We consider the commutative $S$-algebra given by the topological cyclic homology of a point. The induced Dyer–Lashof operations in mod $p$ homology are shown to be non-trivial for $p = 2$, and an explicit formula is given. As a part of the calculation, we are led to compare the fixed point spectrum $S^G$ of the sphere spectrum and the algebraic $K$-theory spectrum of finite $G$-sets, as structured ring spectra.

55S12; 19D55, 55P43, 55P92

Introduction

Let $A(\ast) = K(\mathbb{S})$ denote Waldhausen’s algebraic $K$-theory of a point [23]. It is a commutative $S$-algebra, in the sense of Elmendorf, Kriz, Mandell and May [7], and the algebraic $K$-theory $A(X)$ of any space $X$, or more generally the algebraic $K$-theory $K(R)$ of any $S$-algebra $R$, is a module spectrum over it. Hence it makes sense to carefully study the commutative $S$-algebra structure of $A(\ast)$, or equivalently its structure as an $E_\infty$ ring spectrum. To the eyes of mod $p$ homology, the primary incarnation of this structure is the Pontryagin algebra structure on $H_*(A(\ast))$, together with the multiplicative Dyer–Lashof operations $Q^i: H_*(A(\ast)) \to H_{*+i}(A(\ast))$, as defined by Bruner, May, McClure and Steinberger [2]. Here and elsewhere we write $H_*(E)$ for the mod $p$ homology $H_*(E; \mathbb{F}_p)$ of a spectrum $E$.

The additive structure of $H_*(A(\ast))$ is known for $p = 2$ and for $p$ an odd regular prime, by the second author’s papers [18] and [19], but at present the Pontryagin product and Dyer–Lashof operations are not known for this $E_\infty$ ring spectrum. There is, however, a very good approximation to Waldhausen’s algebraic $K$-theory, given by the cyclotomic trace map to the topological cyclic homology of Bökstedt, Hsiang and Madsen [1]. This is a natural map $trc: K(R) \to TC(R; p)$, which we write as $trc: A(\ast) \to TC(\ast; p)$.
in the special case when \( R = \mathbb{S} \), where \( TC(\ast; p) = TC(\mathbb{S}; p) \) is the topological cyclic homology of a point. By a theorem of Dundas [5], there is a homotopy cartesian square
\[
\begin{array}{ccc}
A(\ast) & \longrightarrow & K(\mathbb{Z}) \\
\downarrow_{trc} & & \downarrow_{trc} \\
TC(\ast; p) & \longrightarrow & TC(\mathbb{S}; p)
\end{array}
\]
(after \( p \)-adic completion) of commutative \( \mathbb{S} \)-algebras [9, Sect. 6], and this square is the basis for our additive understanding of \( H_\ast(A(\ast)) \).

We are therefore led to study the commutative \( \mathbb{S} \)-algebra structure of \( TC(\ast; p) \), including the Pontryagin algebra structure and the Dyer–Lashof operations on its mod \( p \) homology. Like in the case of algebraic \( K \)-theory, the topological cyclic homology \( TC(X; p) \) of any space \( X \), and more generally the topological cyclic homology \( TC(R; p) \) of any \( \mathbb{S} \)-algebra \( R \), is a module spectrum over \( TC(\ast; p) \), and this provides a second motivation for the study of \( TC(\ast; p) \). In the present paper, we determine the Dyer–Lashof operations in \( H_\ast(\ast; p) \) in the case when \( p = 2 \), as explained in Theorem 0.2 and Corollary 0.3 below.

A third motivation stems from ideas of Jack Morava [17], to the effect that there may be a spectral enrichment of the algebro-geometric category of mixed Tate motives, given by \( A \)-theoretic [24] or \( TC \)-theoretic [6] correspondences, followed by stabilization. The trace map \( A(\ast) \rightarrow TC(\ast; p) \rightarrow THH(\ast) = \mathbb{S} \) defines a fiber functor to the category of \( \mathbb{S} \)-modules, with Tannakian automorphism group realized through its Hopf algebra of functions, which will be of the form \( \mathbb{S} \wedge_{A(\ast)} \mathbb{S} \) or \( \mathbb{S} \wedge_{TC(\ast; p)} \mathbb{S} \). Rationally, this is well compatible with Deligne’s results on the Tannakian group of mixed Tate motives over the integers [4]. A calculational analysis of the commutative \( \mathbb{S} \)-algebras \( \mathbb{S} \wedge_{A(\ast)} \mathbb{S} \) or \( \mathbb{S} \wedge_{TC(\ast; p)} \mathbb{S} \) clearly depends heavily on a proper understanding of the commutative \( \mathbb{S} \)-algebra structures of \( A(\ast) \) and \( TC(\ast; p) \).

Let \( \mathbb{T} \) be the circle group and let \( C_{p^n} \subset \mathbb{T} \) be the (cyclic) subgroup of order \( p^n \). The spectrum \( TC(\ast; p) \) is defined as the homotopy inverse limit of a diagram
\[
\cdots \xrightarrow{R} \mathbb{S}^{C_{p^{n+1}}} \xrightarrow{R} \mathbb{S}^{C_{p^n}} \xrightarrow{R} \cdots \xrightarrow{R} \mathbb{S}^{C_2} \xrightarrow{R} \mathbb{S}
\]
of \( E_\infty \) ring spectra, where \( \mathbb{S}^{C_{p^n}} \) denotes the \( C_{p^n} \)-fixed points of the \( \mathbb{T} \)-equivariant sphere spectrum, the maps labeled \( R \) are restriction maps, and the maps labeled \( F \) are Frobenius maps. See Bökstedt, Hsiang and Madsen [1] or Hesselholt and Madsen [10] for the construction of these maps. Similarly, let \( TC^{(1)}(\ast; p) \) denote the homotopy limit of the subdiagram
\[
\cdots \xrightarrow{R} \mathbb{S}^{C_{p^n}} \xrightarrow{R} \mathbb{S}
\]
that is, the homotopy equalizer of $R$ and $F$. The canonical maps

\[(0–3) \quad TC(\ast;p) \xrightarrow{f_1} TC^{(1)}(\ast;p) \xrightarrow{g_1} \mathbb{S}_p\]

are then maps of $E_\infty$ ring spectra.

