We consider the commutative $S$–algebra given by the topological cyclic homology of a point. The induced Dyer–Lashof operations in mod $p$ homology are shown to be nontrivial for $p = 2$, and an explicit formula is given. As a part of the calculation, we are led to compare the fixed point spectrum $S^G$ of the sphere spectrum and the algebraic $K$–theory spectrum of finite $G$–sets, as structured ring spectra.

55S12, 55P43; 19D55, 55P92, 19D10

Introduction

Let $A(\bullet) = K(S)$ denote Waldhausen’s algebraic $K$–theory of a point [23]. It is a commutative $S$–algebra, in the sense of Elmendorf, Kriz, Mandell and May [7], and the algebraic $K$–theory $A(X)$ of any space $X$, or more generally the algebraic $K$–theory $K(R)$ of any $S$–algebra $R$, is a module spectrum over it. Hence it makes sense to carefully study the commutative $S$–algebra structure of $A(\bullet)$, or equivalently its structure as an $E_\infty$ ring spectrum. To the eyes of mod $p$ homology, the primary incarnation of this structure is the Pontryagin algebra structure on $H_*(A(\bullet))$, together with the multiplicative Dyer–Lashof operations $Q^i: H_*(A(\bullet)) \to H_{*+i}(A(\bullet))$, as defined by Bruner, May, McClure and Steinberger [2]. Here and elsewhere we write $H_s(E)$ for the mod $p$ homology $H_*(E; \mathbb{F}_p)$ of a spectrum $E$.

The additive structure of $H_*(A(\bullet))$ is known for $p = 2$ and for $p$ an odd regular prime, by the second author’s papers [18; 19], but at present the Pontryagin product and Dyer–Lashof operations are not known for this $E_\infty$ ring spectrum. There is, however, a very good approximation to Waldhausen’s algebraic $K$–theory, given by the cyclotomic trace map to the topological cyclic homology of Bökstedt, Hsiang and Madsen [1]. This is a natural map $\text{trc}: K(R) \to \text{TC}(R; p)$, which we write as $\text{trc}: A(\bullet) \to \text{TC}(\bullet; p)$ in the special case when $R = S$, where $\text{TC}(\bullet; p) = \text{TC}(S; p)$ is the topological cyclic
homology of a point. By a theorem of Dundas [5], there is a homotopy cartesian square

\[
\begin{array}{ccc}
A(\bullet) & \longrightarrow & K(\mathbb{Z}) \\
\downarrow \text{trc} & & \downarrow \text{trc} \\
\text{TC}(\bullet; p) & \longrightarrow & \text{TC}(\mathbb{Z}; p)
\end{array}
\]

(after \(p\)-adic completion) of commutative \(S\)-algebras (see Geisser and Hesselholt [9, Section 6]), and this square is the basis for our additive understanding of \(H_\ast(A(\bullet))\).

We are therefore led to study the commutative \(S\)-algebra structure of \(\text{TC}(\bullet; p)\), including the Pontryagin algebra structure and the Dyer–Lashof operations on its mod \(p\) homology. Like in the case of algebraic \(K\)-theory, the topological cyclic homology \(\text{TC}(X; p)\) of any space \(X\), and more generally the topological cyclic homology \(\text{TC}(R; p)\) of any \(S\)-algebra \(R\), is a module spectrum over \(\text{TC}(\bullet; p)\), and this provides a second motivation for the study of \(\text{TC}(\bullet; p)\). In the present paper, we determine the Dyer–Lashof operations in \(H_\ast(\text{TC}(\bullet; p))\) in the case when \(p = 2\), as explained in Theorem 0.2 and Corollary 0.3 below.

A third motivation stems from ideas of Jack Morava [17], to the effect that there may be a spectral enrichment of the algebro-geometric category of mixed Tate motives, given by \(A\)-theoretic (see Williams [24]) or \(\text{TC}\)-theoretic (see Dundas and Østvær [6]) correspondences, followed by stabilization. The trace map \(A(\bullet) \rightarrow \text{TC}(\bullet; p) \rightarrow \text{THH}(\bullet) = S\) defines a fiber functor to the category of \(S\)-modules, with Tannakian automorphism group realized through its Hopf algebra of functions, which will be of the form \(S \wedge_{A(\bullet)} S\) or \(S \wedge_{\text{TC}(\bullet; p)} S\). Rationally, this is well compatible with Deligne’s results on the Tannakian group of mixed Tate motives over the integers [4]. A calculational analysis of the commutative \(S\)-algebras \(S \wedge_{A(\bullet)} S\) or \(S \wedge_{\text{TC}(\bullet; p)} S\) clearly depends heavily on a proper understanding of the commutative \(S\)-algebra structures of \(A(\bullet)\) and \(\text{TC}(\bullet; p)\).

