HOMOLOGY OPERATIONS IN THE TOPOLOGICAL CYCLIC HOMOLOGY OF THE CIRCLE

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INTRODUCTION

Algebraic $K$-theory of a ring, fully defined by Quillen in his 1973 paper [Qui73], has become one of the most important invariants of rings, and much effort has gone into computing it. Motivated by his study of manifolds, Waldhausen some years later defined his $A$-theory of spaces [Wal78], and also provided a general framework for algebraic $K$-theory which included both his $A$-theory and Quillen’s $K$-theory as special cases [Wal85]. The introduction of well-behaved categories of structured ring spectra in the 90s allowed for a definition of algebraic $K$-theory of a ring spectrum (suitably interpreted). This construction recovers Quillen’s $K$-theory as $K(R) = K(HR)$, where $HR$ is the Eilenberg-Mac Lane spectrum of the ring $R$. Waldhausen’s $A$-theory of a space $X$ is recovered as $A(X) = K(S[\Omega X])$, where $S[\Omega X]$ is the spherical group ring of the loop space of $X$. Algebraic $K$-theory is notoriously difficult to compute, even for ordinary rings, but several fruitful strategies for its computation has been developed. One such is the so-called trace methods, via topological cyclic homology and topological Hochschild homology.

Topological Hochschild homology of ring spectra was defined by Bökstedt in [Bök87]. It is a topological version of Hochschild homology for rings, in the sense that its construction is essentially to replace tensor products over the integers with smash products over the sphere spectrum. Originally defined for certain “functors with smash products”, Bökstedt’s construction works equally well for the more modern concept of a symmetric ring spectrum. There is a trace map $tr: K(B) \to THH(B)$ from algebraic $K$-theory to topological Hochschild homology which can be used to detect elements in $\pi_* K(B)$, but it is typically very far from being an equivalence.

Topological cyclic homology was defined by Bökstedt, Hsiang and Madsen [BHM93]. It originally takes as input a space, or more generally a functor with smash product, but again the construction also works for more modern categories of structured ring spectra. Topological cyclic homology is a refinement of topological Hochschild homology in the sense that there is a map $TC(B) \to THH(B)$ and the trace map $tr: K(B) \to THH(B)$ lifts to a cyclotomic trace map $trc: K(B) \to TC(B)$. It turns out that the map $trc$ can detect a lot of $K(B)$, and after a suitable completion it can even be an equivalence on connective covers. ($TC$ is in general only $(-2)$-connected.) This is very useful since $TC$, although not easy to compute, is approachable by standard homotopy theoretical methods, such as spectral sequences.

On the geometric side, Waldhausen’s $A$-theory of spaces is related to automorphisms of manifolds. Write $A(X)$ for the spectrum-valued $A$-theory of a space $X$, and $Wh^{CAT}(M)$ for the Whitehead spectrum of a $CAT$-manifold $M$. Here $CAT$ is either $DIFF$, $TOP$ or $PL$, denoting the category of smooth, topological or piecewise linear manifolds, respectively. Results by Burghelea and Lashof [BL74] and Kirby and Siebenmann [KS77] imply that $Wh^{TOP}(M) \simeq Wh^{PL}(M)$, and there is a cofiber sequence

\begin{equation}
A(*) \wedge M_+ \to A(M) \to Wh^{PL}(M) .
\end{equation}

In the smooth case there is a splitting

\begin{equation}
A(M) \simeq \Sigma^\infty_+ M \vee Wh^{DIFF}(M) ,
\end{equation}

Date: April 18, 2016.
which in the case $M = *$ serves to bootstrap the $TOP$ and $PL$ case. (The case $M = *$ for $TOP$ and $PL$ is trivial since $Wh^{PL}(*) \simeq *$ by the so-called Alexander trick.) The underlying infinite loop space of the spectrum $Wh^{CAT}(M)$ is constructed as the double delooping of $C^{CAT}(M)$, the space of stable concordances (also known as pseudo-isotopies) of $M$. By definition, $C^{CAT}(M) = \text{hocolim}_n C^{CAT}(M \times I^n)$, where the concordance space $C^{CAT}(M)$ is the space of automorphisms of $M \times I$ fixing $M \times 0$. Unstably, the $TOP$ and $PL$ case are different in low dimensions, but the natural map $C^{PL}(M) \to C^{TOP}(M)$ becomes a homotopy equivalence as soon as $\dim M \geq 5$.

Since the Whitehead spectra $Wh(M)^{CAT}$ encode information about automorphisms of manifolds, they are of great interest in manifold theory. By the link to algebraic $K$-theory described above, then, it is important to understand the spectra $A(M)$, which then can be approached by studying $TC(M)$. Particularly important examples are $M = *$ and $M = S^1$. This is illustrated by the result of Farrell and Jones [FJ91] which says that the Whitehead spectra $Wh^{TOP}(M)$ are determined by $Wh^{TOP}(S^1)$ when $M$ has non-positive sectional curvature everywhere. The cofiber sequence (1) then motivates the study of the spectra $A(*)$ and $A(S^1)$, and hence $TC(*)$ and $TC(S^1)$. In particular, it is important to understand their homology, with the various extra structure it comes with. Since $*$ and $S^1$ are in fact Lie groups, the spectra $TC(*)$ and $TC(S^1)$ have naturally defined $E_{\infty}$ ring spectrum structures, which induce Pontrjagin products and Dyer-Lashof operations in homology. These are the main additional structures we want to study.

In [BR10] we found explicit formulas for the action of the Dyer-Lashof operations (also called Araki-Kudo operations at the prime 2) on $H_*(TC(*) ; \mathbb{F}_2)$, in addition to describing the Pontrjagin product and dual Steenrod operations. According to the previous paragraph, the next important case concerning symmetries of manifolds is the circle. The main goal of this paper is to determine the homology of $TC(S^1)$, including the Pontrjagin product and the actions of the Steenrod and Dyer-Lashof algebras. Since our proof of this result depends on the corresponding structure for $TC(*)$, we provide an argument for describing the multiplicative structure of $H_*(TC(*) ; \mathbb{F}_p)$ at odd primes. Our proof is modeled on an argument by Mann and Miller in [MM85], where they prove similar formulas for the homology of the underlying $E_{\infty}$ ring space of the fixed point spectrum $S^{C_p}$.

Sections 1 and 2 are of a preliminary nature and recall the most important facts about homology operations and topological cyclic homology, respectively. In Section 3 we give a proof of the following result about $H_*(TC(*) ; \mathbb{F}_p)$ for odd $p$, which complements our previous result at the prime 2.

**Theorem 0.1.** Let $p$ be an odd prime.

1. The homology groups of $TC(*)$ are given as
   
   $$H_*(TC(*) ; \mathbb{F}_p) = \mathbb{F}_p\{1, \Sigma \beta_j \mid j \geq -1\},$$

   with $|\Sigma \beta_j| = 2j + 1$. The Pontrjagin product is trivial, with 1 being the unit.

2. The dual Steenrod operations in $H_*(TC(*) ; \mathbb{F}_p)$ are given by
   
   $$P^{k}_*(\Sigma \beta_j) = \binom{j - k(p - 1)}{k} \Sigma \beta_{j - k(p - 1)}.$$  

   The remaining operations are trivial.

3. The Dyer-Lashof operations are given by
   
   $$Q^i(\Sigma \beta_j) = (-1)^{i+j+1} \binom{i-1}{j} \Sigma \beta_{i(p-1)+j}, \text{ for } i > j \geq 0$$

   $$Q^0(\Sigma \beta_{-1}) = \Sigma \beta_{-1}.$$  

   The remaining operations are trivial.

