Intergenerational Equity and Exhaustible Resources [1,2]

R. M. SOLOW

Massachusetts Institute of Technology

The theory of optimal economic growth, in the form given it by Frank Ramsey and developed by many others, is thoroughly utilitarian in conception. It is utilitarian in the broad sense that social states are valued as a function of the utilities of individuals (individual moments of time, in this case, since individual persons are usually taken as identical and identically treated) with the possibility that a loss of utility to one individual (or generation) can be more than offset by an increment to another. It is also utilitarian in the narrow sense that social welfare is (usually [3]) defined as the sum of the utilities of different individuals or generations.

Recently the whole utilitarian approach to social choice has come under fundamental attack by John Rawls [9]. One particular view advanced by Rawls concerns me here. He argues, in effect, that inequality in the distribution of wealth or utility is justified only if it is a necessary condition for improvement in the position of the poorest individual or individuals. In other words, if social welfare, W, is to be written as a function of individual utilities $U_1, \ldots, U_n$, then Rawls argues for the particular function $W = \min(U_1, \ldots, U_n)$, so that maximizing social welfare amounts to maximizing the smallest U_i.[4] This welfare function is sensitive *only* to gains and losses of utility by the poorest person.

A Theory of Justice contains a section [5] explicitly devoted to equity between generations, i.e. the question that arises in the theory of optimal capital accumulation. Remarkably, the one thing this chapter does not do is to advocate unequivocally the max-min criterion espoused elsewhere in the book. In this context Rawls settles for such ambivalent statements as the following:

" . . . the question of justice between generations . . . subjects any ethical theory to severe if not impossible tests. . . . I believe that it is not possible, at present anyway, to define precise limits on what the rate of savings should be. How the burden of capital accumulation and of raising the standard of civilization is to be shared between generations seems to admit of no definite answer. It does not follow, however, that certain bounds which impose significant ethical constraints cannot be formulated. . . . Thus it seems evident, for example, that the classical principle of utility leads in the wrong direction for questions of justice between generations. . . . Thus the utilitarian doctrine may direct us to demand heavy sacrifices of the poorer generations for the sake of greater advantages for later ones that are far better off. But this calculus of advantages which balances the

1 *First version received June* 1973; *final version accepted October* 1973 (*Eds.*).

2 I thank Joel Yellin for help and advice, and The National Science Foundation for financial assistance.

3 For examples of results involving non-additive social welfare function, see [5, 11, 12].

4 This can be regarded as a limiting special case of utilitarianism because, for instance, if

$$W_p = \left[\frac{1}{n}\Sigma U_i^p\right]^{1/p},$$

then $\lim_{p\to-\infty} W_p = \min(U_1, \ldots, U_n)$. Note that W_1 is the additive welfare function, W_0 the constant-elasticity function, and $\lim_{p\to\infty} W_p$ is the " royalist " $\max(U_1, \ldots, U_n)$. Note also that, if i runs over generations, the rate of time discount has been tacitly set at zero.

5 See pp. 284-293.

losses of some against benefits to others appears even less justified in the case of generations than among contemporaries. . . . It is a natural fact that generations are spread out in time and actual exchanges can take place between them in only one direction. We can do something for posterity but it can do nothing for us. This situation is unalterable, and so the question of justice does not arise. . . . It is now clear why the (max-min criterion) does not apply to the savings problem. There is no way for the later generation to improve the situation of the least fortunate first generation. The principle is inapplicable and it would seem to imply, if anything, that there be no saving at all. Thus, the problem of saving must be treated in another fashion."

For the specific problem of intergenerational equity and savings, Rawls proposes a deliberately vaguer principle, given by the balance between what a typical person feels it is reasonable to ask of his parents and what this same person is prepared to do for his children. He reserves the operation of the max-min principle for intragenerational comparisons: the balance just mentioned should be drawn with special attention to the needs and attitudes of the least favoured group in each generation.[1]

This broader approach says little that is definite within the framework of the standard theory of optimal capital accumulation. There, the intragenerational problem is completely sidestepped by the assumption that all members of any given generation are identical. And so far as the saving-investment decision is concerned, the problem is precisely to start with a presumption about what each generation can legitimately demand of its ancestors and should legitimately leave to its descendants.

In this article I am going to be *plus Rawlsien que le Rawls*: I shall explore the consequences of a straightforward application of the max-min principle to the intergenerational problem of optimal capital accumulation. It will turn out to have both advantages and disadvantages as an ethical principle in this context. The disadvantages are primarily those that led Rawls to shy away from it, though I shall be able to characterize them in more detail. The main advantage is that the max-min principle gives in some respects more sensible precepts than the standard additive-welfare approach.

I shall proceed by starting with the simplest possible case (constant population, no technical progress, no scarce natural resources) and adding complications one at a time. This procedure has the advantage that the simpler cases, where the argument is trivial, serve to illustrate the basic ideas, unencumbered by technical detail.

1. THE MAX-MIN PRINCIPLE AND OPTIMAL ECONOMIC GROWTH

Nothing relevant to this analysis would be gained if I strayed outside the conventional framework of the one-sector economy whose single produced commodity can be either consumed directly or accumulated as a capital good. I shall also hold to the standard assumption that at each instant of time consumption is shared equally by the population (labour force) of the moment. The only equity problem that arises is that between instants of time (i.e. " generations ").

Except possibly for trick cases, the max-min principle requires that consumption per head be constant through time. If consumption per head were higher for a later than for an earlier generation, then social welfare would be increased if the early generation were to save and invest less, or to consume capital, so as to increase its own consumption at the expense of the later generation. If consumption per head were higher for an earlier than for a later generation, then social welfare would be increased if the early generation were to consume less and, correspondingly, save and invest more, so as to permit higher consumption in the future. Thus the max-min principle tells us that consumption per head should be the same for all generations. The exceptions would arise only if there were some technical obstacle to the equalization of consumption over time; that is, they would

[1] One plausible interpretation of just accumulation in the social-contract framework is in terms of an intergenerational Nash equilibrium. This has been explored in an interesting way by Dasgupta [2].

be in the nature of corner solutions. They present no issue of principle, so I shall ignore them.

2. CONSTANT POPULATION, CONSTANT TECHNOLOGY, NO SCARCE NATURAL RESOURCES

Let *net* output Q be produced under constant returns to scale by a stock of homogeneous capital K and a flow of homogeneous labour L according to the well-behaved production function

$$Q = F(K, L) = Lf(k), \qquad \ldots(1)$$

where $k = K/L$. Since Q is net output, we can write

$$Q = C + \dot{K}, \qquad \ldots(2)$$

where C is aggregate consumption. Let K_0 stand for the initial stock of capital at time zero (the present).