Let \(\mathbb{T}\) be the circle group and let \(C_{p^n} \subset \mathbb{T}\) be the (cyclic) subgroup of order \(p^n\). The spectrum \(\text{TC}(\bullet; p)\) is defined as the homotopy inverse limit of a diagram

\[
\begin{array}{cccccccc}
\cdots & \overset{R}{\underset{F}{\longrightarrow}} & S^{C_{p^n+1}} & \overset{R}{\underset{F}{\longrightarrow}} & S^{C_{p^n}} & \overset{R}{\underset{F}{\longrightarrow}} & \cdots & \overset{R}{\underset{F}{\longrightarrow}} & S^{C_p} & \overset{R}{\underset{F}{\longrightarrow}} & S
\end{array}
\]

of \(E_\infty\) ring spectra, where \(S^{C_{p^n}}\) denotes the \(C_{p^n}\)-fixed points of the \(\mathbb{T}\)-equivariant sphere spectrum, the maps labeled \(R\) are restriction maps, and the maps labeled \(F\) are Frobenius maps. See Bökstedt, Hsiang and Madsen [1] or Hesselholt and Madsen [10] for the construction of these maps. Similarly, let \(\text{TC}^{(1)}(\bullet; p)\) denote the homotopy
Homology operations in the topological cyclic homology of a point

limit of the subdiagram

\[(0-2) \quad \mathbb{S} \mathbb{C}_p \xrightarrow{R} \mathbb{S}

that is, the homotopy equalizer of \(R\) and \(F\). The canonical maps

\[(0-3) \quad \text{TC}(\bullet; p) \xrightarrow{f_1} \text{TC}^{(1)}(\bullet; p) \xrightarrow{g_1} \mathbb{S} \mathbb{C}_p

are then maps of \(E_\infty\) ring spectra.

The unit \(\eta: \mathbb{S} \to \text{TC}(\bullet; p)\) and the restriction \(R: \mathbb{S} \mathbb{C}_p \to \mathbb{S}\) let us split off a copy of \(\mathbb{S}\) from each term in (0-3). Let \(\mathbb{C}P_1^\infty\) be the Thom spectrum of the negative tautological complex line bundle \(-\gamma_1\mathbb{C}\) over \(\mathbb{C}P^\infty\). Its suspension \(\Sigma \mathbb{C}P_1^\infty\) is equivalent to the homotopy fiber of the dimension-shifting \(T\)-transfer map \(t_T: \Sigma^\infty \Sigma(\mathbb{C}P^\infty) \to \mathbb{S}\); see Knapp [13, 2.9] or Lemma 1.1 below. We define the spectrum \(L_1\) to be the homotopy fiber of the \(C_p\)-transfer \(t_p: \Sigma^\infty (BP_1)_+ \to \mathbb{S}\). For \(p = 2\), there is an equivalence \(L_1^\infty \simeq \mathbb{R}P_1^\infty\), where \(\mathbb{R}P_1^\infty\) is the Thom spectrum of the negative tautological real line bundle \(-\gamma_1\mathbb{R}\) over \(\mathbb{R}P^\infty\). The mod \(p\) homology groups of these spectra are well known:

\[
H_*(\Sigma \mathbb{C}P_1^\infty) \cong \mathbb{F}_p \{ \Sigma \beta_k \mid k \geq -1 \}
\]

\[
H_*(L_1^\infty) \cong \mathbb{F}_p \{ \alpha_k \mid k \geq -1 \}
\]

\[
H_*(\Sigma^\infty (BP_1)_+) \cong \mathbb{F}_p \{ \alpha_k \mid k \geq 0 \}
\]

Here \(\Sigma \beta_k\) has degree \(2k + 1\) and \(\alpha_k\) has degree \(k\).