In Section 4 we prove our main result about the multiplicative structure of $H_*(TC(S^1))$. We do this by comparing with the second stage in the tower defining $TC(S^1)$, and also by using the
above result for $TC(*)$. We first identify the $p$-adic homotopy type of $TC(S^1)$. The assembly map mentioned in the following result is a certain multiplicative map $TC(A) \wedge BG_+ \rightarrow TC(A[G])$ defined for all symmetric ring spectra $A$ and group-like monoids $G$.

**Proposition 0.2.** There is an equivalence of spectra

$$TC(S^1) \simeq TC(*) \wedge S^1_+ \lor \bigvee_{p \mid n} \left( S^{-1}_+ \bigvee_{k=0}^{\infty} \Sigma^\infty S^p k \right),$$

after $p$-completion. The inclusion of the summand $TC(*) \wedge S^1_+$ coincides with the assembly map.

**Theorem 0.3.** Let $p$ be any prime.

1. The mod $p$ homology of $TC(S^1)$ is

$$H_\ast(TC(S^1); \mathbb{F}_p) = \mathbb{F}_p \{a_{-1}(m), 1, a_0, e_1, a_s(n) \mid m = 0 \text{ or } p \nmid m, s \geq 1, n \in \mathbb{Z} \},$$

with subscripts indicating degree. The Pontrjagin product is trivial, with 1 being the unit.

2. The dual Steenrod operations in $H_\ast(TC(S^1); \mathbb{F}_p)$ are given by

$$Sq^k_s(a_s(n)) = \binom{s-k}{k} a_{s-k}(n)$$

for $p = 2$, and

$$P^k_s(a_s(n)) = \binom{s/2}{k}^{-k(p-1)} a_{s-2k(p-1)}(n)$$

$$\beta_s(a_{2s}(n)) = a_{2s-1}(n) \text{ if } p \nmid n$$

for odd $p$. The remaining operations are trivial.

3. The Dyer-Lashof operations in $H_\ast(TC(S^1); \mathbb{F}_p)$ are given by

$$Q^i_s(a_s(n)) = (-1)^{i+s+1} \binom{i-1}{s} a_{s+i(p-1)}(pn)$$

$$Q^0_s(a_{-1}(0)) = a_{-1}(0).$$

The remaining operations are trivial.

A natural question which arises is to what extent the various structures on the homology of $TC(*)$ and $TC(S^1)$ lift to the homology of the corresponding $A$-theory spectra, via the cyclotomic trace map. Unfortunately, in these cases, the induced map in homology of the trace map is almost trivial. However, in addition to being interesting in itself, the calculations in this paper can be used to provide key inputs to the calculation of multiplicative structures on the topological cyclic homology of other ring spectra, via maps $\mathbb{S}[\mathbb{Z}] \rightarrow B$ or the unit map $\mathbb{S} \rightarrow B$. In particular, they provide crucial information about the multiplicative structures in the homology of $TC(\mathbb{Z})$ and $TC(\mathbb{Z}[x^{\pm 1}])$. This will be addressed in a future paper.

**Notation and conventions.** Given a set $X$, we write $F_p X$ for the $F_p$-vector space with a basis given by $X$. We write $H_\ast(-)$ for mod $p$ homology $H_\ast(-; \mathbb{F}_p)$, and assume all spaces and spectra are $p$-complete, except possibly in Section 1. Where necessary, $p$-completion is implicitly applied. The circle group of complex numbers of modulus 1 is denoted $T$.

### 1. $E_\infty$ ring spectra and homology operations

Unless otherwise noted, spectra will mean Lewis-May spectra and $E_\infty$ ring spectra will mean spectra with an action of an $E_\infty$ operad. We will use these notions both equivariantly and nonequivariantly. For more details, see [LMSM86]. Exceptions are the $K$-theory construction used below, which outputs a symmetric spectrum, and topological Hochschild homology, which takes a symmetric ring spectrum as input. For more about symmetric spectra, see [HSS00].
We use $S$ to denote the sphere spectrum in Lewis-May or symmetric spectra, depending on the context, and we write $S[G]$ for the unreduced suspension spectrum $\Sigma^\infty G$ when $G$ is a topological monoid, or more generally an $A_\infty$ space. The Lewis-May spectrum $S[G]$ is then an $A_\infty$ ring spectrum, which is $E_\infty$ if $G$ is a commutative monoid or merely an $E_\infty$ space. This construction also makes sense for symmetric spectra, provided $G$ is a strict (possibly commutative) monoid.

We briefly recall how the mod $p$ homology of an $E_\infty$ ring spectrum is augmented by Dyer-Lashof operations. Let $\mathcal{O}$ be an $E_\infty$ operad and let $E$ be spectrum with an action of $\mathcal{O}$. This action is given by maps

$$\xi_j : \mathcal{O}(j) \ltimes_{\Sigma_j} E^\wedge j \to E,$$

making certain diagrams commute. Here $\ltimes$ denotes twisted half-smash product in spectra. Let $W_*$ be the standard free $C_p$-resolution of $F_p$ with a single $F_p[C_p]$-basis element $e_i$ in degree $i$. Let $C_*(-)$ denote the cellular chains functor on CW-spaces or CW-spectra; there is then a chain map $W_* \to C_*(\mathcal{O}(p))$ lifting the identity on $F_p$, unique up to homotopy. The image of $e_i$ under this map we also denote by $e_i$. Now let $x \in H_q(E)$ be represented by a cycle $z \in C_q(E)$, and consider the image of $e_i \otimes z^\otimes p$ under the map

$$C_*(\mathcal{O}(p)) \otimes_{\Sigma_p} C_*(E)^{\otimes p} \xrightarrow{\cong} C_*(\mathcal{O}(p) \ltimes_{\Sigma_p} E^\wedge p) \xrightarrow{(\xi_p)^*} C_*(E);$$

denote its image in homology by $Q_i(x)$. Then for $p = 2$ define $Q^i(x) = 0$ when $i < q$, and

$$Q^i(x) = Q_{i-q}(x)$$

when $i \geq q$. For $p > 2$ define $Q^i(x) = 0$ when $2i < q$, and

$$Q^i(x) = (-1)^\nu(q)Q_{(2i-q)(p-1)}(x)$$

when $2i \geq q$, where $\nu(2\ell + \epsilon) = (-1)^\epsilon (m!)^\epsilon$ with $m = \frac{1}{2}(p-1)$ and $\epsilon = 0, 1$. See [BMMS86, Chapter III] for more details.

An important source of $E_\infty$ ring spectra is the algebraic $K$-theory of symmetric bimonoidal categories. The precise construction of the $E_\infty$ ring structure on the $K$-theory space or spectrum of a bipermutative category has historically been a sticky point, with mistakes in several of the older accounts. Here we will use the construction of Elmendorf-Mandell [EM06], but see also [May09].