Since L is constant, by hypothesis, the max-min principle calls for C to be constant in time, and the question is merely: what is the largest aggregate consumption that can be maintained forever? The answer, obviously, is to set $\dot{K} = 0$, $K = K_0$, $C = F(K_0, L)$. In other words, the optimal policy is for each generation to maintain the capital stock intact and consume the whole net national product. For the initial generation to save would make it poorer and the future richer than it; for the initial generation to dissave would make it richer and the future poorer than it. Neither is desirable. And then the same situation reproduces itself for each generation in turn.[1]

3. EXPONENTIALLY GROWING POPULATION

Suppose that $L = L_0e^{nt}$ with no possibility of social control of natural increase. From (1) and (2) it follows that

$$\frac{\dot{K}}{L} = f(k) - c,$$

where $c = C/L$ is consumption per person. From the definitions,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n \text{ or } \dot{k} = \frac{\dot{K}}{L} - nk,$$

and thus, finally,

$$\dot{k} = f(k) - nk - c. \qquad \ldots(3)$$

Any time path $c(t)$ for consumption per head defines a time path for $k(t)$ through the differential equation (3). In the way that growth theory has made familiar, the inherited capital stock and the exogenously given supply of labour determine current full-employment output; once consumption is specified, the rest of full-employment output is net investment, and is added to the inherited stock of capital to give the next instant's stock of capital. The whole future is thus determinate. That is the content of (3).

A time path $c(t)$ is feasible provided the solution of (3) satisfies $k(t) \geqq 0$; i.e. provided it leaves enough net investment to keep the stock of capital from disappearing. The optimum problem according to the max-min principle is to choose the largest constant c_0 such that the $k(t)$ defined by (3) with $c = c_0$ is non-negative for all $t \geqq 0$, given that $k(0) = k_0$ is the initial capital per worker.

Under the usual assumptions about $f(k)$,[2] the function $f(k) - nk$ will appear as in

[1] For Rawls, with his interest in the social contract to be agreed upon before society has any history, it might be natural to ask how the initial capital stock could be accumulated. Under these assumptions, that question has no good answer.

[2] Increasing, concave, $f(0) = 0$, $f'(0) > n$, $f'(\infty) < n$.

Figure 1. The initial k_0 has been marked and a horizontal line has been drawn at height c_0. From (3), $\dot{k}$ is given by the vertical distance between $f(k)-nk$ and c_0, positive if the curve is above c_0, negative if below. If $c_0 = f(k_0)-nk_0 = c^*$, then clearly $\dot{k} = 0$ for all t and k remains equal to k_0. This value of c_0 is feasible. Smaller values of c_0 are also feasible, and will cause k to increase from k_0 to the larger root of $c_0 = f(k)-nk$. But these smaller values of c_0 are not optimal. If c_0 is chosen larger than c^*, so that

$$f(k_0)-nk_0-c_0<0,$$

it is clear from the diagram that k will decrease steadily from k_0, reaching zero in finite time, with $\dot{k}$ strictly negative at that time. So $c_0>c^*$ is not feasible. It follows that c^* is optimal.[1]

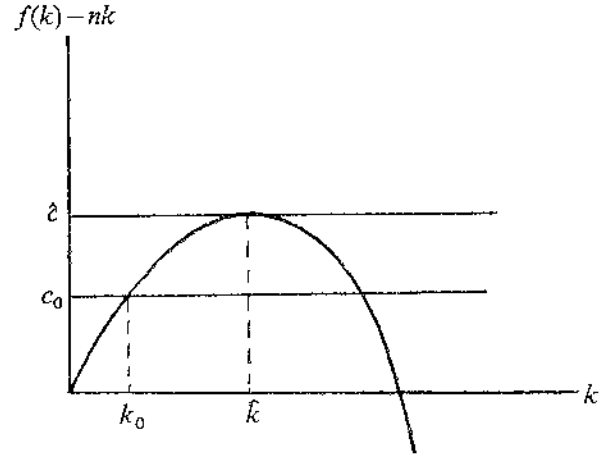

FIGURE 1

The max-min rule says: the initial generation should invest only enough to provide capital for the increase in population at the initial capital-labour ratio. Then each succeeding generation should do the same. That is: widen, but don't deepen. This contrasts with the outcome of the conventional utilitarian theory: [2] if there is no pure time preference, capital should be deepened to approach the Golden-Rule $\hat{k}$ (at which the curve in Figure 1 attains the maximum). Earlier generations consume less than c^* per person, but later generations surpass c^* and ultimately approach the maximum sustainable consumption $\hat{c} = f(\hat{k})-n\hat{k}$. The Rawlsian principle refuses this trade-off, even though c would be less than $c^*-\varepsilon$ for a finite time and greater than $c^*-\varepsilon$ forever after.

4. TECHNICAL PROGRESS

Suppose there is labour-augmenting technical progress, so that (1) becomes

$$Q = F(K, e^{at}L) = e^{at}Lf(z) \qquad \ldots(1a)$$

where z stands for capital per worker in efficiency units, i.e. $z = K/e^{at}L$. Combining (1a) and (2) one finds

$$\frac{\dot{K}}{L} = e^{at}f(z)-c;$$

[1] In Figure 1, $\hat{k}$, defined by $f'(\hat{k}) = n$, is the Golden-Rule capital-stock-per-worker, providing the largest sustainable consumption-per-worker. I am assuming that $k_0<\hat{k}$. If in some Eden $k_0>\hat{k}$, then consuming capital is good for everyone and the optimal max-min policy is to set $c = \hat{c} = f(\hat{k})-n\hat{k}$. Then $k(t)$ will fall from k_0 to $\hat{k}$ and stay there.

[2] See, for instance, [6].

notice that c is still defined as consumption per worker in natural units, because there is no ethical significance whatever in consumption per worker in efficiency units. Now

$$\dot{z} = z\left(\frac{\dot{K}}{K} - n - a\right) = \frac{\dot{K}}{e^{at}L} - (n+a)z$$

so that

$$\dot{z} = f(z) - (n+a)z - ce^{-at}, \qquad \ldots(4)$$

which reduces to (3) when $a = 0$. The problem now is to find the largest constant c which generates a solution $z(t)$ of (4), starting from $z(0) = z_0$, which is non-negative for all $t \geqq 0$.