**Lemma 0.1** After \(p\)-completion, diagram (0-3) is homotopy equivalent to a diagram

\[
\mathbb{S} \vee \Sigma \mathbb{C}P_1^\infty \xrightarrow{1\vee f} \mathbb{S} \vee L_1^\infty \xrightarrow{1\vee g} \mathbb{S} \vee \Sigma^\infty (BP_1)_+
\]

In particular, the Pontryagin product on \(H_*(\text{TC}(\bullet; p))\) is trivial.

Applying homology gives a sequence

\[
H_*(\mathbb{S}) \oplus H_*(\Sigma \mathbb{C}P_1^\infty) \xrightarrow{1\oplus f_*} H_*(\mathbb{S}) \oplus H_*(L_1^\infty) \xrightarrow{1\oplus g_*} H_*(\mathbb{S}) \oplus H_*(\Sigma^\infty (BP_1)_+).
\]

Here \(f_*\) sends \(\Sigma \beta_k\) to \(\alpha_{2k+1}\) for \(k \geq -1\), and \(g_*\) is the identity on \(\alpha_k\) for \(k \geq 0\), while \(\alpha_{-1}\) maps to zero.

We now state our main result, which concerns the Dyer–Lashof operations in the mod \(p\) spectrum homology \(H_*(\text{TC}(\bullet; p))\) for \(p = 2\). The calculations will be done in the auxiliary \(E_\infty\) ring spectra \(\mathbb{S} \mathbb{C}_2\) and \(\text{TC}^{(1)}(\bullet; 2)\).

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Theorem 0.2  The Dyer–Lashof operations $Q^i$ in $H_* (\text{TC}^{(1)}(\bullet ; 2))$ and $H_* (\mathbb{S}^2)$ are given by the formula

$$Q^i (\alpha_j) = \binom{2N + i - 1}{2N + j} \alpha_{i+j},$$

where $j \geq -1$ and $i$ is any integer, and $N$ is sufficiently large.

Corollary 0.3  The Dyer–Lashof operations $Q^{2i}$ in $H_* (\text{TC}(\bullet ; 2))$ are given by the formula

$$Q^{2i} (\Sigma \beta_j) = \binom{2N + i - 1}{2N + j} \Sigma \beta_{i+j},$$

where $j \geq -1$ and $i$ is any integer, and $N$ is sufficiently large. The operations $Q^{2i+1}$ are all zero for degree reasons.

Note that the binomial coefficients used in the theorem and corollary can be evaluated to

$$\binom{2N + i - 1}{2N + j} = \begin{cases} (i-1) & \text{for } i > j \geq 0, \\ 1 & \text{for } (i, j) = (0, -1), \\ 0 & \text{otherwise} \end{cases}$$

modulo 2, for all sufficiently large $N$. In particular $Q^0 (\alpha_{-1}) = \alpha_{-1}$ and $Q^0 (\Sigma \beta_{-1}) = \Sigma \beta_{-1}$.

We prove Lemma 0.1 in Section 1 and Theorem 0.2 in Section 3, after a homological comparison of $E_\infty$ ring structures in Section 2. Corollary 0.3 follows immediately from the lemma and the theorem.

1  Topological cyclic homology of a point

In this preliminary section we review the calculation of $\text{TC}(\bullet ; p)$ from Bökstedt, Hsiang and Madsen [1, 5.17], in order to describe the map $f_1$ to $\text{TC}^{(1)}(\bullet ; p)$.

For each $n \geq 1$ the Segal–tom Dieck splitting tells us that the norm-restriction homotopy cofiber sequence

$$\Sigma^\infty (BC_{p^n})_+ \xrightarrow{N} \mathbb{S}C_{p^n} \xrightarrow{R} \mathbb{S}C_{p^{n-1}}$$

is canonically split. The homotopy limit $\text{TR}(\bullet ; p) = \text{holim}_{n,R} \mathbb{S}C_{p^n}$ of the $R$–maps in $(0-1)$ thus factors as $\text{TR}(\bullet ; p) \simeq \prod_{n \geq 0} \Sigma^\infty (BC_{p^n})_+$. Let

$$\text{pr}_n : \text{TR}(\bullet ; p) \to \Sigma^\infty (BC_{p^n})_+$$

denote the $n$–th projection, and let $\widetilde{\text{TR}}(\bullet; p) \simeq \prod_{n \geq 1} \Sigma^\infty (BC_{p^n})_+$ be the homotopy fiber of $pr_0$. There is a vertical map of horizontal homotopy fiber sequences

\[
\begin{array}{cccccc}
\text{TC}(\bullet; p) & \xrightarrow{\pi} & \text{TR}(\bullet; p) & \xrightarrow{F-R} & \text{TR}(\bullet; p) \\
\downarrow f_1 & & \downarrow p_1 & & \downarrow p_0 \\
\text{TC}^{(1)}(\bullet; p) & \xrightarrow{g_1} & \Sigma C_p & \xrightarrow{F-R} & S
\end{array}
\]