First let $\mathcal{C}$ be a bipermutative category. Here we use the term bipermutative category as originally defined in [May77], not the more general notion defined in [EM06]. From $\mathcal{C}$, the construction in [EM06] produces a symmetric spectrum in simplicial sets with an action of the simplicial Barratt-Eccles operad. After geometric realization this results in a symmetric spectrum in spaces which we denote $K(\mathcal{C})$. The operad action is given by maps

$$\kappa_j : E\Sigma_j \ltimes_{\Sigma_j} K(\mathcal{C})^{\wedge j} \to K(\mathcal{C}),$$

where $E\Sigma_j$ is the classifying space of the translation category $\Sigma_j$. Recall that the translation category of a group (or even a set) $G$ has as objects the elements in $G$, with a unique morphism between each (ordered) pair of objects. The maps $\kappa_j$ are induced by the bipermutative structure in the following way. Let $\otimes$ denote the product in the bipermutative structure on $\mathcal{C}$, and define a functor

$$(2) \quad \tilde{\lambda}_j : \Sigma_j \ltimes_{\Sigma_j} C^j \to \mathcal{C}$$
on objects by sending $(\sigma; a_1, \ldots, a_j)$ to $a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(j)}$. A morphism

$$(\tau \sigma^{-1}; f_1, \ldots, f_j) : (\sigma; a_1, \ldots, a_j) \to (\tau; b_1, \ldots, b_j)$$

is mapped by $\tilde{\lambda}_j$ to the composition

$$a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(j)} \xrightarrow{T_{\tau}} a_1 \otimes \cdots \otimes a_j \xrightarrow{f_1 \otimes \cdots \otimes f_j} b_1 \otimes \cdots \otimes b_j \xrightarrow{T_{\tau^{-1}}} b_{\tau^{-1}(1)} \otimes \cdots \otimes b_{\tau^{-1}(j)},$$
where $T_\sigma$ denotes the twist isomorphisms given by the bipermutative structure. Applying the classifying space functor $B(-)$ to $\tilde{\lambda}_j$ yields a map

$$\lambda_j : E\Sigma_j \ltimes_{\Sigma_j} BC^{\wedge_j} \to BC.$$ 

The zeroth space in $K(C)$ is $BC$, the classifying space of $C$; let $\Sigma^\infty BC \to K(C)$ be the induced map of symmetric spectra. The relationship between $\kappa_j$ and $\lambda_j$ is described by the commutative diagram

$$\begin{array}{ccc}
E\Sigma_j \ltimes_{\Sigma_j} K(C)^{\wedge_j} & \xrightarrow{\kappa_j} & K(C) \\
\uparrow & & \uparrow \\
E\Sigma_j \ltimes_{\Sigma_j} (\Sigma^\infty BC)^{\wedge_j} & \xrightarrow{\sim} & \Sigma^\infty(E\Sigma_j \ltimes_{\Sigma_j} BC^{\wedge_j}) \xrightarrow{\Sigma^\infty \lambda_j} (\Sigma^\infty BC).
\end{array}$$

See [EM06, Section 8] for more details.

Now let $C$ be a symmetric monoidal category. By [May77, VI.3.5] there is a natural equivalence $C \to \Phi C$ of symmetric monoidal categories, where $\Phi C$ is bipermutative. The $K$-theory of $C$ is defined to be the $K$-theory of $\Phi C$, and by precomposing with the map $BC \to B(\Phi C)$ we get that (3) still holds for symmetric bimonoidal $C$.

Let $G$ be a finite group, and let $\mathcal{E}^G$ the category of finite $G$-sets and $G$-equivariant bijections. This is a symmetric bimonoidal category under disjoint union and cartesian product. If we let $C = \mathcal{E}^G$ in (3), the maps $\lambda_j$ are induced by the functors defined by

$$\sigma; X_1, \ldots, X_j \mapsto X_{\sigma^{-1}(1)} \times \cdots \times X_{\sigma^{-1}(j)}$$

on objects, and similarly for morphisms. The following result, proved in [BR10, 2.1], is essential for the computation in Section 3.

**Theorem 1.1.** Let $G$ be a finite group. There is an equivalence $S^G \simeq K(\mathcal{E}^G)$ of spectra such that the induced isomorphism $H_*(S^G) \cong H_*(K(\mathcal{E}^G))$ commutes with the Dyer-Lashof operations.

## 2. Topological cyclic homology

### 2.1. Topological Hochschild and cyclic homology of ring spectra

We recall Bökstedt’s topological Hochschild homology construction. For published accounts with more details, see [BHM93] and [HM97]. Let $B$ be a symmetric ring spectrum, and let $T$ denote Bökstedt’s category of ... numbers and injective functions. Write $i = (i_0, \ldots, i_n) \in T^{n+1}$ for elements in the product category $T^{n+1}$. For each finite-dimensional $T$-representation $V$, define a space

$$\text{thh}(B; V)^n = \text{holim}_{i \in T^{n+1}} F(S^{i_0} \wedge \cdots \wedge S^{i_n}, B_{i_0} \wedge \cdots \wedge B_{i_n} \wedge S^V),$$

where the maps in the directed system $\text{thh}(B; V)^{n+1}$ assemble to a cyclic space $\text{thh}(B; S^V)$, i.e., a cyclic object in the category of spaces. Recall that a cyclic object is a contravariant functor from the cyclic category $\Lambda$, which has the same objects as the simplicial category $\Delta$, but an extra “cyclic” operator $t_n : [n] \to [n]$ satisfying certain relations including $t_n^0 = id$. For $\text{thh}(B; V)$, the face and degeneracy maps are given by ..., and $t_n$ is given by cyclically permuting ...

The geometric realization $\text{thh}(B; V) = \{\text{thh}(B; V)| \}$ has a $T$-action coming from the cyclic structure. It also has a $T$-action coming from the $T$-space $S^V$, resulting in a $T \times T$-action. The restriction of this action to the diagonal $T \subset T \times T$ is the $T$-action we want on $\text{thh}(B; V)$. The association $V \mapsto \text{thh}(B; S^V)$ is an equivariant $T$-prespectrum, with structure maps

$$S^W \wedge \text{thh}(B; V) \to \text{thh}(B; W \oplus V).$$

Let $\text{THH}(B) = L(\text{thh}(B))$ be the associated $T$-spectrum.

In addition to being an equivariant $T$-spectrum, $\text{THH}(B)$ has the structure of a cyclotomic spectrum. This is given by suitably compatible $T$-equivalences

$$r_c : \rho_C \Phi C \text{THH}(B) \to \text{THH}(B)$$

where $t^\infty$ denotes the twist isomorphisms given by the bipermutative structure.
for each finite subgroup $C \subset T$. Here $\rho^C: T \to T/C$ is the $|C|'$th root isomorphism, and $\rho^C_*$ pulls back a $T/C$-spectrum to a $T$-spectrum via $\rho_C$. The cyclotomic structure maps give rise to restriction maps

$$R: THH(B)^D \to THH(B)^C$$

for each pair of subgroups $C \subset D \subset T$ with $|C|$ dividing $|D|$. There are also the so-called Frobenius maps

$$F: THH(B)^D \to THH(B)^C$$

which by definition is the inclusion of fixed points.

Henceforward we will work at a prime $p$, and we only consider finite $p$-subgroups $C \subset T$. The topological cyclic homology of $B$ (at a prime $p$) can now be defined as

$$TC(B; p) = \text{holim}(\ldots \xrightarrow{R_{p^k}} THH(B)^C_{p^k} \xrightarrow{R_{p^k}} THH(B)^C_{p^k-1} \xrightarrow{R_{p^k}} \ldots)$$

We will suppress the prime $p$ in the notation and simply write $TC(B)$ for $TC(B; p)$.

### 2.2. Topological cyclic homology of spaces

Let $X$ be a space, and $B = S[\Omega X]$. In this case we write $TC(X)$ for $TC(B)$. Note that for a topological monoid $X$, the spectrum $S[\Omega X]$ is a commutative symmetric ring spectrum, so $THH(S[\Omega X])$ and $TC(X)$ are $E_\infty$ ring spectra. A homomorphism $X \to Y$ of monoids induces a map $TC(X) \to TC(Y)$ of $E_\infty$ ring spectra.

An important result for us is the following, proved in [BHM93]. See also [Rog02, Section 1] for clarifying points. Let $\Lambda X$ denote the free loop space of $X$, and let $\Delta_p: \Lambda X \to \Lambda X$ be the map that precomposes a loop $S^1 \to X$ with the degree $p$ map $S^1 \to S^1$. The circle group $T$ acts on $\Lambda X$ by rotating loops. There is a homotopy cartesian square

$$TC(X) \xrightarrow{\gamma} \Sigma\Sigma_+^\infty (\Lambda X)_{hT}$$

$$\downarrow \varepsilon$$

$$\Sigma_+^\infty \Lambda X \xrightarrow{1-\Delta_p} \Sigma_+^\infty \Lambda X,$$

where $t$ is the dimension shifting transfer associated to the $T$-bundle $\Lambda X \to (\Lambda X)_{hT}$. The map $\varepsilon$ is the natural map $TC(X) \to THH(S[\Omega X]) \simeq \Sigma_+^\infty \Lambda X$.