The unit $\eta: \mathbb{S} \to TC(\ast;p)$ and the restriction $R: \mathbb{S}_p \to \mathbb{S}$ let us split off a copy of $\mathbb{S}$ from each term in $(0–3)$. Let $\mathbb{C}P^\infty_-$ be the Thom spectrum of the negative tautological complex line bundle $-\gamma^1_0$ over $\mathbb{CP}^\infty$. Its suspension $\Sigma\mathbb{C}P^\infty_-$ is equivalent to the homotopy fiber of the dimension-shifting $T$-transfer map $t_T: \Sigma^\infty(\mathbb{CP}^\infty) \to \mathbb{S}$, see Knapp [13, 2.9] or Lemma 1.1 below. We define the spectrum $L^\infty_1$ to be the homotopy fiber of the $C_p$-transfer $t_p: \Sigma^\infty(BC_p)_+ \to \mathbb{S}$. For $p = 2$, there is an equivalence $L^\infty_1 \simeq \mathbb{RP}^\infty_1$, where $\mathbb{RP}^\infty_1$ is the Thom spectrum of the negative tautological real line bundle $-\gamma^1_1$ over $\mathbb{RP}^\infty$. The mod $p$ homology groups of these spectra are well known:

\[
\begin{align*}
H_*(\Sigma\mathbb{C}P^\infty_-) & \cong \mathbb{F}_p\{\beta_k \mid k \geq -1\} \\
H_*(L^\infty_1) & \cong \mathbb{F}_p\{\alpha_k \mid k \geq -1\} \\
H_*(\Sigma^\infty(BC_p)_+) & \cong \mathbb{F}_p\{\alpha_k \mid k \geq 0\}
\end{align*}
\]

Here $\beta_k$ has degree $2k + 1$ and $\alpha_k$ has degree $k$.

**Lemma 0.1** After $p$-completion, diagram $(0–3)$ is homotopy equivalent to a diagram

\[
\mathbb{S} \vee \Sigma\mathbb{C}P^\infty_- \xrightarrow{1\vee f} \mathbb{S} \vee L^\infty_1 \xrightarrow{1\vee g} \mathbb{S} \vee \Sigma^\infty(BC_p)_+.
\]

In particular, the Pontryagin product on $H_*(TC(\ast;p))$ is trivial.

Applying homology gives a sequence

\[
H_*(\mathbb{S}) \oplus H_*(\Sigma\mathbb{C}P^\infty_-) \xrightarrow{1\oplus f_*} H_*(\mathbb{S}) \oplus H_*(L^\infty_1) \xrightarrow{1\oplus g_*} H_*(\mathbb{S}) \oplus H_*(\Sigma^\infty(BC_p)_+).
\]

Here $f_*$ sends $\beta_k$ to $\alpha_{2k+1}$ for $k \geq -1$, and $g_*$ is the identity on $\alpha_k$ for $k \geq 0$, while $\alpha_{-1}$ maps to zero.

We now state our main result, which concerns the Dyer–Lashof operations in the mod $p$ spectrum homology $H_*(TC(\ast;p))$ for $p = 2$. The calculations will be done in the auxiliary $E_\infty$ ring spectra $\mathbb{S}_{C^2}$ and $TC^{(1)}(\ast;2)$.

**Theorem 0.2** The Dyer–Lashof operations $Q^i$ in $H_*(TC^{(1)}(\ast;2))$ and $H_*(\mathbb{S}_{C^2})$ are given by the formula

\[
Q^i(\alpha_j) = \binom{2N + i - 1}{2N + j} \alpha_{i+j},
\]

where $j \geq -1$ and $i$ is any integer, and $N$ is sufficiently large.
Corollary 0.3  The Dyer–Lashof operations $Q^{2i}$ in $H_*(TC(\star; 2))$ are given by the formula

$$Q^{2i}(\Sigma\beta_j) = \binom{2N + i - 1}{2N + j} \Sigma\beta_{i+j},$$

where $j \geq -1$ and $i$ is any integer, and $N$ is sufficiently large. The operations $Q^{2i+1}$ are all zero for degree reasons.

Note that the binomial coefficients used in the theorem and corollary can be evaluated to

$$\binom{2N + i - 1}{2N + j} \equiv \begin{cases} 
(i-j) & \text{for } i > j \geq 0 \\
1 & \text{for } (i, j) = (0, -1) \\
0 & \text{otherwise}
\end{cases}$$

modulo 2, for all sufficiently large $N$. In particular $Q^0(\alpha_{-1}) = \alpha_{-1}$ and $Q^0(\Sigma\beta_{-1}) = \Sigma\beta_{-1}$.

We prove Lemma 0.1 in Section 1 and Theorem 0.2 in Section 3, after a homological comparison of $E_\infty$ ring structures in Section 2. Corollary 0.3 follows immediately from the lemma and the theorem.

1 Topological cyclic homology of a point

In this preliminary section we review the calculation of $TC(\star; p)$ from Bökstedt, Hsiang and Madsen [1, 5.17], in order to describe the map $f_1$ to $TC^{(1)}(\star; p)$.

For each $n \geq 1$ the Segal–tom Dieck splitting tells us that the norm–restriction homotopy cofiber sequence

$$\Sigma^\infty(BC_{p^n})_+ \xrightarrow{N} \mathbb{S}^C_{p^n} \xrightarrow{R} \mathbb{S}^C_{p^{n-1}}$$

is canonically split. The homotopy limit $TR(\star; p) = \varprojlim_{n,R} \mathbb{S}^C_{p^n}$ of the $R$-maps in (0–1) thus factors as $TR(\star; p) \simeq \prod_{n \geq 0} \Sigma^\infty(BC_{p^n})_+$. Let $pr_n : TR(\star; p) \to \Sigma^\infty(BC_{p^n})_+$ denote the $n$-th projection, and let $\tilde{TR}(\star; p) \simeq \prod_{n \geq 1} \Sigma^\infty(BC_{p^n})_+$ be the homotopy fiber of $pr_0$. There is a vertical map of horizontal homotopy fiber sequences

$$\begin{array}{ccc}
TC(\star; p) & \xrightarrow{\pi} & TR(\star; p) \\
\downarrow f_1 & & \downarrow p_1 \\
TC^{(1)}(\star; p) & \xrightarrow{g_1} & \mathbb{S}^C_p
\end{array}$$

$$\begin{array}{ccc}
 & & TR(\star; p) \\
\downarrow p_1 & & \downarrow p_{0} \\
 & & \mathbb{S}
\end{array}$$

$$\begin{array}{ccc}
TC^{(1)}(\star; p) & \xrightarrow{g_1} & \mathbb{S}^C_p \\
\downarrow f_1 & & \downarrow F_{-R} \\
TC(\star; p) & \xrightarrow{\pi} & TR(\star; p)
\end{array}$$