In the case without technical progress, we could choose a c_0 such that the right-hand side of (3) was zero at $t = 0$. Then (3) would remain zero forever after. In (4), however, maintaining capital intact is not a proper strategy. Ongoing technical progress would favour the future over the present, and unfairly according to the Rawlsian criterion. The proper strategy must be to consume capital from the beginning, allowing technical progress to maintain future consumption standards. For instance, mimicking the earlier procedure would suggest setting $c = c_0 = f(z_0) - (n+a)z_0$ in (4). This would make $\dot{z}(0) = 0$. but still $\dot{z}(t) > 0$ for all $t > 0$. Thus this value of c_0 is possible, but hardly the largest possible.

It is easy to find a c_0 that is too large. Define $\hat{z}$ by $f'(\hat{z}) = n+a$ so that $\hat{z}$ maximizes $f(z) - (n+a)z$ and set $\hat{d} = f(\hat{z}) - (n+a)\hat{z}$. Then, from (4) $\dot{z} \leqq \hat{d} - c_0e^{-at}$. Choose any $T > 0$; then if $\dot{z} < -z_0/T$ for $0 \leqq t \leqq T$, clearly $z(T) < 0$, i.e. z will have become negative before time T. But to ensure that $\dot{z} < -z_0/T$, it is only necessary that $\hat{d} - c_0e^{-at} < -z_0/T$ or $c_0 > e^{at}(\hat{d} + (z_0/T))$ for $0 \leqq t \leqq T$, which can be achieved by choosing $c_0 > e^{aT}(\hat{d} + (z_0/T))$.

Some constant c's are feasible, for instance $c = 0$ or $c = f(z_0) - (a+n)z_0$, and others are not. There will be a largest feasible constant consumption per person, say c_0. For this optimal c_0, the differential equation (4) will imply that $\lim_{t\to\infty} z(t) = 0$. That is to say, when society consumes steadily at the largest permanently maintainable level per person, it will asymptotically consume the entire capital stock. This conclusion depends on the assumed unboundedness of technical progress, and also on the assumption that $f(0) = 0$, i.e. no production without capital.

There are in fact three classes of constant-consumption policies. Setting $c = c_0$ is one of them. If c is maintained always greater than c_0 (constant or not), then the stock of capital will be driven to zero in finite time and the economy will come to an end. If c is maintained always less than c_0 (constant or not), then the capital stock, although it may fall at first, must eventually turn around and tend in the limit to the root in z of

$$f(z) - (a+n)z = 0,$$

which is the highest permanently maintainable capital stock per worker in efficiency units. This is clearly a wasteful programme of over-investment. These statements are demonstrated in Appendix A.

5. SUMMARY SO FAR

My impression is that, in the situations considered so far, the max-min criterion does not function very well as a principle of intergenerational equity. (No doubt his perception of the difficulties led Rawls to abandon it in his discussion of saving.) It calls, as I have mentioned, for zero net saving with stationary technology, and for negative net saving with advancing technology. That is by itself not off-putting. What is less satisfactory is the fact that the max-min criterion is so much at the mercy of the initial conditions. If the initial capital stock is very small, no more will be accumulated and the standard of living will be low forever.[1]

[1] The economist who has most strongly and perceptively argued that ethical principles must be revised or criticized in view of the sensibleness of their implications is, of course, Tjalling Koopmans [7].

Of course, this result follows from the basic principle itself. Capital could be accumulated and consumption increased subsequently, but only at the cost of a lower standard of living for earlier generations. It is part of Rawls's general argument for the max-min criterion that we should regard earlier and later generations as facing each other contemporaneously when the social contract is being drawn up. But then it is hardly surprising that the preferred strategy refuses to make some people poorer than others in order to make the others richer, just because the first group can be given the essentially arbitrary label of " earlier ". From this point of view the distinction by time is merely a trick played on us by posterity and by us on our ancestors. But it can be argued that it is a useful trick, reflecting the " physical " fact that there is no way the past can be compensated by the future *after* the saving has taken place and the productivity of capital goods exploited.[1]

If this productive time-asymmetry were absent, there would be no element of surprise or incongruity. For instance, if the problem were simply to ration out a given finite stock of grain over a finite interval, with no possibility of production, the obvious solution would be an equal division of what there is. We can add just such an element to the problem by allowing for a finite pool of non-reproducible natural resources which has to be used up in production. That is the next step.

6. EXHAUSTIBLE RESOURCES

Suppose we extend the model of production to read

$$Q = F(K, L, R),$$

where R is a rate of flow of a natural resource, extracted from a pre-existing pool. For the problem to be interesting and substantial, R must enter in a certain way. For example, if production is possible without natural resources, then they introduce no new element. Presumably the initial stock would be used up early in the game to shore up consumption while a stock of capital is accumulated, which will then be maintained intact while the same level of consumption goes on even after the natural resource pool is all gone. This is, in fact, proved in Proposition 4 of [3] in a very similar model. On the other hand, if the average product of resources is bounded, then only a finite amount of output can ever be produced from the finite pool of resources; and the only level of aggregate consumption maintainable for infinite time is zero.

The interesting case is one in which $R = 0$ entails $Q = 0$, but the average product of R has no upper bound. (This is what Dasgupta and Heal [3] call an " essential " resource.) The Cobb-Douglas has this property, or a function like

$$Q = F(K, L)R^h \quad (0<h<1)$$

with F homogeneous of degree $1-h$.[2] This being so, I shall carry on the rest of the analysis using the Cobb-Douglas explicitly, especially because that will simplify the treatment of technical progress too. Any extra generality hardly seems worth striving for.

[1] Rawls remarks ([9], pp. 290-291), and attributes a similar thought to Herzen and to Kant, that, at least on the surface, the process of saving is unfair in the sense that later generations fatten on the sacrifices of earlier and offer nothing in return. He argues that this is just a physical asymmetry and to talk about justice is as futile and inappropriate, say, as to discuss the justice of the fact that the earth rotates in one direction, so the sun rises in Boston before it does in San Francisco. But I think this puts the matter too simply. There *is* something the future can do for the past: it can inherit less capital. All the more so, if technical progress favours later periods over earlier, the later generation can compensate the earlier by inheriting even less capital than that. The asymmetry is more subtle. If capital goods were not productive, I think there would appear no difficulty of principle. The problem arises because capital formation exploits nature; if the earlier generation dissaves for the sake of equity, the future pays more—in output foregone—than the past has gained. If the initial standard of consumption were high enough, then the principle of diminishing marginal utility might suggest that the future does not pay more, not in terms of the coin that really counts.