### 3. Topological cyclic homology of a point

Here we recall previous results about the multiplicative structure on the homology of $TC(\ast)$ at the prime 2, and provide an argument for odd primes. We start off by describing the additive structure of $TC(\ast)$.

If we let $X = \ast$ in (6), then $\Sigma_+^\infty \Lambda X \simeq S$ and the map $\Delta_p$ is trivial. It follows that

$$TC(\ast) \simeq S \vee \text{hobif}(t) \simeq S \vee \Sigma CP_2^\infty,$$

where $\text{hobif}(t)$ is the fiber of the circle transfer $t: \Sigma \Sigma_+^\infty CP^\infty \to S$, and $CP_2^\infty$ is the Thom spectrum of the negative canonical complex line bundle over $CP^\infty$. The augmentation $THH(S) \to S$ is an equivalence of $F$-cycloptic spectra (as is the unit map, see Lemma ??), and the map $TC(\ast) \to THH(S)$ factors through a multiplicative map $f: TC(\ast) \to S^{C_p}$. By the Segal-tom Dieck splitting, $S^{C_p} \simeq S \vee S[BC_p]$; write

$$H_*(S^{C_p}) = \mathbb{F}_p \oplus \mathbb{F}_p \{\alpha_j \mid j \geq 0\}$$

for its homology, with $|\alpha_j| = j$. By (7) the homology of $TC(\ast)$ is

$$H_*(TC(\ast)) = \mathbb{F}_p \oplus \mathbb{F}_p \{\beta_j \mid j \geq -1\},$$

with $|\beta_j| = 2j + 1$. The induced map

$$f_*: H_*(TC(\ast)) \to H_*(S^{C_p})$$

in homology is given by $f_*(\beta_j) = \alpha_{2j+1}$ in positive degrees. See [BR10, Section 1] for more details.
The following is the main result of [BR10]. The action of the dual Steenrod operations is not explicitly stated there, but can be extracted from the proof of [BR10, Theorem 0.2].

**Theorem 3.1.** Let $p = 2$. The dual Steenrod operations in $H_*(TC(\ast))$ are given by

$$Sq_*^{2k}(\Sigma \beta_j) = \left(\frac{2j - 2k + 1}{2k}\right) \Sigma \beta_{j-k}.$$

The Pontrjagin product is trivial, and the Dyer-Lashof operations are given by

$$Q^{2i}(\Sigma \beta_j) = \left(\frac{i - 1}{j}\right) \Sigma \beta_{i+j}, \text{ for } i > j \geq 0$$

$$Q^0(\Sigma \beta_{-1}) = \Sigma \beta_{-1}.$$

The remaining operations are zero.

We will now prove the corresponding result for odd primes. The proof goes along the same lines as for the prime 2, but extra care is needed to keep track of the behavior of the relevant elements and maps in group homology. We follow [MM85] and utilize the Evens transfer map to describe the homomorphisms that capture the multiplicative structure in homology. In fact Theorem 3.2 below should follow from [MM85, Theorem 6.8] and stabilization, which is a corresponding result for the underlying $E_\infty$ ring space of $S^{C_p}$. However, they only state and prove the result for the prime 2, noting that the argument for odd primes is similar. We find that the formulas for odd primes become significantly more complicated; for this reason, and for completeness, we provide a full proof.

**Theorem 3.2.** Let $p$ be an odd prime. The dual Steenrod operations in $H_*(S^{C_p})$ are given by

$$P_*^k(\alpha_j) = \left(\frac{j/2 - k(p-1)}{k}\right) \alpha_{j-2k(p-1)}$$

$$\beta_*^k(\alpha_{2j}) = \alpha_{2j-1}, \beta_*^k(\alpha_{2j+1}) = 0.$$

The Pontrjagin product is trivial, and the Dyer-Lashof operations are given by

$$Q^i(\alpha_j) = (-1)^{i+j+1} \left(\frac{i - 1}{j}\right) \alpha_{i(p-1)+j}.$$

The remaining operations are zero.

**Corollary 3.3.** Let $p$ be an odd prime. The dual Steenrod operations in $H_*(TC(\ast))$ are given by

$$P_*^k(\Sigma \beta_j) = \left(\frac{j - k(p-1)}{k}\right) \Sigma \beta_{j-k(p-1)}.$$

The Pontrjagin product is trivial, and the Dyer-Lashof operations are given by

$$Q^i(\Sigma \beta_j) = (-1)^{i+j+1} \left(\frac{i - 1}{j}\right) \Sigma \beta_{i(p-1)+j}, \text{ for } i > j \geq 0$$

$$Q^0(\Sigma \beta_{-1}) = \Sigma \beta_{-1}.$$

The remaining operations are zero.

**Proof.** By Theorem 3.2 and the map (8), only the formulas involving $\Sigma \beta_{-1}$ need verification. The statements about dual Steenrod operations and the Pontrjagin product also easily follow directly, since $TC(\ast) \simeq \mathbb{S} \vee \Sigma \mathbb{C} \mathbb{P}_\infty^1$. Except for the unit summand, $H_*(TC(\ast))$ is concentrated in odd degrees, so the product is automatically trivial. Formulas for the dual Steenrod operations follow by the corresponding formulas for $H_*(\mathbb{C} \mathbb{P}_\infty^\infty)$ and James periodicity. See [Rog03, Section 2] for details.

For the Dyer-Lashof operations on $\Sigma \beta_{-1}$, the only option which is consistent with Theorem 3.2 and the Nishida relations is $Q^i(\Sigma \beta_{-1}) = \Sigma \beta_{-1}$ and $Q^i(\Sigma \beta_{-1}) = 0$ for all $i \geq 1$. We refrain from
writing out the Nishida relations in their full generality, referring to [BMMS86, III.1] for more details. The fact that \( Q^0(\Sigma_{\beta-1}) = \Sigma_{\beta-1} \) follows from the relation
\[
P_i^*Q^{i-1}(\Sigma_{\beta-2}) = Q^0P_1^1(\Sigma_{\beta-2}).
\]
Here \( P_i^* (\Sigma_{\beta^* -p-1}) = \Sigma_{\beta-1} \) and \( P_1^1 (\Sigma_{\beta-2}) = \Sigma_{\beta-1} \). Now the relation
\[
P_i^*Q^i(\Sigma_{\beta-1}) = (-1)^i \left( \frac{p^N}{p^N - i} \right) Q^0P_1^0(\Sigma_{\beta-1}),
\]
where \( N \) is sufficiently large, implies that \( Q^i(\Sigma_{\beta-1}) = 0 \) for \( i \geq 1 \).

As preparation for the proof of Theorem 3.2 we recall the Evens transfer map and some of its properties, originally defined by Evens in [Eve63]. For a group \( H \), write \( \Sigma_n \triangleleft H \) for the wreath product \( \Sigma_n \ltimes H^{\times n} \), where multiplication
\[
(\sigma; h_1, \ldots, h_n) \cdot (\tau; h_1', \ldots, h_n') = (\sigma \tau; h_{\tau(1)}h_1', \ldots, h_{\tau(n)}h_n').
\]
Let \( H \subset G \) be a subgroup with finite index \( n = [G : H] \). Choose coset representatives \( g_i \) such that
\[
G = \bigcup_{i=1}^n g_i H.
\]
For each \( g \in G \) and \( 1 \leq i \leq n \), write \( gg_i = g_{\sigma_x(i)}h_i(g) \), where \( \sigma_g \in \Sigma_n \) and \( h_i(g) \in H \). This defines functions \( \sigma : G \to \Sigma_n \) and \( h_i : G \to H \). The Evens transfer
\[
\tau : G \to \Sigma_n \triangleleft H
\]
is given by \( \tau(g) = (\sigma(g); h_1(g), \ldots, h_n(g)) \); it is easily seen to be a group homomorphism. The map \( \tau \) depends on the choice of coset representatives, but only up to an inner automorphism of \( \Sigma_n \triangleleft H^{\times n} \).

