Technology under R. M. Solow, F. M. Fisher, and P. A. Samuelson. M. E. Yaari discussed an earlier version at the Winter, 1963 meeting of the Econometric Society. Financial support was provided by the Ford Foundation. I wish to thank all of the above, while retaining responsibility for the remaining errors.

[2] Ramsey, F. P., "A Mathematical Theory of Saving," *Economic Journal*, 1928.

[3] For a detailed discussion of the problems of this approach, see Chakravarty, S., "The Existence of an Optimum Savings Program," *Econometrica*, 1962.

[4] Koopmans, T. C., "Stationary Ordinal Utility and Impatience," *Econometrica*, 1960.

Alternatively, preferences may be such that it is possible to derive a one period utility function defined over one period consumption bundles. By making one period utilities rather than one period consumption bundles, the argument of the preferences, certain types of intertemporal complementarity are ruled out. Thus, while the fact is ignored that the type of good one consumes today affects the enjoyment from consuming the same good tomorrow, the possibility of the level of today's consumption affecting the addition to total utility of a given level of consumption tomorrow is included within this context.

The one period utility levels, u_t, will be assumed to lie in the closed unit interval. This implies that the one period utility function has a maximum and can assume it.

The set X, then, of all utility streams will be the infinite Cartesian product of the unit interval. In order to be able to discuss the continuity of a utility function on the set of utility streams it is necessary to define convergence of sequences of points and thus a topology on X. This will be done by defining a distance function or metric over X which gives the distance between pairs of points in X. Two such metrics, d, will be considered in this paper: the sup metric:

$$d(U, U') = \sup_t |u_t - u'_t| ,$$

and the product metric:[5]

$$d(U, U') = \sum_{t=1}^{\infty} 2^{-t}|u_t - u'_t| .$$

Preferences over utility streams will be denoted by $\succ$ and $\sim$ for preferred and indifferent to. The vector inequality $U \geq U'$ will mean $U \geqslant U'$ and $U \neq U'$. The vector with a constant one period utility level, u, will be denoted by $(u_{\text{con}}) = (u, u, \ldots)$. An infinite vector $({}_1u_t, U')$ will mean $(u_1, u_2, \ldots, u_t, u'_1, u'_2, \ldots, u'_t, \ldots)$. By a utility function is meant a real, continuous, order preserving function from X to the real line.

A natural topology for X is a topology in which the sets $\{U \text{ in } X | U \succsim U'\}$ and $\{U \text{ in } X | U' \succsim U\}$ are closed for all U' in X. The assumption that the preferences are such that the topology under discussion is a natural topology will be made repeatedly below. This assumption is equivalent to the assumption that a convergent sequence of points, all of which are preferred or indifferent (all of which are inferior or indifferent) to a given point have a limit point which is also preferred or indifferent (inferior or indifferent) to the given point. The acceptability of this assumption rests critically on the choice of a metric, for this determines which points are or are not limit points of a given sequence. For a finite horizon the choice of Euclidean space with its convergence implications seems reasonable. For an infinite horizon, however, there is no metric which seems so natural. The assumption of naturalness

[5] This is a metric for the product topology on X. The product topology has the property that the reversal of the numbering of two time periods does not alter the topology.

for a given topology means that streams that are not very different from each other (that are close in X) have utility levels that are close. It is then necessary to decide which streams are not very different. Consider the sequence which has a one in the kth place and zeros everywhere else. In the product topology this sequence converges to (0_{con}), while with the sup metric the distance between every pair of streams in this sequence is one. Since any sequence which converges with the sup metric also converges in the product topology, whenever the latter is a natural topology, so, too, is the former.

3. THE EXISTENCE OF A UTILITY FUNCTION

In order that a preference ordering over utility streams be interesting, it must exhibit some degree of sensitivity to changes in the one period utility levels. Two different axioms expressing this sensitivity will be presented in this section. The first sensitivity axiom is that

(S1) $U \geqq U'$ implies $U \succsim U'$,
$U > U'$ implies $U \succ U'$.

This axiom states that a utility stream which is greater than or equal to a second stream in every time period is preferred or indifferent to the second stream, while a utility stream greater than a second stream in every time period is strictly preferred to it. For utility streams, the assumption that more is better seems reasonable.

The existence theorem will make use of a lemma of Debreu[6] which will be stated first.