(1-1)

and the augmentation $\text{TC}(\bullet; p) \to S$ factors as $R \circ p_1 \circ \pi = pr_0 \circ \pi$. Replacing the left hand square by the homotopy fibers of the augmentations to $S$, we get a second vertical map of horizontal homotopy fiber sequences

\[
\begin{array}{cccccc}
\widetilde{\text{TC}}(\bullet; p) & \to & \widetilde{\text{TR}}(\bullet; p) & \xrightarrow{T-I} & \text{TR}(\bullet; p) \\
\downarrow f_2 & & \downarrow pr_1 & & \downarrow pr_0 \\
\widetilde{\text{TC}}^{(1)}(\bullet; p) & \xrightarrow{g_2} & \Sigma^\infty (BC_p)_+ & \xrightarrow{t_p} & S
\end{array}
\]

(1-2)

In the upper row we have used that $F - R$ restricted along the inclusion $I: \widetilde{\text{TR}}(\bullet; p) \to \text{TR}(\bullet; p)$ is homotopic to $T - I$, where $T$ is the product of the $C_p$–transfer maps $\Sigma^\infty (BC_{p^n})_+ \to \Sigma^\infty (BC_{p^{n-1}})_+$ for all $n \geq 1$. See Bökstedt, Hsiang and Madsen [1, (5.18)]. In the lower row we have used that $F - R$ restricted along $N: \Sigma^\infty (BC_p)_+ \to \Sigma C_p$ is homotopic to the $C_p$–transfer map $t_p$.

There is a third vertical map of horizontal homotopy fiber sequences

\[
\begin{array}{cccccc}
\text{holim}_n \Sigma^\infty (BC_{p^n})_+ & \to & \text{TR}(\bullet; p) & \xrightarrow{T-I} & \text{TR}(\bullet; p) \\
\downarrow f_3 & & \downarrow p_1 & & \downarrow p_0 \\
\text{hofib}(F - 2R) & \xrightarrow{g_3} & \Sigma C_p & \xrightarrow{F-2R} & S
\end{array}
\]

(1-3)

Replacing its left hand square by the homotopy fibers of the augmentations to $S$, we also recover diagram (1-2).

**Lemma 1.1** There are equivalences

$\widetilde{\text{TC}}(\bullet; p) \simeq \text{hofib}(t_T: \Sigma^\infty \Sigma(CP_+^\infty) \to S) \simeq \Sigma CP_{-1}^\infty$

(after $p$–completion) and

$\widetilde{\text{TC}}^{(1)}(\bullet; p) \simeq \text{hofib}(t_p: \Sigma^\infty (BC_p)_+ \to S) = L_{-1}^\infty$.

When $p = 2$, $L_{-1}^\infty \simeq \mathbb{RP}_{-1}^\infty$. 

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The dimension-shifting $\mathbb{T}$–transfer maps for the bundles $\text{BC}_{p^n} \to B\mathbb{T}$ induce an equivalence $\Sigma^\infty \Sigma(\mathbb{CP}_+^\infty) \simeq \text{holim}_n \Sigma^\infty (\text{BC}_{p^n})_+$ after $p$–completion [1, 5.15]. The augmentation $\text{holim}_n \Sigma^\infty (\text{BC}_{p^n})_+ \to \mathbb{S}$ then gets identified with $t_\mathbb{T}$, which implies the first claim.