Let \( \text{Aut}^G(X) \) denote the group of \( G \)-equivariant automorphisms of a (right) \( G \)-set \( X \), and consider the map \( f : \text{Aut}^G(G) \to \text{Aut}^H(G) \) that forgets part of the equivariance. The coset decomposition (9) specifies an isomorphism
\[
\Sigma_n \triangleleft H \overset{\cong}{\longrightarrow} \text{Aut}^H(G)
\]
given by sending \( (\sigma; h_1, \ldots, h_n) \) to the \( H \)-automorphism of \( G \) which restricts to maps \( g_i H \to g_{\sigma(i)}H \) that acts by \( h_i \). Composing \( f \) with this isomorphism and the canonical identification \( G = \text{Aut}^G(G) \) results in a map \( G \to \Sigma_n \triangleleft H \), which is easily seen to coincide with the Evens transfer \( \tau \).

Now let \( G \) be a group with \( n \) elements and let \( \Delta \subset G^{\times j} \) be the diagonal subgroup, where \( j \) is any positive integer. Fix an ordering \( G = \{g_1, \ldots, g_n\} \) of the elements of \( G \). This induces the lexicographic ordering on \( G^{\times j} \). We let
\[
\tau_j : G^{\times j} \to \Sigma_{n^j - 1} \triangleleft G
\]
be the Evens transfer associated to the coset decomposition
\[
G^{\times j} = \bigcup_{1 \leq i_1 \leq \cdots \leq i_j \leq n} (1, g_{i_2}, \ldots, g_{i_j}) \Delta,
\]
where we identify \( G \) with \( \Delta \).

For \( \sigma \in \Sigma_j \), let \( a_{\sigma} \in \text{Aut}^G(G^{\times j}) \) be the automorphism that sends \( (x_1, \ldots, x_j) \) to \( (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(j)}) \). It corresponds to an element \( c_{\sigma} \in \Sigma_{n^j - 1} \triangleleft G \) under the isomorphism (10). We can now extend \( \tau_j \) to a map
\[
\psi_j : \Sigma_j \triangleleft G \to \Sigma_{n^j - 1} \triangleleft G
\]
by defining \( \psi_j(\sigma; x_1, \ldots, x_j) = c_{\sigma} \tau(x_1, \ldots, x_j) c_{\sigma}^{-1} \); we call it the extended Evens transfer. Let
\[
\phi_j : \Sigma_j \triangleleft \text{Aut}^G(G^{\times j}) \to \text{Aut}^G(G^{\times j})
\]
be the homomorphism defined similarly to morphism part of the functor $\tilde{\alpha}_j$ in (2). Thus
$$\phi_j(\sigma; f_1, \ldots, f_j) = a_{\sigma} \circ (f_1 \times \cdots \times f_j) \circ a_{\sigma}^{-1}.$$  
Comparing the definitions we immediately get the following.

**Lemma 3.4.** Under the identification $G = \text{Aut}\,\mathbb{Z}(G)$ and the isomorphism (10), the map $\phi_j$ coincides with the extended Evens transfer $\psi_j$.

From now on we will only consider the case $G = C_p$, and we also restrict $\psi_j$ to $C_j \lhd C_p \subset \Sigma_j \lhd C_p$. Here the inclusion $C_j \subset \Sigma_j$ is given by letting the generator $c$ of $C_j$ permute $(1, \ldots, j)$ as $(j, 1, \ldots, j - 1)$. We define two homomorphisms needed for the next lemma. Let
$$\mu: C_p^{\times j} \to \Sigma_{p^{j-1}} \times C_p$$
be defined by $\mu(x_1, \ldots, x_j) = (\sigma_x, x_1)$, where $\sigma_x$ is the permutation sending $(i_2, \ldots, i_j)$ to $(i_2 + x_2 - x_1, \ldots, i_j + x_j - x_1)$. For the second, let $\lambda \in \Sigma_{p^{j-1}}$ be the permutation given by sending $(i_2, \ldots, i_j)$ to $(i_3 - i_2, \ldots, i_j - i_2, -i_2)$. Now let
$$\nu: C_j \times C_p \to C_p^{\times p^{j-1}}$$
be defined by $\nu(c; x) = (\nu_1, \ldots, \nu_{p^j})$, where $\nu_i = x + i_2 - \tau^{-1}(i)$ for $i = (i_2, \ldots, i_j)$, and $c \in C_j$ is the generator.

**Lemma 3.5.** The diagram

$$\begin{array}{c}
C_p^{\times j} \xrightarrow{i} C_j \lhd C_p \leftarrow d C_j \times C_p \\
\downarrow \mu \quad \quad \quad \quad \quad \downarrow \psi_j \quad \quad \quad \quad \quad \downarrow \nu \\
\Sigma_{p^{j-1}} \times C_p \xrightarrow{d} \Sigma_{p^{j-1}} \lhd C_p \leftarrow i C_p^{\times p^{j-1}}
\end{array}$$

commutes.

**Proof.** The first claim follows directly from the definitions. For the second, we only have to check commutativity on elements of the form $(c, x) \in C_j \times C_p$, where $c \in C_j$ is the generator. To identify the image $\psi_j d(c, x) = b_c \tau_j(x, \ldots, x)b_c^{-1}$, we first find the element $b_c$. The automorphism $a_c \in \text{Aut}\,\mathbb{Z}(C_p^{\times j})$ maps the coset $(1, i_2, \ldots, i_j) + \Delta$ to $(1, i_3 - i_2, \ldots, i_j - i_2, -i_2) + \Delta$ by adding $i_2$ to each coordinate. Hence under the isomorphism (10) it corresponds to the element
$$b_c = (\tau; 0, \ldots, 0, 1, \ldots, 1, p - 1, \ldots, p - 1) \in \Sigma_{p^{j-1}} \lhd C_p,$$
each number repeated $p^{j-2}$ times. Here $\tau$ is the permutation used in the definition of $\nu$ above. Using the definition of $\tau_j$, we have $\tau_j(x, \ldots, x) = (1; x, \ldots, x).$ \hfill $\square$

The following lemma is the main calculation we need for the proof of Theorem 3.2. Before we state it, we recall some standard group homological facts. Recall that $H_* (C_p) = \mathbb{F}_p \{\alpha_j \mid j \geq 0\}$ and
$$H_*(\Sigma_p) = \mathbb{F}_p \{\alpha_j \mid j = 2k(p - 1) \text{ or } j = 2k(p - 1) - 1, k \geq 0\},$$
with the induced map $H_*(C_p) \to H_*(\Sigma_p)$ being the canonical projection. For any $\pi \subset \Sigma_j$, with $j$ a positive integer, we have
$$H_* (\pi \lhd C_p) \cong H_* (\pi; H_*(C_p)^{\otimes j}).$$
When $\pi = C_p$, the $C_p$-module $H_* (C_p)^{\otimes p}$ can be decomposed as
$$H_* (C_p)^{\otimes p} \cong A \oplus (\mathbb{F}_p[C_p] \otimes B),$$
where $A = \mathbb{F}_p \{\alpha_j^{\otimes p} \mid j \geq 0\}$ and $B = \mathbb{F}_p \{\alpha_{\gamma(j_1)} \otimes \cdots \otimes \alpha_{\gamma(j_p)} \mid \gamma \in C, j_s \leq j_{s+1}, j_1 < j_p\}$. Here $C$ is a set of coset representatives for $C_p$ in $\Sigma_p$. This implies that
$$H_* (C_p; H_*(C_p)^{\otimes p}) \cong (H_* (C_p) \otimes A) \oplus B.$$
For the map
\[(13) \quad j_* : H_*(C_p; H_*(C_p)^{\otimes p}) \to H_*(\Sigma_p; H_*(C_p)^{\otimes p})\]
induced by the inclusion \(C_p \subset \Sigma_p\), \(j_*(\alpha_i \otimes \alpha_j^{\otimes p})\) is zero unless
(i) \(j\) is even and \(i = 2t(p-1)\) or \(i = 2t(p-1) - 1\).
(ii) \(j\) is odd and \(i = (2t+1)(p-1)\) or \((2t+1)(p-1) - 1\).