LEMMA: *Let X be a completely ordered set, $Z=(Z_0, Z_1, \ldots)$ a countable subset of X. If for every pair U, U' of elements of X such that $U \succ U'$, there is an element Z_i of Z such that $U \succsim Z_i \succsim U'$, then there exists on X a real, order-preserving function, continuous in any natural topology.*

The existence theorem will also use a lemma which states that every utility stream U is indifferent to some constant stream (u_{con}) if the preferences satisfy the sensitivity axiom (S1) and are such that the sup metric generates a natural topology.

LEMMA: *Let X be completely ordered by a preference ordering satisfying (S1) and for which the sup metric generates a natural topology. Then for any U in X there exists a one period utility level, u, such that $U \sim (u_{\text{con}})$.*

Proof: Let D be the subspace of constant utility streams, (u_{con}). D is connected. Assume there exists a U^* in X such that there does not exist a (u_{con}) indifferent to it.

[6] Debreu, G., "Representation of a Preference Ordering by a Numerical Function," Chapter XI in R. M. Thrall, C. H. Coombs, R. L. Davis (eds.), *Decision Processes*, New York, 1954.

Define $A=\{U \text{ in } X|U \succsim U^*\} \cap D$, $B=\{U \text{ in } X|U^* \succsim U\} \cap D$. From the assumption about U^*, $A \cap B = \phi$. Since the ordering is complete, $A \cup B = D$. By (S1), $(0_{\text{con}}) \precsim U^* \precsim (1_{\text{con}})$. Therefore $A \neq \phi$, $B \neq \phi$. Since the sup metric generates a natural topology, both A and B are closed relative to D. This contradicts the connectedness of D.

The existence theorem can now be proved by showing that the set of constant utility streams with rational one period utility levels satisfies the conditions to be the set Z in Debreu's lemma.[7]

EXISTENCE THEOREM: *Let X be completely ordered by a preference ordering satisfying* (S1) *and for which the sup metric generates a natural topology. Then there exists a utility function from X with the sup metric to the real line.*

Proof: For any pair U, U' in X such that $U \succ U'$, there exists a pair u, u' in the unit interval such that $u > u'$, $(u_{\text{con}}) \sim U$, and $(u'_{\text{con}}) \sim U'$. For any such pair u, u' there exists a rational number r such that $u > r > u'$. Therefore, $U \succ (r_{\text{con}}) \succ U'$. Thus the set $Z = \{(r_{\text{con}}) | r \text{ rational}, 0 \leqslant r \leqslant 1\}$ satisfies the conditions of Debreu's lemma.

By the same proofs, the existence theorem could be shown to hold when the product metric is used in place of the sup metric. The theorem will also continue to be true for both metrics if the sensitivity axiom (S1) is replaced by the following sensitivity axiom which implies (S1).

(S2) $\quad U \geqslant U'$ implies $U \succ U'$.

This axiom states that a utility stream that has at least as high one period utility levels in all periods as a second stream and a higher level in at least one period, is preferred to the second.

4. EVENTUAL IMPATIENCE

Preference for earlier timing of utility can be expressed by a preference for a utility stream $U=(u_1, u_2, u_3, \ldots, u_{t-1}, u_t, u_{t+1}, \ldots)$ over a stream $U^t=(u_t, u_2, u_3, \ldots, u_{t-1}, u_1, u_{t+1}, \ldots)$ if $u_1 > u_t$. In other words, reversing the timing of the first and tth period utility levels raises the utility of the entire stream if it places the larger one period level in the first period. This is impatience for the first over the tth period. In the case where the preferences conform to a utility function $f(U) = \sum_{t=1}^{\infty} w_t u_t$, this would mean $w_1 > w_t$. Eventual impatience means impatience for the

[7] An alternative proof of the Existence Theorem, not employing Debreu's Lemma, is due to M. E. Yaari: Consider the function f from X to the real line such that each U in X is mapped into the real value u such that $U \sim (u_{\text{con}})$. The lemma shows that this function exists, and it is clearly order preserving. Since open sets in the unit interval are unions of open intervals, for showing the continuity of f it is sufficient to prove that the inverse image of an open interval is open. But this is equivalent to the naturalness of the topology, since the inverse image of the open interval (u^*, u^{**}) is the intersection of the two sets $\{U \text{ in } X|U \succ (u^*_{\text{con}})\}$, $\{U \text{ in } X|U \prec (u^{**}_{\text{con}})\}$, both of which are open.

first period over the tth period for all t sufficiently far in the future. In the example above, this would be satisfied if $w_1 > 0$ and w_t went to zero as t goes to infinity. The following two theorems examine the conditions for eventual impatience in the two cases of the product and sup metrics.