There is a $\mathbb{T}$–equivariant homotopy cofiber sequence $S^0 \xrightarrow{z} S^\mathbb{C} \xrightarrow{t} \mathbb{T}_+ \wedge S^1$, where $z$ is the zero-inclusion. The right hand map $t$ is the Pontryagin–Thom collapse associated to the standard embedding $\mathbb{T} \subset \mathbb{C}$, as in Lewis, May and Steinberger [14, II.5.1]. The dimension-shifting $\mathbb{T}$–transfer $t_\mathbb{T} : \Sigma^\infty \Sigma(\mathbb{CP}_+^\infty) \to \mathbb{S}$ for $E\mathbb{T} \to B\mathbb{T}$ is constructed as the balanced smash product

$$1 \wedge_\mathbb{T} \Sigma^1 - \mathbb{C}(t) : E\mathbb{T}_+ \wedge_\mathbb{T} \Sigma^1 - \mathbb{C}(S^\mathbb{C}) \to E\mathbb{T}_+ \wedge_\mathbb{T} \Sigma^1 - \mathbb{C}(\mathbb{T}_+ \wedge S^1)$$

(see Lewis, May and Steinberger [14, II.7.5]). Hence its homotopy fiber is $E\mathbb{T}_+ \wedge_\mathbb{T} \Sigma^1 - \mathbb{C}(S^0) \cong \Sigma \mathbb{CP}_+^\infty$.

The proof that $\mathbb{RP}_{\infty}^1$ is the homotopy fiber of $t_p$ for $p = 2$ is essentially the same. □

**Proof of Lemma 0.1** Under the identifications of Lemma 1.1, the maps $f : \Sigma \mathbb{CP}_+^\infty \to L^\mathbb{C}_1$ and $g : L^\mathbb{C}_1 \to \Sigma^\infty (\text{BC}_p)_+$ correspond to the maps $f_2$ and $g_2$ in diagram (1-2), respectively.

The $\mathbb{C}_p$–transfer map $t_p$ induces multiplication by $p$ on $\pi_0$, and the zero map in mod $p$ homology, so $\pi_{-1}(f)$ is surjective, $f_*$ maps $\Sigma \beta_{-1}$ to $\alpha_{-1}$, and $g_*$ maps $\alpha_k$ to $\alpha_k$ for all $k \geq 0$. It remains to see that $g_* f_*$ maps $\Sigma \beta_k$ to $\alpha_{2k+1}$ for $k \geq 0$. This is clear from diagram (1-3), since $g_3 f_3$ agrees in positive degrees with the $\mathbb{T}$–transfer map $\Sigma^\infty \Sigma(\mathbb{CP}_+^\infty) \to \Sigma^\infty (\text{BC}_p)_+$, which has this behavior on homology. □

## 2 Algebraic $K$–theory of finite $G$–sets

In this section we will compare the algebraic $K$–theory spectrum of finite $G$–sets with the $G$–fixed points of the sphere spectrum, as structured ring spectra. Before we state the result we recall some of the definitions involved.

The $K$–theory construction we use is that of Elmendorf and Mandell [8]. When the input category is a bipermutative category $\mathcal{C}$, their machine produces a symmetric spectrum $K(\mathcal{C})$, in the sense of Hovey, Shipley and Smith [11], with an action of the simplicial Barratt–Eccles operad. We will use the same notation for the geometrically realized symmetric spectrum in topological spaces, which has an action

$$\kappa_j : E \Sigma_j \simeq K(\mathcal{C})^\wedge_j \to K(\mathcal{C})$$
of the operad $E \Sigma$ consisting of the contractible $\Sigma_j$–free spaces $E \Sigma_j$. As usual, $E \Sigma_j$ can be defined as the nerve $N \bar{\Sigma}_j$ of the translation category $\bar{\Sigma}_j$, for $j \geq 0$. The $K$–theory construction itself is somewhat involved, but all we need to know is that the zeroth space $K(C)_0$ is the nerve $NC$ of $C$, so the zeroth space of $E \Sigma_j \vee \Sigma_j K(C)^{\wedge j}$ is the nerve of $\bar{\Sigma}_j \vee \Sigma_j C$ and the action of $E \Sigma$ on $K(C)_0$ is given by the maps $\lambda_j: E \Sigma_j \vee \Sigma_j NC^{\wedge j} \to NC$ that are induced by the functors that take an object $(\sigma; a_1, \ldots, a_j)$ in $\bar{\Sigma}_j \vee \Sigma_j C$ to the object $a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(j)}$ in $C$ (see Elmendorf and Mandell [8, Section 8]). Here $\otimes$ denotes the product in the bipermutative structure on $C$. Hence there is a commutative diagram

$$
\begin{array}{ccc}
E \Sigma_j \vee \Sigma_j K(C)^{\wedge j} & \xrightarrow{\kappa_j} & K(C) \\
\downarrow \quad 1_{K(C)} & & \downarrow 1_{K(C)} \\
E \Sigma_j \vee \Sigma_j (\Sigma^\infty NC)^{\wedge j} & \xrightarrow{\lambda_j} & \Sigma^\infty (E \Sigma_j \vee \Sigma_j NC^{\wedge j}) \\
\end{array}
$$

for each $j \geq 0$.