where $F_{-R}$ is the map from $\mathbb{S}^C_p$ to $\mathbb{S}$ induced by $\mathbb{S}^C_1$.
and the augmentation $\text{TC}(\ast; p) \to S$ factors as $R \circ p_1 \circ \pi = pr_0 \circ \pi$. Replacing the left hand square by the homotopy fibers of the augmentations to $S$, we get a second vertical map of horizontal homotopy fiber sequences

\[
\begin{array}{ccc}
\widetilde{\text{TC}}(\ast; p) & \longrightarrow & \widetilde{\text{TR}}(\ast; p) \xrightarrow{T-I} \text{TR}(\ast; p) \\
\downarrow f_2 & & \downarrow pr_1 \\
\widetilde{\text{TC}}(1)(\ast; p) & \xrightarrow{g_2} & \Sigma^\infty(\text{BC}_p)_+ \\
\downarrow \text{hofib}(F - 2R) & & \downarrow pr_0 \\
\end{array}
\]

In the upper row we have used that $F - R$ restricted along the inclusion $I: \widetilde{\text{TR}}(\ast; p) \to \text{TR}(\ast; p)$ is homotopic to $T - I$, where $T$ is the product of the $C_p$-transfer maps $\Sigma^\infty(\text{BC}_{p^n})_+ \to \Sigma^\infty(\text{BC}_{p^{n-1}})_+$ for all $n \geq 1$. See [1, (5.18)]. In the lower row we have used that $F - R$ restricted along $N: \Sigma^\infty(\text{BC}_p)_+ \to \Sigma C_p$ is homotopic to the $C_p$-transfer map $t_p$.

There is a third vertical map of horizontal homotopy fiber sequences

\[
\begin{array}{ccc}
\text{holim}_n \Sigma^\infty(\text{BC}_{p^n})_+ & \longrightarrow & \text{TR}(\ast; p) \xrightarrow{T-I} \text{TR}(\ast; p) \\
\downarrow f_3 & & \downarrow pr_1 \\
\text{hofib}(F - 2R) & \xrightarrow{g_3} & \Sigma C_p - F - 2R \\
\downarrow \text{hofib}(F - 2R) & & \downarrow pr_0 \\
\end{array}
\]

Replacing its left hand square by the homotopy fibers of the augmentations to $S$, we also recover diagram (1–2).

**Lemma 1.1** There are equivalences

\[
\widetilde{\text{TC}}(\ast; p) \simeq \text{hofib}(t_T: \Sigma^\infty \Sigma(CP^\infty_+) \to S) \simeq \Sigma CP^\infty_+ \text{\ (after } p\text{-completion)}
\]

and

\[
\widetilde{\text{TC}}(1)(\ast; p) \simeq \text{hofib}(t_p: \Sigma^\infty(\text{BC}_p)_+ \to S) = L^\infty_{-1}.
\]

When $p = 2$, $L^\infty_{-1} \simeq \mathbb{R}P^\infty_{-1}$.

**Proof** The dimension-shifting $T$-transfer maps for the bundles $\text{BC}_{p^n} \to BT$ induce an equivalence $\Sigma^\infty \Sigma(CP^\infty_+) \simeq \text{holim}_n \Sigma^\infty(\text{BC}_{p^n})_+$ after $p$-completion [1, 5.15]. The augmentation $\text{holim}_n \Sigma^\infty(\text{BC}_{p^n})_+ \to S$ then gets identified with $t_T$, which implies the first claim.

There is a $T$-equivariant homotopy cofiber sequence $S^0 \xrightarrow{z} \Sigma \xrightarrow{t} T_+ \wedge S^1$, where $z$ is the zero-inclusion. The right hand map $t$ is the Pontryagin–Thom collapse associated to the standard embedding $T \subset \mathbb{C}$, as in Lewis, May and Steinberger [14, II.5.1]. The
dimension-shifting $T$-transfer $t_T : \Sigma^\infty \Sigma(\mathbb{C}P^\infty_+) \to \mathbb{S}$ for $ET \to BT$ is constructed as the balanced smash product

$$1 \wedge_T \Sigma^{1-C}(t) : ET_+ \wedge_T \Sigma^{1-C}(S^C) \to ET_+ \wedge_T \Sigma^{1-C}(T_+ \wedge S^1),$$

see [14, II.7.5]. Hence its homotopy fiber is $ET_+ \wedge_T \Sigma^{1-C}(S^0) \cong \Sigma \mathbb{C}P^\infty_-$.

The proof that $\mathbb{R}P^\infty_-$ is the homotopy fiber of $t_p$ for $p = 2$ is essentially the same. □

**Proof of Lemma 0.1** Under the identifications of Lemma 1.1, the maps $f : \Sigma \mathbb{C}P^\infty_- \to L^\infty_-$ and $g : L^\infty_- \to \Sigma^\infty(BC_p)_+$ correspond to the maps $f_2$ and $g_2$ in diagram (1–2), respectively.

The $C_p$-transfer map $t_p$ induces multiplication by $p$ on $\pi_0$, and the zero map in mod $p$ homology, so $\pi_{-1}(f)$ is surjective, $f_*$ maps $\Sigma \beta_{-1}$ to $\alpha_{-1}$, and $g_*$ maps $\alpha_k$ to $\alpha_k$ for all $k \geq 0$. It remains to see that $g_3f_*$ maps $\Sigma \beta_k$ to $\alpha_{2k+1}$ for $k \geq 0$. This is clear from diagram (1–3), since $g_3f_3$ agrees in positive degrees with the $T$-transfer map $\Sigma^\infty \Sigma(\mathbb{C}P^\infty_+) \to \Sigma^\infty(BC_p)_+$, which has this behavior on homology. □

## 2 Algebraic $K$-theory of finite $G$-sets

In this section we will compare the algebraic $K$-theory spectrum of finite $G$-sets with the $G$-fixed points of the sphere spectrum, as structured ring spectra. Before we state the result we recall some of the definitions involved.