With the assumptions of (S2) and the naturalness of the product topology, it can be shown that for any given utility difference there is a point in the future, depending on the given difference, such that there is impatience for the first period over all time beyond that point for all utility differences at least as large as the given difference.

THEOREM: *Let X be completely ordered by a preference ordering satisfying (S2) and for which the product metric generates a natural topology. Then for any $\varepsilon > 0$ there exists an s such that, for all U in X, for all $t \geqslant s$ for which $|u_1 - u_t| \geqslant \varepsilon$,*

$$U\begin{Bmatrix}\succ\\ \prec\end{Bmatrix}U^t \quad as \quad u_1\begin{Bmatrix}>\\ <\end{Bmatrix}u_t. \tag{1}$$

Proof: By the existence theorem, there exists a utility function f. Given $\varepsilon > 0$ define, for all U in X, $K_1(U) = \{k | u_k \geqslant u_1 + \varepsilon,\ U \succsim U^k\}$. Then $K_1(U)$ is the set of indices k, for which (1) is not true when $u_k \geqslant u_1 + \varepsilon$.
Define

$$K_2(U) = \{k | u_k \leqslant u_1 - \varepsilon,\ U^k \succsim U\},$$

$$K_1 = \{k | \text{there exists } U \text{ in } X \text{ such that } k \text{ in } K_1(U)\},$$

and

$$K_2 = \{k | \text{there exists } U \text{ in } X \text{ such that } k \text{ in } K_2(U)\}.$$

It is sufficient to show, for any $\varepsilon > 0$, that both K_1 and K_2 have only a finite number of elements. The two cases are symmetric and it will be shown that K_1 has a finite number of elements.

Assume K_1 has infinitely many elements. For each k in K_1 select a $U(k)$ such that k is in $K_1(U(k))$. Since X is compact the sequence $\{U(k) | k \text{ in } K_1\}$ has a convergent subsequence $(U(k_i))$ converging to U^*. For each $U(k_i)$ in the convergent subsequence consider $U^{k_i}(k_i)$. This set has a convergent subsequence converging to U^{**}.

By comparing the elements of the two convergent subsequences it is seen that $u_1^{**} \geqslant u_1^* + \varepsilon$, $u_j^{**} = u_j^*$ for $j = 2, 3, 4, \ldots$. Thus $U^{**} \geqslant U^*$ and, by (S2), $U^{**} \succ U^*$. However for each k_i $U(k_i) \succsim U^{k_i}(k_i)$. This and the continuity of f imply $U^* \succsim U^{**}$, which is a contradiction.

The case with the sup metric is complicated by the fact that infinite sequences of utility streams need not have convergent subsequences since this space is not compact. Thus, to show eventual impatience for a stream U in X it is not sufficient to look at the infinite sequence (U^t). The proof involves constructing for each U a sequence which does converge and which, without eventual impatience, would violate the continuity of the utility function. This difficulty results in a need for two additional axioms and a somewhat weaker statement of the theorem in that the point in the future after which there is impatience depends on the utility difference and on the utility stream being considered.

(NC1) For all u, u', U, U', $(u, U)\gtrsim(u, U')$ implies $(u', U)\gtrsim(u', U')$.

(NC2) For all ${}_1u_t, {}_1u_t', U, U'$, $({}_1u_t, U)\gtrsim({}_1u_t', U)$ implies $({}_1u_t, U')\gtrsim({}_1u_t', U')$.

The axioms express a certain type of noncomplementarity of the preferences over time or that the "preferences" over part of the time horizon are independent of the utility levels achieved in other times.

THEOREM: *Let X be completely ordered by a preference ordering satisfying (S2), (NC1), and (NC2) and for which the sup metric generates a natural topology. Then for each U in X and* $\varepsilon>0$, *there exists an s such that for all* $t\geqslant s$ *for which* $|u_1-u_t|\geqslant\varepsilon$,

$$(1)\qquad U\begin{Bmatrix}\succ\\ \prec\end{Bmatrix}U^t \quad as \quad u_1\begin{Bmatrix}>\\ <\end{Bmatrix}u_t .$$

Proof: By the existence theorem there exists a utility function f. Given U and ε, define $K_1=\{k|u_k\geqslant u_1+\varepsilon,\ U\gtrsim U^k\}$, $K_2=\{k|u_k\leqslant u_1-\varepsilon,\ U^k\gtrsim U\}$. It is sufficient to show that both sets have a finite number of elements. The two cases are symmetric and only K_1 will be considered.