[2] Only the Cobb-Douglas will do among CES functions. If the elasticity of substitution between resources and other factors exceeds one, then resources are not indispensable to production. If it is less than one, then the average product of resources is bounded. So only the Cobb-Douglas remains.

Suppose, therefore, that

$$Q = e^{mgt}L^g R^h K^{1-g-h}, \qquad \ldots(6)$$

where mg is the rate of Hicks-neutral technical progress or, equivalently, m is the rate of labour-augmenting technical progress. Combining (6) with (2), letting $y = R/Le^{mt}$, $z = K/Le^{mt}$ and $c = C/L$, and proceeding as before, we get the differential equation

$$\dot{z} = z^{1-g-h}y^h - (n+m)z - ce^{-mt}, \qquad \ldots(7)$$

which generates $z(t)$ and therefore $K(t)$ starting from given $z(0)$ or $K(0)$, and given also time paths for c and y (or C and R).

Formally, the optimum problem with the max-min criterion is to find the largest constant c_0 for which there exists a function $y(t) \geqq 0$ for all $t \geqq 0$ constrained by

$$L_0 \int_0^\infty y(t)e^{(m+n)t}dt \leqq \bar{R} \qquad \ldots(8)$$

such that when this $y(t)$ and $c(t) = c_0$ are inserted in (7), the solution $z(t)$, $z(0) = z_0$, of the differential equation is non-negative for all $t \geqq 0$. That is to say, we must find the largest constant consumption per head which can be maintained forever with account taken of the finiteness of the pool of exhaustible resource and of the fact that we can not consume capital that isn't there.

7. EXHAUSTIBLE RESOURCES: REFORMULATION

This is an unusual sort of maximum problem and I do not see any obvious direct approach. Fortunately, there is an alternative way to go about it. Choose an arbitrary c_0 in (7) and solve the more conventional problem of minimizing $\int_0^\infty y(t)e^{(m+n)t}dt$ subject to (7) and $y(t) \geqq 0$, $z(t) \geqq 0$. If the minimized value of the integral is greater than $\bar{R}/L_0$, the c_0 chosen was too high and must be diminished; if the minimized value of the integral is less than $\bar{R}/L_0$, the c_0 chosen was too small and can be increased. When a c_0 is found for which the minimized value of the integral is just $\bar{R}/L_0$, the original problem is solved.

A necessary condition for a minimum of this modified problem is the existence of a shadow-price of capital (in terms of the natural resource) $p(t)$ with the properties

$$phz^{1-g-h}y^{h-1} = 1 \qquad \ldots(9a)$$

$$\dot{p}/p = -(1-g-h)z^{-g-h}y^h; \qquad \ldots(9b)$$

and the constraint (7) must hold.[1]

The economic meaning of (9*a*) and (9*b*) is straightforward, given the interpretation of $p(t)$ as the efficiency price of capital in terms of the natural resource. The content of (9*a*) is simply that the resource should be drawn down in such a way that its marginal value product is kept equal to its own efficiency price. In effect, (9*b*) says that the rate of change of the shadow price of resources should equal the sum of the rate of change of the shadow price of a unit of capital and the own rate of return from using capital (optimally) to produce itself; in other words, a rational investor, calculating with efficiency prices, should be at all times indifferent at the margin between holding capital goods and holding mineral deposits as earning assets.

These shadow-price relationships can be thrown into an unusual form, not limited to the Cobb-Douglas case: they say that along an optimal path the proportional *rate of change* of the marginal productivity of the resource should always equal the *level* of the

[1] The applicable version of the Pontryagin Principle is nicely laid out in [1], Proposition 7, pp. 48-49. But it is evident from (7) that neither y nor z can be zero at any finite time along a feasible path, so ordinary Euler-Lagrange methods will do.

marginal productivity of reproducible capital. The mysteriousness evaporates as soon as one notices that the first of these quantities is a capital gain, the instantaneous return from leaving a dollar's worth of the resource in the ground, and the second is the instantaneous return from owning and using or renting a dollar's worth of reproducible capital.

It is clear from this interpretation that some of the properties of an optimal path are easily achieved through market processes. But I can see no force—other than " Rawlsian conviction "—that will steer a market economy to constant consumption and the right $y(0)$. " Rawlsian conviction " here means something more than current legislation; there needs to be a kind of social contract to bind the next Congress, and the next.

Together (7), (9*a*) and (9*b*) are three equations in the three unknown time-functions $p(t)$, $y(t)$ and $z(t)$. Two of them are first-order differential equations, so a solution of the system will contain two arbitrary constants. One of these is used up in making

$$z(0) = z_0 = K_0/L_0.$$

The remaining degree of freedom is available to choose $p(0)$ (or $y(0)$, the other being determined by (9*a*)) so that the resulting path is actually optimal. All this is with arbitrarily chosen c_0, which must then be varied until (9) holds with equality.

To get further, take the logarithm of (9*a*), differentiate with respect to time and use the result to eliminate $\dot{p}/p$ from (9*b*). The result is

$$\dot{y}/y = -\left(\frac{1-g-h}{1-h}\right)\left(m+n+\frac{ce^{-mt}}{z}\right), \qquad \ldots(10)$$

which, with (7), gives two equations in y and z. They contain time explicitly, so can not be fully described in the usual sort of phase diagram. For this and for other reasons, I take up some special cases first. It will turn out that I can find out most of what I want to know without tackling the full problem.

8. ZERO POPULATION GROWTH AND ZERO TECHNICAL PROGRESS AGAIN

One reason for choosing the Cobb-Douglas function (6) is that it makes output per unit of natural-resource input go to infinity as the flow of resources diminishes toward zero. Otherwise, as I pointed out, the total output that can possibly be built on a finite pool of resources is finite and therefore the only aggregate output flow maintainable forever is zero. Even with the Cobb-Douglas,

$$\frac{Q}{L} = e^{mgt}\left(\frac{R}{L}\right)^{h}\left(\frac{K}{L}\right)^{1-g-h},$$

so that at any given time, with given capital per worker, output *per worker* goes to zero as the resource flow per worker goes to zero. With a finite initial pool, the annual resource flow must eventually go to zero. In the absence of technical progress, the only way a positive consumption flow can be maintained is through fast enough capital accumulation to drive K/L toward infinity as R/L drops toward zero. But, in the absence of technical progress, conventional growth theory tells us that there is a largest maintainable stock of capital per worker *if the labour force grows geometrically*. (It is the right-hand intersection of the curve in Figure 1 with the k-axis.) And that is without worrying about resource exhaustion. This suggests that continued technological progress is likely to be necessary for a positive consumption flow to be maintainable.