The $K$-theory construction we use is that of Elmendorf and Mandell [8]. When the input category is a bipermutative category $\mathcal{C}$, their machine produces a symmetric spectrum $K(\mathcal{C})$, in the sense of Hovey, Shipley and Smith [11], with an action of the simplicial Barratt–Eccles operad. We will use the same notation for the geometrically realized symmetric spectrum in topological spaces, which has an action

$$\kappa_j : E\Sigma_j \times_{\Sigma_j} K(\mathcal{C})^{\vee j} \to K(\mathcal{C})$$

of the operad $E\Sigma$ consisting of the contractible $\Sigma_j$-free spaces $E\Sigma_j$. As usual, $E\Sigma_j$ can be defined as the nerve $N\Sigma_j$ of the translation category $\tilde{\Sigma}_j$, for $j \geq 0$. The $K$-theory construction itself is somewhat involved, but all we need to know is that the zero space $K(\mathcal{C})_0$ is the nerve $NC$ of $\mathcal{C}$, so the zeroth space of $E\Sigma_j \times_{\Sigma_j} K(\mathcal{C})^{\vee j}$ is the nerve of $\tilde{\Sigma}_j \times_{\Sigma_j} \mathcal{C}^j$, and the action of $E\Sigma$ on $K(\mathcal{C})_0$ is given by the maps $\lambda_j : E\Sigma_j \times_{\Sigma_j} NC^{\wedge j} \to NC$ that are induced by the functors taking an object $(\sigma ; a_1, \ldots, a_j)$ in $\tilde{\Sigma}_j \times_{\Sigma_j} \mathcal{C}^j$ to the
Homology operations in $TC(\ast; p)$

object $a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(j)}$ in $C$ (see [8, Sect. 8]). Here $\otimes$ denotes the product in the bipermutative structure on $C$. Hence there is a commutative diagram

$$
\begin{array}{ccc}
E\Sigma_j \ltimes_{\Sigma_j} K(C)^{\vee_j} & \xrightarrow{\kappa_j} & K(C) \\
\downarrow & & \downarrow \\
E\Sigma_j \ltimes_{\Sigma_j} (\Sigma^\infty NC)^{\vee_j} & \xrightarrow{\cong} & \Sigma^\infty (E\Sigma_j \ltimes_{\Sigma_j} NC^{\vee_j}) \xrightarrow{\Sigma^\infty \lambda_j} \Sigma^\infty NC
\end{array}
$$

for each $j \geq 0$.

Let $G$ be a finite group, and let $E^G$ denote the category of finite $G$-sets and $G$-equivariant bijections. This is a symmetric bimonoidal category under disjoint union and cartesian product, taking $(X, Y)$ to $X \coprod Y$ and $X \times Y$, respectively. We give $X \times Y$ the diagonal $G$-action. There is a functorially defined bipermutative category $\Phi E^G$, and a natural equivalence $E^G \rightarrow \Phi E^G$ [16, VI.3.5]. It follows that there is a homotopy commutative diagram

$$
(2-1)
\begin{array}{ccc}
E\Sigma_j \ltimes_{\Sigma_j} K(\Phi E^G)^{\vee_j} & \xrightarrow{\kappa_j} & K(\Phi E^G) \\
\downarrow 1 \otimes \epsilon^{\vee_j} & & \downarrow \epsilon \\
E\Sigma_j \ltimes_{\Sigma_j} (\Sigma^\infty NE^G)^{\vee_j} & \xrightarrow{\cong} & \Sigma^\infty (E\Sigma_j \ltimes_{\Sigma_j} (NE^G)^{\vee_j}) \xrightarrow{\Sigma^\infty \lambda_j} \Sigma^\infty NE^G
\end{array}
$$

for each $j \geq 0$, where

$$
\lambda_j: E\Sigma_j \ltimes_{\Sigma_j} (NE^G)^{\vee_j} \rightarrow NE^G
$$

is induced by the functor $\tilde{\Sigma}_j \ltimes_{\Sigma_j} (E^G)^{\vee_j} \rightarrow E^G$ that takes $(\sigma; X_1, \ldots, X_j)$ to the cartesian product

$$
X_{\sigma^{-1}(1)} \times (X_{\sigma^{-1}(2)} \times \cdots \times (X_{\sigma^{-1}(j-1)} \times X_{\sigma^{-1}(j)}) \ldots).
$$

Let $U$ be a complete $G$-universe, and let $L$ denote the linear isometries operad with spaces $L(j)$ consisting of linear isometries $U^j \rightarrow U$, where $U^j$ denotes the direct sum of $j$ copies of $U$. There is an action of $G$ on each $L(j)$ given by conjugation, and this gives $L$ the structure of an $E_\infty$ $G$-operad in the sense of Lewis, May and Steinberger [14, VII.1.1]. The $E_\infty$ ring structure on the $G$-equivariant sphere spectrum $S^G = \Sigma^\infty S^0$ is given by an action

$$
\zeta_j: L(j) \ltimes_{\Sigma_j} S^G \rightarrow S^G
$$

of this operad (where, for once, $\ltimes$ denotes the twisted half-smash product in Lewis–May spectra). It is compatible with a corresponding action

$$
\omega_j: L(j) \ltimes_{\Sigma_j} QG(S^0)^{\vee_j} \rightarrow QG(S^0)
$$
on the underlying infinite loop space \( Q_G(S^0) = \Omega^\infty S_G = \text{colim}_{V \subseteq \mathcal{U}} \Omega^V S^V \), in the sense that the following diagram commutes.

\[
\begin{array}{cccccc}
\mathcal{L}(j) \times_{\Sigma_j} (\Sigma^\infty Q_G(S^0))^{\wedge j} & \xrightarrow{\cong} & \Sigma^\infty (\mathcal{L}(j) \times_{\Sigma_j} Q_G(S^0)^{\wedge j}) & \xrightarrow{\Sigma^\infty \omega_j} & \Sigma^\infty Q_G(S^0) \\
1 \times e & & & & e \\
\mathcal{L}(j) \times_{\Sigma_j} S_G^{\wedge j} & \xrightarrow{\zeta} & S_G
\end{array}
\]

Here \( \omega_j \) sends an element in \( \mathcal{L}(j) \times_{\Sigma_j} Q_G(S^0)^{\wedge j} \) represented by \((f; g_1, \ldots, g_i)\), where \( f : \mathcal{U} \to \mathcal{U} \) and \( g_i : S^V \to S^V \), to the element represented by the composite of the following maps.

\[
S^V(V_1 \oplus \cdots \oplus V_j) \xleftarrow{f_*} S^V(V_1 \oplus \cdots \oplus V_j) \xrightarrow{g_1 \wedge \cdots \wedge g_i} S^V(V_1 \oplus \cdots \oplus V_j) \xrightarrow{f_*} S^V(V_1 \oplus \cdots \oplus V_j)
\]

