Introduction

In multivariate polynomial interpolation theory, the properties of polynomial interpolants depend very much on the configuration of the interpolation points in space. An important class of such configurations is made up by the generalized principal lattices, which form a corner stone in the classification of the meshes with simple Lagrange formula and can be viewed as a generalization of the triangular meshes. It is known that every cubic curve in the plane can be used to construct a generalized principal lattice in the plane. In [2, 3], Carnicer and Gasca developed the idea to unify the three linear pencils in a plane.

The above triangular mesh can be constructed by taking the points of triple intersection of the families of 3 lines. In [2, 3], Carnicer and Gasca showed that the lines

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For any choice of real function values \( f_{ij} \) at the points \( S \), these lines immediately yield the Lagrange interpolating polynomial

\[
L(x) = \sum_{i+j \leq m} f_{ij} L_{ij}(x),
\]

where the Lagrange polynomials \( L_{ij} \) are defined as

\[
L_{ij}(x,y) = \frac{\prod_{k=0}^{i-1} x - k}{\prod_{k=0}^{i-1} (x - y - k)} \cdot \frac{\prod_{k=0}^{j-1} y - k}{\prod_{k=0}^{j-1} (x - y - k)}.
\]

Within each family, the lines are parallel and therefore meet at a point at infinity. In [6], Lee and Phillips generalized the triangular meshes to 3-pencil lattices, for which the defining lines in each family do not need to be parallel, but have to meet in some point in the projective plane \( \mathbb{P}^2 \).

From triangular meshes to generalized principal lattices

Triangular meshes have been studied extensively in the literature, appearing as early as 1903 in the work of Otto Biermann [1]. A triangular mesh of degree \( m \) is a mesh of points in the plane that is, after scaling by an appropriate factor, of the form

\[
x \geq 0, \quad y \geq 0, \quad x + y \leq m.
\]

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Definition (generalized principal lattice in \( \mathbb{P}^2 \))

Let \( m \geq 0 \), \( n \geq 2 \), and consider \( n + 1 \) families of \( m + 1 \) hyperplanes in \( \mathbb{P}^n \):

\[
H_{ij}^1, H_{ij}^2, \ldots, H_{ij}^{m+1},
\]

for which any two of the \((n+1)(m+1)\) hyperplanes are distinct. Suppose that, for \( 0 \leq i < \cdots < n \leq m \), every intersection \( H_{ij}^1 \cap \cdots \cap H_{ij}^{m+1} \) consists of precisely one point. Moreover, suppose that the intersection \( H_{ij}^1 \cap \cdots \cap H_{ij}^{m+1} \neq \emptyset \) whenever \( i_0 + \cdots + i_k = \emptyset \).

The set

\[
S = \{ (i_0, \ldots, i_k) : H_{ij}^1 \cap \cdots \cap H_{ij}^{m+1} = \{x_{i_0, \ldots, i_k}\}, \quad i_0 + \cdots + i_k = m \}
\]

is a generalized principal lattice of degree \( m \) in \( \mathbb{P}^n \), if, for any \( i_0, \ldots, i_k \in \{0, 1, \ldots, m\} \),

\[
H_{ij}^1 \cap \cdots \cap H_{ij}^{m+1} \cap S = \emptyset \quad \implies \quad i_0 + \cdots + i_k = m.
\]

Classification of planar generalized principal lattices

Carnicer and Godés showed that the lines \( \{H_i\} \) that define a generalized principal lattice in the plane always correspond to points in a cubic curve in the dual plane [6]. Defining, on the set of nonsingular points of any cubic curve, a group law that encodes collinearity of its points, Carnicer and Gasca obtained a classification of all generalized principal lattices in the plane [4].

\begin{tabular}{|c|c|c|}
\hline
Type of cubic & Type of mesh & GPL group \\
\hline
elliptic, two components & tangents to a conic with three real cusps and two components & \( S \times \mathbb{Z}^2 \) \\
elliptic, one component & tangents to a conic with three real cusps and one component & \( S \) \\
irreducible, cuspidal singularity & tangents to a semicubical parabola & \( R \) \\
irreducible, nodal singularity & tangents to a triscubical quartic with one real cusp & \( R \times \mathbb{Z}^2 \) \\
irreducible, isolated point & tangents to a triscubical quartic with three real cusps & \( S \) \\
union of a conic and a septic & tangents to a conic and lines through a vertex inside this conic & \( R \times \mathbb{Z}^2 \times \mathbb{Z}^2 \) \\
union of a conic and a tangent & tangents to a conic and lines through a vertex outside this conic & \( R \times \mathbb{Z}^2 \times \mathbb{Z}^2 \) \\
disjoint union of a conic and a line & tangents to a conic and lines through a vertex inside a conic & \( S \times \mathbb{Z}^2 \) \\
union of 3 generic lines & 3-pencil lattice & \( R \times \mathbb{Z}^2 \times \mathbb{Z}^3 \) \\
union of 3 concurrent lines & principal lattice & \( R \times \mathbb{Z}^3 \) \\
\hline
\end{tabular}

New results on generalized principal lattices in \( \mathbb{P}^3 \)

Similar to the planar case, the hyperplanes \( \{H_i\} \) in each example in [5] correspond to points of a curve in \( \mathbb{P}^3 \) of degree 4 and arithmetic genus 1. In [9], it is shown that the nonsingular points of any cubic curve \( C \subseteq \mathbb{P}^3 \) that is the complete intersection of two quadrics – except for the elliptic normal curves – can be used to construct generalized principal lattices of arbitrary degree in dual space \( \mathbb{P}^3 \). Moreover, two continuous families of elliptic normal curves with this property were exhibited.

This is accomplished by transforming each curve \( C \) to a normal form by means of a real projectivity, finding parameterizations of the components of \( C \) that together encode coplanarity of the points of \( C \), and apply a theorem from [5].

Example

Let \( \phi : \mathbb{R} \rightarrow C_{nm}, \ t \rightarrow [1 : t : t^2 : t^3] \) be a parametrization of the nonsingular points \( C_{nm} \) of a cuspidal quartic \( C \subseteq \mathbb{P}^3 \). Four distinct points \( \phi(t_0), \phi(t_1), \phi(t_2), \phi(t_3) \) are coplanar if and only if \( t_0 + t_1 + t_2 + t_3 = 0 \). Let \( t_0, t_1, t_2, t_3, \delta \in \mathbb{R} \) be such be such that \( t_0 + t_1 + t_2 + t_3 + \delta m = 0 \) and that

\[
p (t_0, t_1, t_2, t_3, \delta) \equiv \phi(t_0 + t_1), \quad m = 1, 0, 1, 2, 3 \quad \text{are distinct.}
\]

Then the hyperplanes \( H_i \subseteq \mathbb{P}^3 \) dual to the points \( p_i \) define a generalized principal lattice of degree \( m \) in \( \mathbb{P}^3 \).

References