In a model with finite natural resources, it seems ridiculous to hold to the convention of exponentially growing population. We all know that population can not grow forever, if only for square-footage reasons. The convention of exponential population growth makes excellent sense as an approximation so long as population is well below its limit. On a time-scale appropriate to finite resources, however, exponential growth of population is an inappropriate idealization.[1] But then we might as well treat the population as constant. So suppose for now that $n = 0$ in (7) and (10).

Suppose in addition that there is no technical progress, i.e. that $m = 0$. Then (7) and (10) become

$$\dot{z} = z^{1-g-h}y^h - c \qquad \text{...(7}a\text{)}$$

$$\dot{y} = -\left(1 - \frac{g}{1-h}\right)cy/z. \qquad \text{...(10}a\text{)}$$

Now z, y and c are now essentially capital per worker, resource flow per worker and consumption per worker, since the population is constant in both natural and efficiency units, so L_0 can be normalized at one. For temporary notational reasons, I set $1-g-h = a$ and $h = b$ so that

$$\dot{z} = z^a y^b - c \qquad \text{...(7}a'\text{)}$$

$$\dot{y} = -\frac{a}{1-b}cy/z. \qquad \text{...(10}a'\text{)}$$

In this notation, it is taken for granted that $0 < a$, $b < 1$, indeed that $a+b < 1$. Moreover, for reasons that will become clear, I assume that $a > b$, i.e. that the elasticity of output with respect to reproducible capital exceeds that with respect to exhaustible resources. This seems quite safe: from factor shares, a would be at least three times b.[2]

One important preliminary remains to be checked before formal analysis. It would be reassuring to know that under the present assumptions there does indeed exist an indefinitely-maintainable positive level of consumption per head. That is to say, I would like to exhibit a function $y(t)$ and a positive constant c_0 such that (7a) has a forever non-negative solution $z(t)$ with $z(0) = z_0$, and such that $y(t) \geqq 0$ for all $t \geqq 0$ and

$$\int_0^\infty y(t)dt = \bar{R}/L_0.$$

It is clear from (7a) that any such solution $z(t)$ must necessarily go to infinity with t. For if $z(t)$ is bounded then, because $y(t) \to 0$ as $t \to \infty$ so that its integral can converge, the RHS of (7a) must eventually become and remain negative and bounded away from zero. Thus eventually any bounded $z(t)$ must become negative.

There does, in fact, exist an admissible solution to (7a).[3] For example,

$$z(t) = z_0 + ut, \; y(t) = (c_0+u)^{1/b}(z_0+ut)^{-a/b}$$

can be verified to be a solution of (7a'). Clearly $z(t)$ is unbounded if $u > 0$, and the integral of $y(t)$ converges if $a > b$. Calculation of the integral gives

$$c_0 = u^b\left(\frac{a-b}{b}\frac{\bar{R}}{L_0}\right)^b z_0^{a-b} - u,$$

so that it is always possible to choose c_0 and u positive. As one would expect, the range of feasible c_0 is larger, the larger $\bar{R}$ and z_0.

If, however, $a < b$, then it can be shown that there is no positive c_0 such that (7a) has a non-negative solution for any admissible $y(t)$. I owe a neat proof of this proposition

[1] That may be a very long time scale. For a delightful combination of science and imagination see [4].

[2] See W. Nordhaus and J. Tobin [8]. Nordhaus and Tobin's calculations suggest much stronger conclusions, that the elasticity of substitution between natural resources and the labour-capital composite exceeds one, or that there is rapid resource-saving technical progress, or both. In either case, the exhaustible-resource problem disappears as a *fundamental* problem. For my purposes, however, I must stick to my moderately pessimistic assumptions.

[3] I am very grateful to Professor Frederic Y-M Wan of the MIT Mathematics Department for having provided this example and set me on the right track.

to Professor Frederic Wan and Louis Howard of the MIT mathematics department; it is given in Appendix B.

9. ZPG AND ZTP, DETAILED SOLUTION

Now (7a') and (10a') are necessary conditions for an optimal choice of $y(t)$, given c_0. Since they are autonomous equations, they can be analysed in a phase diagram, as in Figure 2. The locus along which $\dot{z} = 0$ is obviously $z^a y^b = c_0$. The locus $\dot{y} = 0$ is the z-axis and, in addition, for any y, $\dot{y} \to 0$ as $z \to \infty$. The general shape of Figure 2 differs from that of the usual hyperbolic phase diagram, because Figure 2 is " trying " to have a saddle point at $y = 0$, $z = \infty$.

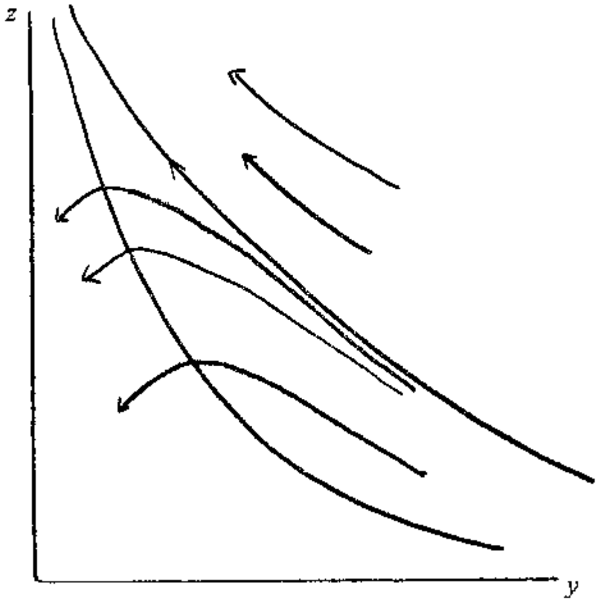

FIGURE 2

Any trajectory in Figure 2 is an integral curve of the differential equations. Since $z(0)$ is fixed by the initial capital stock at z_0, a candidate optimal path must start on the horizontal $z = z_0$. The initial y_0 has to be chosen (optimally). Clearly if y_0 is too small, the corresponding trajectory will reach a maximum and turn down; and we know that is not permanently feasible, let alone optimal. The analogy to a saddle-point at $y = 0$, $z = \infty$ suggests looking for a " separatrix ", a trajectory that does not ever turn down but instead heads for the singular point at infinity.

It turns out that the curve defined by the equation

$$z^a y^b = \frac{c_0}{1-b} \qquad \ldots(11)$$

is an integral curve of (7a') and (10a'). If this equation is substituted into (7a'), it follows easily that

$$z = z_0 + \frac{bc_0}{1-b}\, t \qquad \ldots(12)$$

$$y = \left(\frac{c_0}{1-b}\right)^{1/b} \left(z_0 + \frac{bc_0}{1-b}\, t\right)^{-a/b} \qquad \ldots(13)$$

provides a solution to (7a') and (10a') that lies on the curve (11) for all $t \geqq 0$.