By taking \( G \)-fixed points we get the non-equivariant \( E_\infty \) ring spectrum \( S^G = (S_G)^G \) with an action

\[
\xi_j : \mathcal{L}^G(j) \times_{\Sigma_j} (S^G)^{\wedge j} \to S^G
\]

of the non-equivariant \( E_\infty \) operad \( \mathcal{L}^G \) of \( G \)-equivariant isometries. The corresponding infinite loop space \( \Omega^\infty(S^G) \) is the space \( Q_G(S^0)^G = \text{colim}_{V \subseteq \mathcal{U}} (\Omega^V S^V)^G \), with the inherited \( \mathcal{L}^G \)-action

\[
\eta_j : \mathcal{L}^G(j) \times_{\Sigma_j} (Q_G(S^0)^G)^{\wedge j} \to Q_G(S^0)^G.
\]

Next we recall the definition of the Dyer–Lashof operations \( Q^i \). Let \( C_*(-) \) denote the cellular chains functor, from either CW complexes or CW spectra to chain complexes. Let \( E \) be a spectrum with an action of an \( E_\infty \) operad \( \mathcal{O} \), and let \( W_\ast \) be the standard free \( C_\ast \)-resolution of \( \mathbb{F}_p \) with basis elements \( e_i \) in degree \( i \). There is a chain map \( W_\ast \to C_\ast(\mathcal{O}(p)) \) lifting the identity on \( \mathbb{F}_p \), unique up to homotopy, and we also denote the image of \( e_i \) under this map by \( e_i \). Let \( x \in H_q(E) \) be represented by a cycle \( z \in C_q(E) \). Now consider the image of the cycle \( e_i \otimes z^{\otimes p} \) under the map

\[
C_\ast(\mathcal{O}(p)) \otimes_{C_\ast(p)} C_\ast(E)^{\otimes p} \xrightarrow{\cong} C_\ast(\mathcal{O}(p) \times_{\Sigma_p} E^{\wedge p}) \xrightarrow{\xi_p} C_\ast(E),
\]

and denote its image in homology by \( Q_i(x) \). Here \( \xi_p \) is the \( E_\infty \) structure map. Then for \( p = 2 \) define \( Q^i(x) = 0 \) when \( i < q \), and

\[
Q^i(x) = Q_{i-q}(x)
\]

when \( i \geq q \). For \( p > 2 \) define \( Q^i(x) = 0 \) when \( 2i < q \), and

\[
Q^i(x) = (-1)^i \nu(q) \cdot Q_{(2i-q)(p-1)}(x)
\]
Homology operations in \( TC(*; p) \)

when \( 2i \geq q \), where \( \nu(q) = (-1)^{q-1} (p-1)^{q-1} (\frac{1}{2}(p-1)!)^q \). See Bruner, May, McClure and Steinberger [2, Ch. III] for more details.

The spectra \( S^G \) and \( K(\Phi E^G) \) should be equivalent as \( E_\infty \) ring spectra, but we will only need the following weaker result.

**Lemma 2.1** There is an equivalence \( S^G \simeq K(\Phi E^G) \) of spectra such that the induced isomorphism \( H_* (S^G) \sim H_* (K(\Phi E^G)) \) commutes with the Dyer–Lashof operations.

**Proof** Our first goal is to construct a commutative diagram

\[
\begin{align*}
E \Sigma_j \times \Sigma_j (N E^G)^{\wedge j} & \xrightarrow{\sim} (E \Sigma_j \times L G(j)) \times \Sigma_j D_U^{\wedge j} \xrightarrow{\sim} D_U \xrightarrow{\sim} N E^G_p \\
E \Sigma_j \times \Sigma_j (N E^G)^{\wedge j} & \xrightarrow{\sim} (E \Sigma_j \times L G(j)) \times \Sigma_j C_U^{\wedge j} \xrightarrow{\sim} C_U \xrightarrow{\sim} Q G(S^0)^G.
\end{align*}
\]

We start by describing the space \( C_U \). Let \( V \) be an indexing space in \( U \). For each finite \( G \)-set \( X \), consider the space \( E_V(X) \) of \( X \)-tuples of distance-reducing embeddings of \( V \) in \( V \), closed under the action of \( G \). More precisely, this is the space of \( G \)-equivariant maps \( \coprod_X V \to V \) such that the restriction to each summand \( g: V \to V \) is an embedding that satisfies \( |g(v) - g(w)| \leq |v - w| \) for all \( v, w \in V \). Let \( K_V(X) \) be the space of paths \( [0, 1] \to E_V(X) \) such that the embeddings at the endpoint 0 are identities, and the embeddings at 1 have disjoint images. Now let

\[
K_U(X) = \text{colim}_{V \subseteq U} K_V(X).
\]

These are \( G \)-equivariant versions of the spaces in the Steiner operad [22]. The group \( \text{Aut}^G(X) \) acts freely on \( K_U(X) \) by permuting the embeddings, and the space \( C_U \) is to be the disjoint union

\[
C_U = \coprod_{[X]} K_U(X) / \text{Aut}^G(X)
\]

where \( X \) ranges over all isomorphism classes of finite \( G \)-sets.

The action of the operad \( L G \) on \( C_U \) is defined as follows. Let \( f: U^i \to U \) be a \( G \)-linear isometry, and let \([g_i], 1 \leq i \leq j\), be elements in \( C_U \), represented by paths
of $X_i$-tuples of embeddings $g_i \in K_U(X_i)$. Denote the component paths of embeddings that constitute $g_j$ by $g_{i,x_i}$, where $x_i \in X_i$. The resulting element $\nu_j(f;[g_1],\ldots,[g_j])$ in $C_U$ is represented by an element in $K_U(X_1 \times \cdots \times X_j)$, which on the summand indexed by $(x_1,\ldots,x_j)$ is given by $f \circ (g_{1,x_1} \times \cdots \times g_{j,x_j}) \circ f^{-1}$.

There is a map $C_U \to Q_G(S^0)^G$, given by evaluating a Steiner path in $E_V(X)$ at 1 to get a $G$-equivariant embedding $e : \prod_X V \to V$, and then applying a folded Pontryagin–Thom construction to obtain a $G$-equivariant map $q : S^V \to S^V$, which is a point in $Q_G(S^0)^G$. Given the distance-reducing embedding $e$, let $S^V \to \bigvee_X S^V$ be the $G$-equivariant map that is given by $e^{-1}$ on the image of $e$ in $V \subset S^V$ and maps the remainder of $S^V$ to the base point of $\bigvee_X S^V \supset \prod_X V$. Let $\bigvee_X S^V \to S^V$ be the fold map that is the identity on each summand. The folded Pontryagin–Thom construction of these two $G$-maps. If we permute the embeddings indexed by $X$ we get the same element in $Q_G(S^0)^G$, so our map is well-defined. A comparison of definitions shows that this construction is compatible with the $L^G$-actions on $C_U$ and $Q_G(S^0)^G$, so the lower square in (2–3) commutes.