See [May70] for more details.

The classifying space \(B\mathcal{E}C_p\) has the structure of an \((\text{additive}) \ E_\infty\) space coming from disjoint union of \(C_p\)-sets, and as such has Dyer-Lashof operations in homology which we denote by \(\mathcal{Q}_a^i\), to differentiate them from the multiplicative operations in \(H_*(K(\mathcal{E}C_p))\).

**Lemma 3.6.**

(i) The induced map
\[ (\psi_2)_* : H_*(C_2 \wr C_p) \to H_*(B\mathcal{E}C_p) \]
satisfies
\[ (\psi_2)_*(1 \otimes \alpha_j \otimes \alpha_k) = \sum c_i \mathcal{Q}_a^i(\alpha_j \otimes k - 2t(p-1)) + d_i \mathcal{Q}_a^i(\alpha_j \otimes k + 2t(p-1)+1), \]
for some constants \(c_i\) and \(d_i\).

(ii) The induced map
\[ (\psi_p)_* : H_*(C_p \wr C_p) \to H_*(\Sigma_p \wr C_p) \]
satisfies
\[ (\psi_p)_*(c_i \otimes \alpha_a^{\otimes p}) = \sum \]

\[ \text{Proof.} \] For (i), we use the left-hand square in (12) with \(j = 2\). In this case \(\mu\) is given by \(\mu(c_1, c_2) = (c_2 - c_1, c_1)\), where \(c_2 - c_1 \in C_p \subset \Sigma_p\) is considered as a cyclic permutation. In homology we now have the commutative diagram
\[
\begin{array}{ccc}
H_*(C_p \times C_p) & \xrightarrow{i_*} & H_*(C_2 \wr C_p) \\
\downarrow{\mu_*} & & \downarrow{(\psi_2)_*} \\
H_*(\Sigma_p \times C_p) & \xrightarrow{d_*} & H_*(\Sigma_p \wr C_p),
\end{array}
\]
and to prove the stated formula we evaluate the effect of \(d_* \mu_*\) on the element \(\alpha_j \otimes \alpha_k\). Since \(\mu\) takes values in \(C_p \times C_p\), the composition \(d \circ \mu\) factors as
\[
C_p \times C_p \xrightarrow{\mu} C_p \times C_p \xrightarrow{d} C_p \wr C_p \xrightarrow{j} \Sigma_p \wr C_p.
\]
In homology we have
\[
\mu_*(\alpha_j \otimes 1) = (\chi \otimes 1) \circ \Delta(\alpha_j) = \sum_{s=0}^j (-1)^{j-s} \alpha_{j-s} \otimes \alpha_s,
\]
where \(\chi\) and \(\Delta\) are the conjugation and coproduct in \(H_*(C_p)\), respectively. Combined with the formula \(\mu_*(1 \otimes \alpha_k) = \alpha_k \otimes 1\), this yields
\[
\mu_*(\alpha_j \otimes \alpha_k) = \sum_{s=0}^j (-1)^{j+(k-1)s} (\alpha_{j-s} \ast \alpha_k) \otimes \alpha_s = \sum_{s=0}^j (-1)^{j+(k-1)s} \binom{j+k-s}{k} \alpha_{j+k-s} \otimes \alpha_s,
\]
where \(\ast\) denotes the Pontrjagin product in \(H_*(C_p)\). A standard computation gives a formula for \(d_*\) in terms of the dual Steenrod operations, see e.g. [May70, 9.1]. With our notation it is given
by

\[ d_*(\alpha_j \otimes \alpha_k) = \nu(k) \sum_t (-1)^t \alpha_{j+(2\ell-k)(p-1)} \otimes P^t_*(\alpha_k)^{\otimes p} \]

\[ - \delta(j) \nu(k - 1) \sum_t (-1)^t \alpha_{j+p+(2\ell-k)(p-1)} \otimes P^t_* \beta_*(\alpha_k)^{\otimes p}, \]

where \( \nu(2\ell + \epsilon) = (-1)^\ell(m!)^\epsilon, \ m = (p - 1)/2, \) and \( \delta(2\ell + \epsilon) = \epsilon. \) Using the explicit formulas for the \( P^*_* \), we have

\[ d_* \mu_*(\alpha_j \otimes \alpha_k) = \sum_{s=0}^{\infty} \sum_t \left[ c(s, t) \nu(s) \alpha_{j+k-s+(2\ell-s)(p-1)} \otimes \alpha_s^{\otimes p} \right] \]

\[ - \delta(s) \delta(j+s) \nu(s-1) \alpha_{j+k-s+p+(2\ell-s)(p-1)} \otimes \alpha_{s-1}^{\otimes p}, \]

where

\[ c(s, t) = (-1)^{j+(k-1)s+t} \left( \frac{j + k - s}{k} \right) \left( \frac{[s/2] - t(p-1)}{t} \right) \]

and

\[ d(s, t) = (-1)^{j+(k-1)s+t} \left( \frac{j + k - s}{k} \right) \left( \frac{[(s-1)/2] - t(p-1)}{t} \right). \]

Since we are really interested in the image in \( H_*(\Sigma_p \circ C_p) \), i.e., after composing with \( j_* \), the degrees of the \( \alpha \)-elements are subject to the constraints in (13). We consider two cases, according to the parity of \( j + k \). If \( j + k \) is even, all the terms on the second line in (15) vanish because of the \( \delta \)-factors. For the remaining terms, we first consider the terms with \( s \) even. Then \( \alpha_s^{\otimes p} \) is in even degree and \( 2(p-1) \) divides \( j + k - s + (2\ell-s)(p-1) \), hence we can write \( j + k - s = 2\ell(p-1) \) for some \( i \). For odd \( s \),

\[ j + k - s + (2\ell - s)(p-1) = u(p-1) - 1 \]

with \( u \) odd, hence \( j + k - s = 2i(p-1) - 1 \) for some \( i \). Letting \( \ell = i + t \), and noting that \( \nu(s + 4v) = \nu(s) \), we can rewrite the sum (15) for \( j + k \) even, after application of \( j_* \), as

\[ j_* d_* \mu_*(\alpha_j \otimes \alpha_k) = \sum_\ell [c_\ell (-1)^{\ell} \nu(j + k - 2\ell(p-1)) \alpha_{2\ell-\ell-j+k+2\ell(p-1)}(p-1) \otimes \alpha_{j+k-2\ell}(p-1) \]

\[ + d_\ell (-1)^{\ell} \nu(j + k - 2\ell(p-1)) \alpha_{2\ell-\ell-j+k+2\ell(p-1)}(p-1) \otimes \alpha_{j+k-2\ell(p-1)+1} \]

where

\[ c_\ell = \sum_k (-1)^{j+t} \left( \frac{2\ell(p-1)}{j + k - 2\ell(p-1)} \right) \left( \frac{(j + k)/2 - \ell(p-1)}{\ell - t} \right) \]

and

\[ d_\ell = \sum_k (-1)^{j+t} \left( \frac{2\ell(p-1)}{j + k - 2\ell(p-1)} \right) \left( \frac{(j + k)/2 - \ell(p-1)}{\ell - t} \right). \]

For (ii), choose \( j = p \) in the right-hand square in (12).