These are necessary conditions. It takes only a little effort to complete the argument by showing that the choices (12) and (13) do in fact minimize total resource use in maintaining the constant consumption level c_0 per worker.[1] The last step is to set

$$\int_0^\infty y(t)dt = \bar{R}/L_0,$$

to determine the largest feasible c_0. Routine computation gives

$$c_0 = \left(\frac{\bar{R}}{L_0}\right)^{b/1-b} z_0^{\frac{(a-b)}{(1-b)}} (a-b)^{b/1-b}(1-b).$$

10. ZPG AND ZTP: DISCUSSION

As in the corresponding situation without exhaustible resources (treated in Section 2 earlier), there is a well-defined solution to the max-min problem. Again, the allowable consumption per head depends very much on the initial capital stock. Under the assumptions I have made, the allowable c_0 is a concave unbounded function of the initial capital stock per worker, when population is constant. The existence of indispensable exhaustible resources makes no difference to that proposition (provided, of course, that the elasticity of substitution between resources and labour-and-capital is at least one). *Any* level of consumption per worker can be maintained if only the initial capital stock is large enough.

The optimal programme under the Rawlsian criterion calls for capital per worker to grow from the very beginning and for resource use per worker to fall from the very beginning. If a is much bigger than b (as is probably the case) earlier generations should use up the resource pool quite fast, building up the capital stock in return.

A much more precise statement is possible. From (11), along the optimal path, net output is *constant*. Moreover, a fraction $(1-b) = (1-h)$ of output is consumed and the rest is net investment. Since h is the elasticity of output with respect to resource input, it is probably quite small, perhaps near 0·05. Net investment of 5 per cent of net output is enough to maintain output and consumption constant in the face of dwindling resource imputs.

From (9a), the shadow-price of capital in terms of resources is proportional to

$$(z_0 + (bc_0)t/(1-b))^{-a/b}.$$

Thus the shadow-price of resources in terms of the produced commodity rises ultimately like $t^{a/b}$ where $a/b > 1$.

All this contrasts with the Ramseyan or utilitarian approach to the same problem. I have not tracked down all the implications; for what strikes me as the interesting range of time-additive welfare functions, however, some generalizations are possible. Under the technological assumptions of this section, a utilitarian society with vanishing time preference will aim at a path of consumption per head that rises without limit through time. (With zero population growth and Cobb-Douglas production, such a path can be achieved.) It will use up the pool of natural resources more slowly than a Rawlsian society. The utilitarian prescription will call for more saving and, of course, later generations will enjoy a much higher standard of living than earlier ones. Earlier generations will have less than the constant Rawlsian consumption per head and, as already mentioned, consumption will rise without bound over time. This is the sort of conclusion the Ramsey approach typically produces. I do not give the details, but Appendix C provides a bare sketch of the way the analysis goes. Dasgupta and Heal [3] lay out the utilitarian version

[1] According to Proposition 8 on page 49 of [1], the necessary conditions are sufficient provided that (1) the Hamiltonian $-y+p(z^a y^b - c_0)$, maximized with respect to y, is a concave function of z and (2) $\lim_{t\to\infty} p(t) \geqq 0$, $\lim_{t\to\infty} p(t)z(t) = 0$. One can check that (1) is satisfied if $a+b<1$, which is so, and (2) is satisfied if $a>b$, which is assumed to be so.

in greater detail in an almost identical model. Their analysis makes the important point that, even in the utilitarian framework, a positive rate of social time preference causes the optimal consumption path eventually to turn down. Time discount seems inappropriate with the max-min principle, and I have my doubts that it is defensible anyway on the time-scale appropriate here.

11. EXPONENTIAL POPULATION GROWTH WITH LIMITED RESOURCES

I have already said why I do not consider the assumption of unlimited population growth to be sensible in the present context. For the sake of completeness, however, I will merely point out that the basic constraint equation (7) becomes, when $m = 0$ but $n > 0$,

$$\dot{z} = z^{1-g-h}y^h - nz - c,$$

where z, y and c are expressed per worker in natural units. No positive constant consumption per worker is maintainable forever. To see this, observe that if $z(t)$ is bounded then, because $y(t) \to 0$ eventually, $\dot{z}$ must finally become and remain less than some negative number, so that eventually z itself goes negative. But if $z(t)$ is unbounded, exactly the same is true, because the term $-nz$ must eventually dominate the first term on the RHS. There is no surprise in this.

12. CONSTANT POPULATION WITH UNLIMITED TECHNICAL PROGRESS

Unlimited technological progress may be unlikely, but it is not, like unlimited population growth on a finite planet, absurd. A complete analysis of its implications would be laborious, as one can see from (7) and (10), even with $n = 0$. I shall limit myself here to one simple point. For this purpose I return to the extensive equation (6) in the form

$$\dot{K} = e^{mgt}R^hK^{1-g-h} - C, \qquad \text{...(14)}$$

where I have put $L = 1$ with no loss of generality, since population is constant.

If there were no technical change, we would be back to the case already studied in detail. Suppose C_0 were the maximum maintainable consumption, $J(t)$ the corresponding time path for the capital stock and $R(t)$ the optimal resource flow. Then $J(t)$ satisfies

$$\dot{J} = R^hJ^{1-g-h} - C_0.$$

I want to show that C_0e^{mgt} is a possible consumption path for (14). In other words, the differential equation

$$\dot{K} = e^{mgt}(R^hK^{1-g-h} - C_0)$$

starting from $K(0) = J(0)$ has a non-negative solution for all $t \geqq 0$. Take the same resource-use profile $R(t)$ in both situations. Then, first of all, $\dot{K}(0) = \dot{J}(0)$.

Next, by differentiation with respect to time at $t = 0$, one can calculate that

$$\ddot{K}(0) = mg\dot{K}(0) + \ddot{J}(0).$$

From earlier analysis, we know that $\dot{K}(0) = \dot{J}(0) > 0$. Therefore K exceeds J at least for all t in an interval $0 < t < \bar{t}$.