Assume K_1 has infinitely many elements. Label them in order k_i with $i=1, 2, \ldots$. For k in K_1 define:

$$U^{*k}=(u_1^{*k}, u_2^{*k}, \ldots) \quad \text{where} \quad u_j^{*k}=\begin{Bmatrix} u_1;\ j \text{ in } K_1 \text{ and } j\geqslant k; \\ \\ u_j;\ j \text{ in } K_1 \text{ and } j<k, \text{ or } j \text{ not in } K_1 \end{Bmatrix},$$

$$U^{**k}=(u_k, u_2^{*k}, u_3^{*k}, \ldots) .$$

From (S2) we see that $U^{**k}\succ U^{*k}$ and $U^{*k_{i+1}}\succ U^{*k_i}$. By (NC2) $U\gtrsim U^{k_i}$ implies $U^{*k_{i+1}}\gtrsim U^{**k_i}$. Therefore $\lim_{k\to\infty} f(U^{*k})=\lim_{k\to\infty} f(U^{**k})=f^*, f(0_{\text{con}})\leqslant f^* \leqslant f(1_{\text{con}})$. For each k there exists an a_k such that $U^{*k}\sim(u_1, a_k, a_k, a_k, \ldots)$. By (NC1) $U^{**k}\sim(u_k, a_k, a_k, a_k, \ldots)$. Since $U^{*k_{i+1}}\gtrsim U^{*k_i}$, $a_{k_{i+1}}>a_{k_i}$, $\lim_{k\to\infty} a_k$ exists and equals a^*. Then $f^*=f(u_1, a^*, a^*, \ldots)$. There exists a convergent subsequence of $(u_{k_i}|k_i \text{ in } K_1)$ converging to, say, u^*. Then, $f^*=f(u^*, a^*, a^*, \ldots)$ implying $u_1=u^*$. But $u_k\geqslant u_1+\varepsilon$ for all k in K_1; therefore $u^*>u_1$. This is a contradiction.

5. EQUAL TREATMENT FOR ALL GENERATIONS

A preference ordering which treats all generations equally is one that satisfies the condition:

(C) $U\sim U^t$ for all U in X and all $t=1, 2, \ldots$.

This condition is satisfied, for example, by Ramsey's functional, which does not discount future utilities in the integral to be maximized. However, certain axiom sets can be shown to be inconsistent with this condition. The two axiom sets used in Section 4 to derive eventual impatience are clearly inconsistent with (C). For each of the metrics being examined in this paper a weakening of the axiom sets used in Section 4 can be permitted while preserving the inconsistency of the axiom set with (C).

THEOREM: *Let X be completely ordered by a preference ordering satisfying (S1) and for which the product metric generates a natural topology. Then condition (C) is inconsistent with this preference ordering.*

Proof: Assume (C) is satisfied. By the existence theorem there exists a utility function f. For $j \geqq i$ define $a_{ij} = (\underbrace{\overbrace{bbb \ldots b}^{i-1} aaa \ldots ab}_{j\text{th place}} aa \ldots)$, for $1 \geqq b > a \geqq 0$. (a_{ij} has i b's, in the first $i-1$ places and in the jth place.) For $k, j \geqq i$, (C) implies $a_{ij} \sim a_{ik}$.

$\lim_{j\to\infty} a_{ij} = a_{i-1,\, i-1}$. Therefore $f(a_{ij}) = f(a_{i-1,\, i-1})$ for all i, j.

$\lim_{j\to\infty} a_{1j} = (a_{\text{con}})$. Therefore $f(a_{1j}) = f(a_{ij}) = f((a_{\text{con}}))$.

$\lim_{i\to\infty} a_{ii} = (b_{\text{con}})$. Therefore $f(a_{ii}) = f((b_{\text{con}})) = f((a_{\text{con}}))$.

But $(b_{\text{con}}) > (a_{\text{con}})$. This is a contradiction.