Let

$$D_U = \prod_{[X]} (E \text{Aut}^G(X) \times K_U(X))/\text{Aut}^G(X),$$

where $\text{Aut}^G(X)$ acts diagonally on the product. The nerve $N\mathcal{E}^G$ splits as a sum of components

$$(2–4) \quad N\mathcal{E}^G \simeq \prod_{[X]} B\text{Aut}^G(X),$$

where the disjoint union is over the isomorphism classes of $G$-sets $X$. Projection on the first factor in $D_U$ followed by this homotopy equivalence gives the map $D_U \to N\mathcal{E}^G$ in (2–3), while the map $D_U \to C_U$ is the projection on the second factor. There is an induced action of the product operad $E\Sigma \times L^G$ on $D_U$, defined as follows. Let $(e,f) \in E\Sigma_j \times L^G(j)$ and let $(e_i,f_i) \in E\text{Aut}^G(X) \times K_U(X)$ represent elements in $D_U$, for $1 \leq i \leq j$. The image under $\mu_{ij}$ is the element represented by $(\lambda_j(e;e_1,\ldots,e_j),\nu_j(f;f_1,\ldots,f_j))$. This makes the upper and middle squares in (2–3) commute.

Let

$$N\mathcal{E}^G_{tr} = \left( \prod_{(H)} (E \text{Aut}^G(G/H) \times K_U(G/H))/\text{Aut}^G(G/H) \right)_{+},$$

where the coproduct is taken over the conjugacy classes of subgroups $H$ of $G$. The map $N\mathcal{E}^G_{tr} \to D_U$ in (2–3) is the inclusion of the components indexed by the isomorphism classes of transitive $G$-sets.
The maps $\phi$ and $\psi$ are defined by commutativity of the right hand triangles in the diagram. We claim that the adjoints

\[
\Sigma^\infty NE^G_{tr} \to K(E^G) \simeq K(\Phi E^G)
\]

\[
\Sigma^\infty NE^G_{tr} \to S^G
\]

of the maps $\phi$ and $\psi$, respectively, are both equivalences. Here $K(E^G)$ is the additive $K$-theory spectrum of $E^G$, with zeroth space $K(E^G)_0 = NE^G$, which only depends on the additive symmetric monoidal structure of $E^G$. There is an equivalence

\[
\Sigma^\infty NE^G_{tr} \simeq \bigvee_{(H)} \Sigma^\infty BW_G H_+,
\]

where $W_G H = NG_H/H \cong Aut(G/H)$ is the Weyl group of $H$ and the wedge sum is over the conjugacy classes of subgroups of $G$. By Waldhausen’s additivity theorem \[23, 1.3.2\] applied to a suitable filtration of $E^G$ according to stabilizer types, there is a splitting

\[
K(E^G) \simeq \bigvee_{(H)} K(E(W_G H)),
\]

where $E(W_G H)$ is the category of finite free $W_G H$-sets and equivariant bijections. The map (2–5) is equivalent under these identifications to the wedge sum of the maps

\[
\Sigma^\infty BW_G H_+ \to K(E(W_G H))
\]

that are left adjoint to the inclusions $BW_G H_+ \to NE(W_G H) = K(E(W_G H))_0$.

The Barratt–Priddy–Quillen–Segal theorem \[20, 3.6\] says that each of the maps (2–7) is an equivalence, hence (2–5) is an equivalence. The map (2–6) is an equivalence by the Segal–tom Dieck splitting \[14, V.11.2\]. The composition of these two equivalences is the equivalence $S^G \simeq K(\Phi E^G)$ referred to in the statement of the lemma.

We apply the suspension spectrum functor $\Sigma^\infty$ to the diagram (2–3), combine it with diagram (2–1) and the $G$-fixed part of (2–2), take homology, and end up with the following commutative diagram.

\[
\begin{array}{ccc}
H_* (\Sigma j; H_*(\Phi E^G) \otimes j) & \xrightarrow{\kappa_*} & H_* (K(\Phi E^G)) \\
\downarrow \cong & & \downarrow \cong \\
H_* (\Sigma j; H_*(NE^G_{tr}) \otimes j) & \xrightarrow{\eta_*} & H_* (D(\Sigma j)) \\
\downarrow \cong & & \downarrow \cong \\
H_* (\Sigma j; H_*(S^G) \otimes j) & \xrightarrow{\xi_*} & H_* (S^G)
\end{array}
\]
We need the fact that \( \epsilon_1 \) and \( \epsilon_2 \) have the same kernel. In fact, all summands in \( \tilde{H}_*(D_{(k)}) \) indexed by \( G \)-sets with more than one orbit map to zero under both \( \epsilon_1 \) and \( \epsilon_2 \). This follows from the fact that Pontryagin products and additive Dyer–Lashof operations vanish after stabilization. More precisely, a decomposition of a \( G \)-set \( X = \prod_{i=1}^k n_i(G/H_i) \), where the \( H_i \) lie in distinct conjugacy classes, induces a factorization

\[
B \text{Aut}^G(X) \cong \prod_{i=1}^k B(S_{n_i}(\Sigma; W_GH_i)).
\]

The homology group \( H_*(B(S_{n_i}(\Sigma; W_GH_i)) \subset H_*(N\mathcal{E}^G) \) is generated by \( H_*(BW_GH_i) \) under iterated Pontryagin products and Dyer–Lashof operations, see Cohen, Lada and May \[3, 1.4.1\], which all map to zero under \( \epsilon_1 \) and \( \epsilon_2 \) unless \( k = 1 \) and \( n_1 = 1 \).