Finally, after eliminating C_0 between the two differential equations, one can write

$$\dot{K} - \dot{J} = e^{mgt}R^h(K^{1-g-h} - J^{1-g-h}) + (e^{mgt} - 1)\dot{J}.$$

It is known from (12) that $\dot{J} > 0$ always. Thus whenever $K(t) \geqq J(t)$, $\dot{K}(t) > \dot{J}(t)$. Since $K(t)$ starts larger than $J(t)$, it must forever remain larger than $J(t)$. This proves the assertion that C_0e^{gmt} is an admissible consumption path for (14).

Now presumably the problem (14) does admit a largest maintainable constant consumption per head, say C_1, and undoubtedly $C_1 > C_0$. The presence of exponential technological change must permit a permanently higher rate of consumption. Although

I have not proved it, I would guess that when $C = C_1$ and $R(t)$ is optimally chosen, $K(t) \to 0$ as $t \to \infty$. That is to say, society asymptotically consumes its stock of capital as it consumes its pool of resources, relying on technical progress to maintain net output and consumption. But this is far from certain, because in a growing system, any constant capital stock becomes effectively small.

The result proved in this section merely suggests that the Rawlsian criterion may be unsatisfactory when there is limited population but unlimited technical progress. It requires society to choose a constant level of consumption per head when it could have exponentially-growing consumption per head. Even in the absence of social time preference, society must prefer C_1 forever to a history in which consumption is slightly less than C_1 for finite time, between C_1 and $C_2 > C_1$ for finite time, and greater than C_2 for infinite time, where C_2 may be chosen *as large as desired*. That seems rather strong, given the natural asymmetry of time.

13. SUMMARY

Apart from any detail that may be found interesting, there are two main conclusions from the analysis. The first is that the max-min criterion seems to be a reasonable criterion for intertemporal planning decisions except for two important difficulties: (*a*) it requires an initial capital stock big enough to support a decent standard of living, else it perpetuates poverty, but it can not tell us why the initial capital stock should ever have been accumulated; and (*b*) it seems to give foolishly conservative injunctions when there is stationary population and unlimited technical progress.

The second main conclusion is that the introduction of exhaustible resources into this sort of optimization model leads to interesting results—some of which have been sketched—but to no great reversal of basic principles. This conclusion depends on the presumption that the elasticity of substitution between natural resources and labour-and-capital-goods is no less than unity—which would certainly be the educated guess at the moment. The finite pool of resources (I have excluded full recyling) should be used up optimally according to the general rules that govern the optimal use of reproducible assets. In particular, earlier generations are entitled to draw down the pool (optimally, of course!) so long as they add (optimally, of course!) to the stock of reproducible capital.

APPENDIX A

Analysis of equation (4):

Differentiate (4) with respect to time and then substitute from (4) to eliminate ce^{-at}. The result is a second-order equation

$$\ddot{z} = (f'(z)-(2a+n))\dot{z}+a(f(z)-(a+n)z).$$

With the notation $v = \dot{z}$, this becomes the pair of first-order equations

$$\begin{aligned} \dot{z} &= v \\ \dot{v} &= (f'(z)-(2a+n))v+a(f(z)-(a+n)z). \end{aligned} \qquad \ldots(4')$$

It should be carefully remembered that, although every solution of (4) is a solution of (4′), not every solution of the latter is a solution of (4). In particular, (4) and the non-negativity of c guarantees that $\dot{z} \leqq f(z)-(a+n)z$; but (4′) has no way of knowing that.

The phase diagram is as shown. The curve A is the locus $v = f(z)-(a+n)z$. It is a trajectory of the system. This can be seen at once by noticing that it is the solution of (4) with $c = 0$. It can also be verified directly from (4′). The importance of this fact is that since A is a trajectory, no other trajectory can cross it. Any trajectory starting under the curve A must forever remain under it. But the part of the phase plane lying beneath A is precisely the region of non-negative consumption. Thus if we confine ourselves, as we must, to trajectories which start with non-negative consumption we can work solely in terms of (4′) and rest assured that the trajectory will not pass out of the region of non-negative consumption.

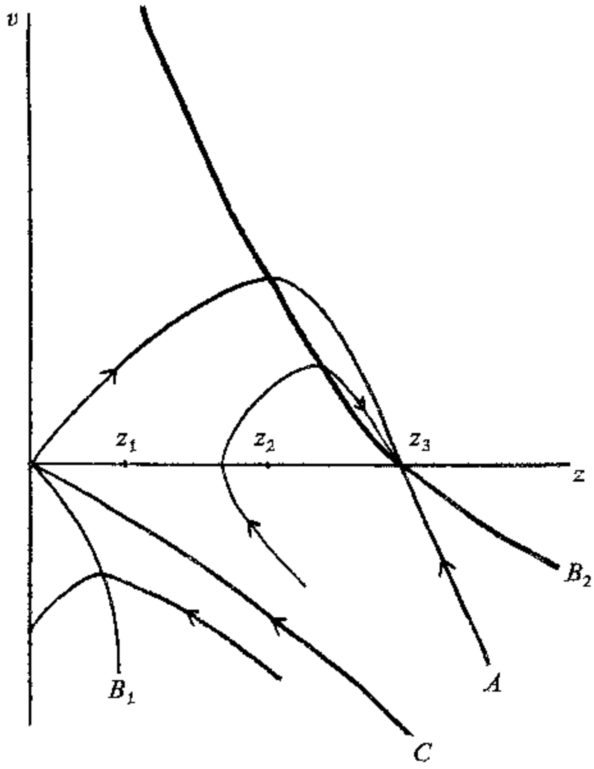

FIGURE 3

There are two singular points: the origin and the point z_3 where A crosses the z-axis. The origin is a saddle-point and z_3 is a stable node. The locus $\dot{z} = 0$ is the z-axis, so trajectories pass through the vertical there. The curves B_1 and B_2 are branches of the locus $\dot{v} = 0$; trajectories pass through the horizontal when they cross B_1 or B_2. B_1 starts at the origin (I am simply ignoring the left half-plane because negative z is meaningless) and falls toward minus infinity at $z = z_1$ (the root of $f'(z) = 2a+n$). B_2 starts at infinity when $z = z_1$ and passes through the maximum point of A and through z_3. Curve C is the separatrix, the trajectory that converges in infinite time to the saddle-point at the origin.

Initially $z = z_0$ is given. A meaningful trajectory must start with $z = z_0$ and a value of v which places the initial point below the curve A. If the initial point is chosen below C, it is evident from the diagram that z goes to zero in finite time. If the initial point is chosen above C but below the z-axis, z decreases at first but eventually turns around and increases, has a point of inflection where it crosses B_2 and tends eventually to z_3 or capital-saturation. If the initial point is above the z-axis the initial decreasing-z phase does not happen; if the initial point is chosen above B_2, there is no inflection and $z(t)$ is concave from the beginning. If the trajectory starts on C, z decreases asymptotically to zero from the very beginning.