THEOREM:[8] *Let X be completely ordered by a preference ordering satisfying (S2) and for which the sup metric generates a natural topology. Then condition (C) is inconsistent with this preference ordering.*

Proof: Assume (C) is satisfied. By the existence theorem, there exists a utility function f. Consider

$$U = \left(0, 1, 0, \tfrac{1}{2}, 1, \ldots, 0, \frac{1}{2^k}, \frac{2}{2^k}, \ldots, \frac{2^k - 1}{2^k}, 1, 0, \ldots\right).$$

Define $U^k = (1, 1, 0, \frac{1}{2}, 1, \ldots, 0, 0, 1/2^k, \ldots, (2^k-2)/2^k, (2^k-1)/2^k, 0, \ldots)$ which is obtained from U by interchanging the initial 0 with the $(k+1)$st 1 appearing in U and then interchanging the sequence $(1/2^k, 2/2^k, \ldots, (2^k-1)/2^k, 0)$ with $(0, 1/2^k, 2/2^k, \ldots, (2^k-2)/2^k, (2^k-1)/2^k)$. Since this involves a finite number of interchanges, (C) implies $U \sim U^k$.

Define $U^* = (1, 1, 0, \frac{1}{2}, 1, \ldots, 0, 1/2^k, 2/2^k, \ldots, (2^k-1)/2^k, 1, 0, \ldots)$. We see that $\lim_{k\to\infty} U^k = U^*$ since $d(U^k, U^*) = 1/2^k$. By the continuity of f this implies $U \sim U^*$. However $U^* \geqslant U$ and thus, by (S2), $U^* \succ U$. This is a contradiction.

[8] This theorem is due to M. E. Yaari.

6. PREFERENCES WITHOUT A UTILITY FUNCTION

Throughout the above sections the preferences were assumed to be such that the existence of a utility function could be shown. This assumption will now be dropped. Preferences over infinite horizon streams, U, will be assumed to have certain relationships with preferences over finite horizon streams, ${}_1u_t$. These relationships may be interpreted in either of two ways. The interpretation may be made that the preferences, $\succsim$, over infinite streams, U, are such that it is possible to construct preference orderings $\succsim_t$ over finite sequences, ${}_1u_t$, for all t. Then these derived preferences, $\succsim_t$, can be assumed to have certain properties and bear certain relationships with the basic preference ordering $\succsim$.

On the other hand, it may be interpreted to mean that persons, when facing decisions about an infinite future, do not directly know their own minds, but they do know their preferences, $\succsim_t$, over finite sequences, ${}_1u_t$, and they have convictions about the relationships of preferences for the entire future and preferences for just parts of it.

It will be assumed that there are complete preference orderings $\succsim$ over U and $\succsim_t$ over ${}_1u_t$, for all t. $\succsim_1$ will be assumed to be the same as $\geqslant$. Define $({}_1u_{t_{rep}})$to be the infinite stream $({}_1u_t, {}_1u_t, {}_1u_t, \ldots)$. There are three axioms that will be assumed about the relationships among the preference orderings.

(A1) For all t, ${}_1u_t$, ${}_1u_t'$, U; ${}_1u_t \succsim_t {}_1u_t'$ implies $({}_1u_t, U)\succsim({}_1u_t', U)$.

(A2) For all t, ${}_1u_t$, ${}_1u_t'$; ${}_1u_t \succsim_t {}_1u_t'$ implies $({}_1u_{t_{rep}})\succsim({}_1u'_{t_{rep}})$.

(A3) There exists a u such that for all U, U', $U\{\gtreqless\}U'$ implies $(u, U)\{\gtreqless\}(u, U')$.

Axiom 1 is a noncomplementarity axiom and is equivalent to (NC2). Axiom 2 has both noncomplementarity and persistence of preferences aspects. Axiom 3 is a stationarity axiom, and is equivalent to the stationarity axiom of Koopmans.[9] It assumes a preservation of preference orderings when the timing of all periods is moved one period into the future, while the present assumes a constant utility level for all streams.

THEOREM: *A set of complete preference orderings* $\succsim$, $\succsim_t$, $t=1, 2, \ldots$ *satisfying* ($A1$), ($A2$), *and* ($A3$) *is inconsistent with* (C).

Proof: Assume u in (A3) is unequal to 0. (If it is equal to zero the proof will hold replacing u and 0 by 0 and 1.) Applying (A3) to $((u, 0)_{rep})$ and $((0, u)_{rep})$ we have $((u, 0)_{rep})\{\gtreqless\}((0, u)_{rep})$ implies $(u, (u, 0)_{rep})\{\gtreqless\}((u, 0)_{rep})$. $u>0$ implies $(u, (u, 0)_{rep})\succ((0, u)_{rep})$. Therefore $(u, 0)_{rep}\succ(0, u)_{rep}$, implying $(u, 0) \succsim_2 (0, u)$ and thus $(u, 0, U)\succ(0, u, U)$, contradicting (C).

University of California, Berkeley

[9] Koopmans, T. C., *op. cit.*