Let \( x \in H_*(K(\Phi \mathcal{E}^G)) \), and let \( y \in H_*(S^G) \) be the element corresponding to \( x \) under \( \psi_* \circ \phi_*^{-1} \), via an element \( z \in H_*(N\mathcal{E}^G) \). We need to show that the image \( Q_i(x) \) of \( e_i \otimes \bar{x}^{\otimes p} \) under the top map corresponds, via the isomorphism, to the image \( Q_i(y) \) of \( e_i \otimes \bar{y}^{\otimes p} \) under the bottom map. The element \( e_i \otimes \bar{z}^{\otimes p} \in H_*(\Sigma_p; H_*(N\mathcal{E}^G)^{\otimes p}) \) maps to an element \( Q_i(z) \in \tilde{H}_*(D_{(k)}) \), which further maps to \( Q_i(x) \) and \( Q_i(y) \) under \( \epsilon_1 \) and \( \epsilon_2 \), respectively. Let \( w \in H_*(N\mathcal{E}^G) \) map to \( Q_i(x) \) under \( \phi_* \). Since the maps \( \epsilon_1 \) and \( \epsilon_2 \) have the same kernel, the elements \( Q_i(z) \) and \( w \) have the same image in \( H_*(S^G) \), which implies the result. \( \square \)

**Remark 2.2** The additive equivalence \( S^G \simeq K(\mathcal{E}^G) \simeq K(\Phi \mathcal{E}^G) \) of spectra can be realized as the \( G \)-fixed part of a \( G \)-equivalence \( S_G \simeq K_G(\mathcal{E}) \) of \( G \)-spectra, for example using Shimakawa’s construction \[21\] of \( G \)-equivariant \( K \)-theory spectra. Presumably this is a \( G \)-equivalence of \( E_\infty \) ring \( G \)-spectra.

## 3 Proof of the main theorem

Recall the \( E_\infty \) structure maps \( \lambda_j : E\Sigma_j \ltimes \Sigma_j (N\mathcal{E}^G)^{\otimes j} \to N\mathcal{E}^G \). We have inclusions \( B \text{Aut}^G(G) \to N\mathcal{E}^G \) and \( \delta : B \text{Aut}^G(G') \to N\mathcal{E}^G \), corresponding to the summands indexed by \( X = G \) and \( X = G' = G \times \cdots \times G \), respectively, in the decomposition (2–4) of \( N\mathcal{E}^G \). Restricting \( \lambda_j \) to these summands, we have a commutative diagram

\[
\begin{array}{ccc}
E\Sigma_j \ltimes \Sigma_j (N\mathcal{E}^G)^{\otimes j} & \xrightarrow{\lambda_j} & N\mathcal{E}^G \\
\downarrow & & \\
E\Sigma_j \ltimes \Sigma_j B \text{Aut}^G(G)^j & \cong \rightarrow & B(\Sigma_j \ltimes \text{Aut}^G(G)^j) \xrightarrow{B\delta} B \text{Aut}^G(G')
\end{array}
\]
where the homomorphism $\phi$ sends an element $(\sigma; f_1, \ldots, f_j)$ in $\Sigma_j \ltimes \text{Aut}^G(G^j)$ to the $G$-automorphism $f_{\sigma^{-1}(1)} \times \cdots \times f_{\sigma^{-1}(j)}$ of $G^j$.

We write $\Sigma_j \ltimes \text{Aut}^G(G) \cong \Sigma_j \ltimes G$ for the wreath product $\Sigma_j \ltimes \text{Aut}^G(G^j)$. The free $G$-set $G^j$ splits into $k = |G|^{j-1}$ orbits, and we fix a $G$-equivariant bijection $G^j \cong \coprod_k G$. This induces an isomorphism $\text{Aut}^G(G^j) \cong \text{Aut}^G(\coprod_k G)$, and we also have $\text{Aut}^G(\coprod_k G) \cong \Sigma_k \ltimes G$. Thus we get a commutative diagram

\[
\begin{array}{ccc}
B(\Sigma_j \ltimes \text{Aut}^G(G^j)) & \xrightarrow{B\phi} & B\text{Aut}^G(G^j) \\
\cong & & \cong \\
B(\Sigma_j \ltimes G) & \xrightarrow{B\phi} & B(\Sigma_k \ltimes G)
\end{array}
\]

where we also write $\phi$ for the induced homomorphism $\Sigma_j \ltimes G \to \Sigma_k \ltimes G$.

Now we specialize to the case $p = 2$. First we study the Dyer–Lashof operation $Q^2$: $H_1(\mathbb{S}^2) \to H_3(\mathbb{S}^2)$.

**Lemma 3.1** The operation $Q^2$ in $H_*\mathbb{S}^2)$ satisfies $Q^2(\alpha_1) = \alpha_3$.

**Proof** Let $C = C_2$. By Lemma 2.1, we may instead compute $Q^2$ in $H_*\text{K}(\Phi \mathcal{E}^C)$.

We let $j = 2$, combine diagrams (2–1), (3–1) and (3–2), apply homology, and end up with the upper half of the diagram

\[
\begin{array}{ccc}
H_*(E\Sigma_2 \ltimes \Sigma_2 K(\Phi \mathcal{E}^C)^\wedge 2) & \xrightarrow{\kappa_2^*} & H_*(\text{K}(\Phi \mathcal{E}^C)) \\
\hvee \circ \delta_* & & \downarrow \\
H_*(B(\Sigma_2 \ltimes C)) & \xrightarrow{B\phi_*} & H_*(B(\Sigma_2 \ltimes C)) \\
\hvee \circ s & & \downarrow B\iota_* \\
H_*(B(\Sigma_2 \ltimes C) \vee B(\Sigma_2 \ltimes C)) & \xrightarrow{B\psi_*} & H_*(B(C \times C)).
\end{array}
\]

The vertical homomorphisms in the lower square are induced by the homomorphism $d = 1 \times \Delta$ that sends $(\sigma, x)$ to $(\sigma; x, x)$, and the inclusion $\iota$ of the subgroup $C \times C = C^2$ in $\Sigma_2 \ltimes C = \Sigma_2 \ltimes C^2$. The homomorphism $\psi$ is the restriction of $\phi$ to $\Sigma_2 \times C$. It is easily checked that $\psi$ takes values in the subgroup $C \times C$ (since $p = 2$) and is given by $\psi(\sigma, x) = (x, \sigma x)$, using the description of $\phi$ given after diagram (3–1). We have $B\psi_*(e_1 \otimes 1) = 1 \otimes \alpha_1$ and

\[
B\psi_*(1 \otimes \alpha_j) = \Delta_*(\alpha_j) = \sum_{s+t=j} \alpha_s \otimes \alpha_t,
\]
which combine to give

\[ B\psi_*(e_i \otimes \alpha_j) = \sum_{s+t=j} \alpha_s \otimes (\alpha_t \ast \alpha_i), \]

where \( \ast \) denotes the Pontryagin product in \( H_*(BC) \) induced by the topological group multiplication \( BC \times BC \to BC \). We recall that \( \alpha_i \ast \alpha_t = \binom{i}{i} \alpha_{i+t} \).