It is now only necessary to observe that at time zero $c = f(z)-(a+n)z-v$. That is to say, the constant level of consumption per head is given geometrically by the vertical distance from curve A to the initial point in the phase plane. The largest maintainable consumption per head is achieved when that distance is as large as it can be without the trajectory reaching the v-axis in finite time. Obviously, then, the optimal constant consumption per person is achieved if the trajectory starts on the curve C, at the point corresponding to the given z_0. (As a minor refinement, it is conceivable that, if the initial z were very large, it might be better to throw away some capital at the very beginning. But that is hardly an eventuality worth discussing.)

APPENDIX B

Proof that (7*a*) has no admissible solution if $a<b$:

If $c_0>0$, it follows from (7*a*) that

$$z^{-a}\dot{z} = \frac{d}{dt}\left(\frac{z^{1-a}}{1-a}\right) < y^b,$$

so that

$$\frac{z(t)^{1-a}-z_0^{1-a}}{1-a} < \int_0^t y^b(s)ds = \int_0^t y^b(s)\,.\,1^{1-b}ds.$$

By Hölder's Inequality,

$$\int_0^t y^b(s)\,.\,1^{1-b}ds \leqq \left[\int_0^t y(s)ds\right]^b \left[\int_0^t 1dt\right]^{1-b} < t^{1-b}\left(\frac{\bar{R}}{L}\right)^{1-b}$$

or

$$z(t)^{1-a} < K_0 t^{1-b} + z_0^{1-a}.$$

Thus there is a positive constant K_1 such that

$$z(t)^{1-a} < K_1 t^{1-b} \text{ for all } t \geqq 1.$$

Going back to the differential equation (7*a*), if $t>1$,

$$\dot{z} = z^a y^b - c_0 < [K_1 t^{1-b}]^{a/1-a} y^b - c_0,$$

which implies, after integration, that

$$z(t)-z(1) < K_2 \int_1^t [s^{a/1-a}]^{1-b} y^b(s)ds - c_0(t-1)$$

$$\leqq K_2 \left[\int_1^t y(s)ds\right]^b \left[\int_1^t s^{a/1-a}ds\right]^{1-b} - c_0(t-1)$$

$$< K_2 \left(\frac{\bar{R}}{L}\right)^b \{[1-a)s^{\,1/1-a}]_1^t\}^{1-b} - c_0(t-1)$$

again by Hölder's Inequality.

Therefore

$$z(t) < z(1) - c_0(t-1) + K_3[t^{1/1-a}-1]^{1-b}$$

$$< z(1) - c_0(t-1) + K_3 t^{(1-b)/(1-a)}.$$

But $b>a$ implies $(1-b)/(1-a)<1$; so the linear term dominates and $z(t)<0$ for sufficiently large t.

APPENDIX C

The Ramsey version of the problem asks for the maximum of

$$\int_0^\infty e^{-pt}\frac{c^\theta}{\theta}dt, \quad \theta<1$$

subject to

$$\int_0^\infty y(t)dt=\bar{R}/L_0$$

where $c(t)=z^a y^b-\dot{z}$. Then let $p\to 0$.

Routine methods lead to

$$\dot{z}=z^a y^b-\left(\frac{b}{\lambda}\right)^{\frac{1}{1-\theta}} z^{\frac{a}{1-\theta}} y^{-\left(\frac{1-b}{1-\theta}\right)}$$

$$\dot{y}=-\frac{a}{1-b}\left(\frac{b}{\lambda}\right)^{\frac{1}{1-\theta}} z^{\frac{a}{1-\theta}-1} y^{\frac{b-\theta}{1-\theta}}$$

where λ is a Lagrange multiplier whose numerical value is to be chosen to satisfy the resource constraint.

The trajectories in the phase plane are the integral curves of

$$\frac{dz}{dy}=\frac{\dot{z}}{\dot{y}}.$$

The picture is qualitatively like Figure 2 provided $\theta<0$, which I now assume to be so. Positive θ appears to lead to quite different sorts of behaviour. The locus $z=0$ is given by

$$z=\left(\frac{\lambda}{b}\right)^{\frac{1}{a\theta}} y^{\frac{1-b\theta}{a\theta}}$$

and the optimal trajectory by

$$z=ky^{\frac{1-b\theta}{a\theta}}$$

where

$$k=\left(\frac{-\theta}{1-\theta}\right)^{\frac{1-\theta}{a\theta}}(1-b)^{\frac{1-\theta}{a\theta}}\left(\frac{\lambda}{b}\right)^{\frac{1}{a\theta}}>\left(\frac{\lambda}{b}\right)^{\frac{1}{a\theta}}$$

Along the optimal trajectory, it is easily calculated that

$$y(t)=(y_0^{1-q}+St)^{\frac{1}{1-q}}$$

where S is a positive constant if $q>1$, which is in fact the case. To be precise

$$q=\frac{-(a+b)\theta^2+(1+b)\theta-(1-a)}{a\theta(1-\theta)}.$$

As $\theta\to-\infty$, $q\to(a+b)/a$, $1/(1-q)\to-a/b$ exactly in accord with (13) for the limiting max-min case. It is easily checked that q increases with θ, and $q\to\infty$ as $\theta\to 0$. But it will turn out that not all negative θ are admissible.

We have that

$$\frac{1}{1-q}=\frac{a\theta(1-\theta)}{b\theta^2-(1+b-a)\theta+(1-a)}$$

If $\int_0^\infty y(t)dt$ is to converge, it is necessary and sufficient that $1/(1-q)<-1$ or $q<2$. This will be the case if $\theta<-(1-a)/(a-b)$.

From the definition,

$$c=\left(\frac{b}{\lambda}\right)^{\frac{1}{1-\theta}} k^{\frac{a}{1-\theta}} y^{\frac{1}{\theta}}.$$

Since $y(t)\to 0$, $c(t)\to\infty$. But notice that c^θ is proportionate to y, so the utility integral converges.

Finally, the ratio of consumption to output turns out to be $(b/\lambda)^{\frac{1}{1-\theta}} k^{\frac{a\theta}{1-\theta}}$ and this, using the definition of k, is simply $-\theta(1-b)/(1-\theta)$. Since $\theta<0$, the ratio of consumption to output is smaller than $(1-b)$, which is its limiting value in the max-min case. In other words, the optimal saving ratio rises from b to a as θ rises from $-\infty$ to its upper limit $-(1-a)/(a-b)$.

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