**Proof of Theorem 3.2.** The idea of the proof is to use the equivalence in Theorem 1.1 and study the maps of \( C_p \)-sets inducing the multiplicative structure on \( K \)-theory. This in turn entails finding explicit formulas in homology for certain group homomorphisms. The problem is to describe these group homomorphisms in a convenient way so as to be able to extract the relevant information. Fortunately we have already done all the hard work.

Consider the map

\[ \lambda_j: E\Sigma_j \times \Sigma_j (BC^rC^r) \wedge \rightarrow BC^rC^r \]
defined in (4). The classifying space $BE^C p$ decomposes as a disjoint union of components

$$BE^C p \simeq \bigsqcup_{[X]} B \text{Aut}(X),$$

where the union runs over isomorphism classes of finite $C_p$-sets. Let $\iota_j : B \text{Aut}(C_p^{	imes j}) \to BE^C p$ denote the inclusion of the component of $X = C_p^{	imes j}$. We have the commutative diagram

$$E \Sigma_j \rtimes \Sigma_j (BE^C p)^{\lambda_j} \xrightarrow{\lambda_j} BE^C p$$

$$E \Sigma_j \rtimes \Sigma_j (B \text{Aut}(C_p))^{\lambda_j} \xrightarrow{\cong} B(\Sigma_j \wr \text{Aut}(C_p)) \xrightarrow{B\phi} B \text{Aut}(C_p^{	imes j}),$$

$$\Sigma_j \wr \text{Aut}(G) \xrightarrow{\phi} \text{Aut}(G^{	imes j})$$

$$C_j \wr G \xrightarrow{\psi} \Sigma_{n+1} \wr G$$

\[\square\]

4. Topological cyclic homology of the circle

Here we compute the homology and induced multiplicative structure of $TC(S^1)$. As in the case $TC(*)$, the result will follow from a computation of the corresponding structure for the second stage in the inverse system defining $TC$. Since $TC(S^1) = TC(S[Z])$, this inverse system consists of cyclic fixed points of $THH(S[Z])$. To ease the notation we will often write $T(-)$ for $THH(-)$.

4.1. The homology of $T(S[Z])^C p$. First we need some general results about $THH$ of group rings. Recall that a topological monoid $G$ is called group-like if $\pi_0(G)$ is a group.

**Proposition 4.1.** Let $G$ be a topological group-like monoid.

(i) Let $A$ be a symmetric ring spectrum. There is an equivalence of cyclotomic spectra

$$T(A[G]) \simeq T(A) \wedge S[B^\varphi(G)].$$

(ii) There is an equivalence of cyclotomic spectra

$$S[\Lambda BG] \simeq S[B^\varphi(G)].$$

If $G$ is commutative, both equivalences are equivalences of $E_\infty$ ring spectra.

The inclusion of constant loops $BG \to \Lambda BG$ induces a map $T(A) \wedge BG_+ \to T(A) \wedge \Lambda BG_+$, which combined with the equivalence in Proposition 4.1 gives the assembly map

$$a_T : T(A) \wedge BG_+ \to T(A[G]).$$

The inclusion $BG \to \Lambda BG$ is $T$-equivariant when we give $BG$ the trivial action, and the map $a_T$ induces an assembly map

$$a_{TC} : TC(A) \wedge BG_+ \to TC(A[G]).$$

We now specialize to the case $A = S, G = Z$.

**Lemma 4.2.** The unit map

$$\iota : S \to T(S)$$

is a map of cyclotomic spectra which is a $C$-equivalence for all finite subgroups $C \subset T$.

Combining 4.1 and 4.2 we get the following.
Corollary 4.3. There is an $E_\infty$ ring map of cyclotomic spectra
\[ \mathbb{S}[\Lambda S^1] \to T(\mathbb{S}[\mathbb{Z}]) \]
which is a $C$-equivalence for all finite subgroups $C \subset T$.

Additively, the fixed point spectra $T(\mathbb{S}[\mathbb{Z}])^{C_{p^k}} \simeq \mathbb{S}[\Lambda S^1]^{C_{p^k}}$ can be identified by applying the Segal-tom Dieck splitting. We will need a stronger multiplicative statement, which is a reformulation of a result by Hesselholt and Madsen [HM04].

Proposition 4.4. There is an $E_\infty$ ring map $f$ which is part of a split cofiber sequence
\[ T(A)^{C_{p^k}}[\Lambda S^1] \xrightarrow{f} T(A)[\Lambda S^1]^{C_{p^k}} \longrightarrow \bigvee_{s=1}^{n} \bigoplus_{p^r \mid n} (\rho^s T(A)^{C_{p^k-r}} \wedge S^1(n)_+)^{C_{p^r}}. \]

We turn to the homological analysis. First note that $H_*(\mathbb{S}[\mathbb{Z}]) = \mathbb{F}_p[x, x^{-1}]$, concentrated in degree zero. If we forget the $\sigma$ induced from the $T$-equivariant $\mathbb{S}$, we have an equivalence of topological groups
\[ \Lambda S^1 \simeq \mathbb{Z} \times S^1. \]

Hence we get an equivalence of $E_\infty$ ring spectra $\mathbb{S}[\Lambda S^1] \simeq \mathbb{S}[\mathbb{Z}] \wedge \mathbb{S}[S^1]$, with homology algebra
\[ H_*(\mathbb{S}[\Lambda S^1]) = \mathbb{F}_p[x, x^{-1}] \otimes E(e). \]

Here $e$ is in degree 1. The action of the dual Steenrod algebra on $H_*(\mathbb{S}[\Lambda S^1])$ is trivial, and the only non-trivial Dyer-Lashof operation is $Q^0(x^k) = x^{pk}$.

Next we consider the homology of the fixed point spectrum $\mathbb{S}[\Lambda S^1]^{C_p}$. Proposition 4.4 immediately follows from the above result.

Proposition 4.5. Let $p$ be any prime.

1. There is a split short exact sequence
\[ 0 \longrightarrow H_*(\mathbb{S}^{C_p}) \otimes \mathcal{P}(x^{\pm 1}) \otimes E(e) \xrightarrow{f_*} H_*(\mathbb{S}[\Lambda S^1]^{C_p}) \longrightarrow \mathbb{F}_p\{e_0(n), e_1(n) \mid n \in \mathbb{Z}\} \longrightarrow 0, \]
of right $\mathcal{A}_e$-modules, with $|x| = 0$, $|e| = 1$ and $|e_s(n)| = s$. The map $f_*$ is an inclusion of algebras.

2. The dual Steenrod operations in $H_*(\mathbb{S}[\Lambda S^1]^{C_p})$ are trivial on $x^{\pm 1}$, $e$ and $e_s(n)$. The action on the $H_*(\mathbb{S}^{C_p})$ factor is given by Theorem 3.2.

3. The Dyer-Lashof operations in $H_*(\mathbb{S}[\Lambda S^1]^{C_p})$ are trivial on $x^{\pm 1}$ and $e$, with the exception that $Q^0(x^k) = x^{pk}$. The action on the $H_*(\mathbb{S}^{C_p})$ factor is given by Theorem 3.2.