By May’s paper [15, 9.1] the map \( B\delta_* \) is given by

\[ B\delta_*(e_i \otimes \alpha_j) = \sum_k e_{i+2k-j} \otimes Sq^k_*(\alpha_j) \otimes Sq^k_*(\alpha_j). \]

Recall that \( Sq^k_*(\alpha_j) = \binom{j-k}{k} \alpha_{j-k} \), where \( Sq^k_\ast \) denotes the dual of the Steenrod operation \( Sq^k \). In particular \( B\delta_*(e_1 \otimes \alpha_2) = e_1 \otimes \alpha_1 \otimes \alpha_1 \), which further maps to \( Q_1(\alpha_1) \) in the upper right hand corner of (3–3). But now \( Q_1(\alpha_1) \) is also the image of \( e_1 \otimes \alpha_2 \) under \( \epsilon_* \circ \delta_* \circ B\delta_* \circ B\psi_* \). Using the description of \( B\psi_* \) above, we see that \( Q_1(\alpha_1) \) equals

\[ \sum_{s+t=2} \epsilon_* \delta_*(1 \otimes \alpha_s \otimes (\alpha_1 \ast \alpha_t)). \]

The map \( \epsilon_* \) vanishes on decomposables with respect to the product in \( H_*(N\mathcal{E} C) \) induced by the additive symmetric monoidal structure on \( \mathcal{E}_C \). The element \( \delta_*(1 \otimes \alpha_s \otimes (\alpha_1 \ast \alpha_t)) \) is the image of \( \alpha_s \otimes (\alpha_1 \ast \alpha_t) \in H_*(B \text{ Aut}^C(C) \times B \text{ Aut}^C(C)) \) in \( H_*(B \text{ Aut}^C(C \coprod C)) \subset H_*(N\mathcal{E} C) \) under the map induced by disjoint union, thus the only non-zero term in (3–4) is the one with \( s = 0 \) and \( t = 2 \), and \( Q^2(\alpha_1) = Q_1(\alpha_1) = \epsilon_*(1 \otimes \alpha_0 \otimes (\alpha_1 \ast \alpha_2)) = \alpha_3 \).

**Proof of Theorem 0.2** We now turn to the operations in \( H_*(TC^{(1)}(\ast; 2)) = \mathbb{F}_2 \oplus H_*(\mathbb{R}P^{\infty}_1) \). The general formula for the \( Q^i \) will follow from Lemma 3.1 and the Nishida relations, which say in particular (see [2, III.1.1]) that

\[ Sq^*_s^{i+j+1}Q^i(\alpha_j) = \sum_k \binom{2N-j-1}{2N-i-2j-2+2k} Q^{k-j-1} Sq^k_*(\alpha_j), \]

where \( N \) is sufficiently large. When \( k \geq j + 2 \) the element \( Sq^k_*(\alpha_j) \) is zero for degree reasons, and when \( k \leq j \) the fact that \( Q^{k-j-1} \) vanishes on classes in degree higher than \( k-j-1 \) implies that \( Q^{k-j-1} Sq^k_*(\alpha_j) = 0 \). Hence the sum in (3–5) simplifies to the single term

\[ Sq^*_s^{i+j+1}Q^i(\alpha_j) = \binom{2N-j-1}{2N-i} Q^0 Sq^j_*(\alpha_j) \]

for \( k = j + 1 \), where \( N \) is large.
Bob Bruner has observed that

\[
\binom{2^N - j - 1}{2^N - i} \equiv \binom{2^N + i - 1}{2^N + j} \quad \text{mod } 2,
\]

for large \( N \). Here is a quick proof. Let \( x_k \) denote the \( k \)'th bit in the binary expansion of a natural number \( x \). Then \( \binom{2^N - j - 1}{2^N - i} \equiv 1 \) if and only if \( (2^N - i)_k = 1 \) implies \( (2^N - j - 1)_k = 1 \) for all \( k \), and \( \binom{2^N + i - 1}{2^N + j} \equiv 1 \) if and only if \( (2^N + j)_k = 1 \) implies \( (2^N + i - 1)_k = 1 \) for all \( k \). But for \( N \) large compared to \( i, j \) and \( k \) the bit \( (2^N - i)_k \) is complementary to \( (2^N + i - 1)_k \), and \( (2^N - j - 1)_k \) is complementary to \( (2^N + j)_k \), so \( \binom{2^N - j - 1}{2^N - i} \equiv 1 \) if and only if \( \binom{2^N + i - 1}{2^N + j} \equiv 1 \).

The operations \( Sq^k_n \) in \( H_*(\mathbb{P}^{\infty}_n) \) are given by the formula

\[
Sq^k_n(\alpha_j) = \binom{j - k}{k} \alpha_{j-k}.
\]

This follows by the corresponding formula for \( \mathbb{P}^{\infty} \) and James periodicity. More precisely, a theorem of James [12] says that given \( m \leq n \), there is a positive integer \( M \), depending only on \( n - m \), such that \( \mathbb{P}^{n+\ell}_m \simeq \Sigma^\ell \mathbb{P}^m \) when \( \ell \) is a positive multiple of \( 2^M \). The space \( \mathbb{P}^m_n \) is the stunted projective space \( \mathbb{P}^n/\mathbb{P}^{n-1} \). If we now define the spectrum \( \mathbb{P}^{n+\ell}_m \) to be \( \Sigma^{-\ell} \mathbb{P}^{\infty} \mathbb{P}^{n+\ell}_m \) for such \( \ell \) (depending on \( n \)), we have that \( \mathbb{P}^{\infty}_m = \text{colim}_n \mathbb{P}_m^{n+1} \). The Steenrod operations in \( H_*(\mathbb{P}^{\infty}_m) \) can now be calculated from the operations in \( H_*(\mathbb{P}^{n+1}_m) \), and the stated formula follows by noting that the relevant binomial coefficients are \( 2^M \)-periodic in the numerator.

In particular \( Sq^{i+1}_n(\alpha_j) = \alpha_{-1} \) for all \( j \geq -1 \), and we have

\[
Sq^{i+1}_n(\alpha_j) = \binom{2^N + i - 1}{2^N + j} Q^0(\alpha_{-1}).
\]

If \( Q^0(\alpha_{-1}) \) were zero, it would follow that \( Q^j(\alpha_j) = 0 \) for all \( i \) and \( j \), since \( Sq^{i+j+1}_n \) is an isomorphism to dimension \(-1\). But this contradicts Lemma 3.1. Hence \( Q^0(\alpha_{-1}) = \alpha_{-1} \), and the formula stated in the theorem follows.

\[
\square
\]

References


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