4.2. The homotopy of $TC(S^1)$. Here we use the defining cartesian square (6) to calculate the homotopy type and homology of $TC(S^1)$. As a $T$-space
\[ (\Lambda S^1)_{hT} \simeq \bigsqcup_{n \in \mathbb{Z}} S^1(n), \]
where $S^1(n)$ is $S^1$ with a $T$-action given by $t \cdot s = t^n s$. Write
\[ H_*(\Lambda S^1) = \bigoplus_{n \in \mathbb{Z}} \mathbb{F}_p\{e_0(n), e_1(n)\} \]
for its homology, where $e_i(n)$ is in degree $i$. The map
\[ \sigma_* : H_*(\Lambda S^1) \to H_{*+1}(\Lambda S^1) \]
induced from the $T$-action is given by $\sigma_0(e_0(n)) = ne_1(n)$. We now have
\[ (\Lambda S^1)_{hT} \simeq \bigsqcup_{n \in \mathbb{Z}} ET \times_T S^1(n) \simeq \bigsqcup_{n \neq 0} BC_n \bigsqcup CP^\infty \times S^1, \]
where we define $C_n = C_{-n}$ for negative $n$, and we can easily write down the homology
\[ H_*(((\Lambda S^1)_{hT})) = \bigoplus_{p \mid n} \mathbb{F}_p\{\alpha_0(n)\} \bigoplus_{p \nmid n \neq 0} \mathbb{F}_p\{\alpha_j(n) \mid j \geq 0\} \bigoplus \mathbb{F}_p\{\beta_j, \beta'_j \mid j \geq 0\}. \]
Here $\alpha_j(n)$ has degree $j$, $\beta_j$ has degree $2j$ and $\beta_j'$ has degree $2j+1$. We have chosen the generators so that

$\pi_\ast(e_0(n)) = \begin{cases} 
\alpha_0(n) & \text{if } n \neq 0 \\
\beta_0 & \text{if } n = 0 
\end{cases},$

where $\pi: \Lambda S^1 \to (\Lambda S^1)_{hT}$ is the canonical projection. Note that the summands with $p \nmid n$ reduce to just an $\mathbb{F}_p$ in degree zero, or in other words, $(BC_n)_p^\wedge \simeq *$.

**Lemma 4.6.** The induced map in homology

$t_\ast: H_\ast(\Sigma(\Lambda S^1)_{hT}) \to H_\ast(\Lambda S^1)$

of the circle transfer is given by

$t_\ast(\Sigma \alpha_0(n)) = ne_1(n) \quad t_\ast(\Sigma \beta_0) = 0. $

The remaining classes are mapped to zero by degree reasons.

**Proof.** We will use the “degenerate double coset formula” from [MMM86, Lemma 2.7], which for us takes the form of a commutative diagram

$S^1 \wedge \Sigma^\infty_+ \Lambda S^1 \xrightarrow{\Sigma \pi} \Sigma \Sigma^\infty_+ (\Lambda S^1)_{hT} \xrightarrow{t} \Sigma \Sigma^\infty_+ \Lambda S^1.$

Here $\pi$ is the projection, $[T] \in \pi^1_+(T_\ast)$ is the fundamental class, and $\alpha$ is the action map. Applying homology we get the following diagram.

$\Sigma H_\ast(\Lambda S^1) \xrightarrow{\Sigma \pi_\ast} \Sigma H_\ast((\Lambda S^1)_{hT}) \xrightarrow{t} \Sigma \Sigma^\infty_+ H_\ast(\Lambda S^1).$

Here $\alpha_\ast$ restricted to $\Sigma H_\ast(\Lambda S^1) \subset H_\ast(T) \otimes H_\ast(\Lambda S^1)$ coincides with the map $\sigma_\ast$ in (18). Tracing the elements $\Sigma e_0(n) \in \Sigma H_\ast(\Lambda S^1)$ around the diagram, using (20) and (18), gives the result. \( \square \)

**Proposition 4.7.** There is an equivalence of spectra

$TC(S^1) \simeq TC(*) \wedge S^1_+ \vee \bigvee_{p \nmid n} \left( S^{-1}_+ \vee \bigvee_{k=0}^{\infty} \Sigma \Sigma^\infty_+ BC_p^k \right)$

such that the inclusion of the summand $TC(*) \wedge S^1_+$ coincides with the assembly map.

**Proof.** Using the splitting (17) of $\Lambda S^1$, the homotopy cartesian square (6) with $X = S^1$ splits as a wedge of simpler squares. Corresponding to $n = 0$ is the square

$TC(*) \wedge S^1_+ \xrightarrow{t} \Sigma \Sigma^\infty_+ \mathcal{P}^\infty_+ \wedge S^1_+ \xrightarrow{1-\Delta_p} \Sigma \Sigma^\infty_+ S^1(0).$
which is the square for $TC(*)$ smashed with $S^1_1$. Note that $1 - \Delta_p \simeq *$ in this case. For each $n \neq 0$ with $p \nmid n$, there is a pullback square

$$\begin{array}{ccc}
C & \rightarrow & \bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n) \\
\downarrow & & \downarrow t \\
\bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n) & \xrightarrow{1 - \Delta_p} & \bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n).
\end{array}$$

To identify $C$ we use the split cofiber sequence

$$\bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n) \xrightarrow{1 - \Delta_p} \bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n) \xrightarrow{\nabla} \Sigma_+ S^1,$$

where $\nabla$ is the identity on each summand. This gives a splitting of the bottom right corner in (21), and induces a splitting of $C$ as the sum of the pullbacks in the two squares

$$\begin{array}{ccc}
S^{-1} & \rightarrow & \Sigma_+ S^1 \\
\downarrow & & \downarrow t \\
\bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n) & \xrightarrow{s} & \Sigma_+ S^1
\end{array}$$

and

$$\begin{array}{ccc}
\bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n) & \rightarrow & \bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n) \\
\downarrow & & \downarrow t \\
\bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n) & \xrightarrow{1} & \bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n).
\end{array}$$

Here we use that $BC_{p^k} \simeq BC_{p^k}$ after $p$-completion. We can now conclude that $C \simeq S^{-1} \bigvee_{k=0}^\infty \Sigma_+ S^1(p^k n)$. \hfill \square

In conclusion we have the following description of the homology of $TC(S^1)$.

**Theorem 4.8.** Let $p$ be any prime.

1. The homology of $TC(S^1)$ is

$$H_*(TC(S^1)) = \mathbb{F}_p \{ a_0(m), 1, a_0, e_1, a_s(n) | m = 0 \text{ or } p \nmid m, s \geq 1, n \in \mathbb{Z} \},$$

with subscripts indicating degree. The Pontrjagin product is trivial, with 1 being the unit.

2. The dual Steenrod operations in $H_*(TC(S^1))$ are given by

$$Sq^k_s(a_s(n)) = \left( s - k \right) a_{s-k}(n)$$

for $p = 2$, and

$$P^k_s(a_s(n)) = \left( \left\lfloor s/2 \right\rfloor - k(p-1) \right) a_{s-2k(p-1)}(n)$$

$$\beta_s(a_{2s}(n)) = a_{2s-1}(n) \text{ if } p \nmid n$$

for odd $p$. The remaining operations are trivial.

3. The Dyer-Lashof operations in $H_*(TC(S^1))$ are given by

$$Q^i(a_s(n)) = (-1)^{i+s+1} \left( i - 1 \right) a_{s+i(p-1)}(pn)$$

$$Q^0(a_{-1}(n)) = a_{-1}(n),$$

The remaining operations are trivial.
Proof. All the statements involving only elements in non-negative degree will follow from Proposition 4.5, once we have determined the map $H_*(TC(S^1)) \rightarrow H_*(S[AS^1]^C_T)$ in homology.

For the remaining formulas involving $a_{-1}(m)$, we first use the map

$$H_*(TC(*)) \rightarrow H_*(TC(S^1))$$

induced from the inclusion of a point in $S^1$. This map respects all the structure in question and maps $\Sigma^j \beta_{-1}$ to $a_{-1}(0)$, so the formulas for $a_{-1}(0)$ follow from Theorem 3.1 and Corollary 3.3.

\[ \Box \]

References


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