Let $G$ be a finite group, and let $\mathcal{E}^G$ denote the category of finite $G$–sets and $G$–equivariant bijections. This is a symmetric bimodal category under disjoint union and cartesian product, taking $(X, Y)$ to $X \coprod Y$ and $X \times Y$, respectively. We give $X \times Y$ the diagonal $G$–action. There is a functorially defined bipermutative category $\Phi \mathcal{E}^G$, and a natural equivalence $\mathcal{E}^G \to \Phi \mathcal{E}^G$ [16, VI.3.5]. It follows that there is a homotopy commutative diagram

$$
\begin{array}{ccc}
E \Sigma_j \vee \Sigma_j (\Sigma^\infty N \mathcal{E}^G)^{\wedge j} & \xrightarrow{\kappa_j} & K(\Phi \mathcal{E}^G) \\
\downarrow \quad 1_{\Phi \mathcal{E}^G} & & \downarrow 1_{\Phi \mathcal{E}^G} \\
E \Sigma_j \vee \Sigma_j (\Sigma^\infty N \mathcal{E}^G)^{\wedge j} & \xrightarrow{\lambda_j} & \Sigma^\infty (E \Sigma_j \vee \Sigma_j (N \mathcal{E}^G)^{\wedge j}) \\
\end{array}
$$

for each $j \geq 0$, where

$$
\lambda_j: E \Sigma_j \vee \Sigma_j (N \mathcal{E}^G)^{\wedge j} \to N \mathcal{E}^G
$$

is induced by the functor $\bar{\Sigma}_j \vee \Sigma_j (\mathcal{E}^G)^j \to \mathcal{E}^G$ that takes $(\sigma; X_1, \ldots, X_j)$ to the cartesian product

$$
X_{\sigma^{-1}(1)} \times (X_{\sigma^{-1}(2)} \times \cdots \times (X_{\sigma^{-1}(j-1)} \times X_{\sigma^{-1}(j)}) \cdots).
$$

Let $\mathcal{U}$ be a complete $G$–universe, and let $\mathcal{L}$ denote the linear isometries operad with spaces $\mathcal{L}(j)$ consisting of linear isometries $\mathcal{U}^j \to \mathcal{U}$, where $\mathcal{U}^j$ denotes the direct sum of $j$ copies of $\mathcal{U}$. There is an action of $G$ on each $\mathcal{L}(j)$ given by conjugation, and this gives $\mathcal{L}$ the structure of an $E_\infty$ $G$–operad in the sense of Lewis, May and

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Steinberger [14, VII.1.1]. The $E_\infty$ ring structure on the $G$–equivariant sphere spectrum $S_G = \Sigma_G^\infty S^0$ is given by an action

$$\zeta_j: \mathcal{L}(j) \times \Sigma_j \Sigma_G^j \to S_G$$

of this operad (where, for once, $\times$ denotes the twisted half-smash product in Lewis–May spectra). It is compatible with a corresponding action

$$\omega_j: \mathcal{L}(j) \times \Sigma_j Q_G(S^0)^\wedge j \to Q_G(S^0)$$

on the underlying infinite loop space $Q_G(S^0) = \Omega^\infty S_G = \operatorname{colim}_{V \subset U} \Omega^V S^V$, in the sense that the following diagram commutes.

$$\begin{align*}
\mathcal{L}(j) \times \Sigma_j (\Sigma^\infty Q_G(S^0)^\wedge j) &\xrightarrow{1 \times e^\wedge j} \Sigma^\infty (\mathcal{L}(j) \times \Sigma_j Q_G(S^0)^\wedge j) \\
&\xrightarrow{\Sigma^\infty \omega_j} \Sigma^\infty Q_G(S^0) \\
\mathcal{L}(j) \times \Sigma_j S_G^j &\xrightarrow{\zeta_j} S_G
\end{align*}$$

Here $\omega_j$ sends an element in $\mathcal{L}(j) \times \Sigma_j Q_G(S^0)^\wedge j$ represented by $(f; g_1, \ldots, g_j)$, where $f: U^j \to U$ and $g_i: S^{V_j} \to S^{V_i}$, to the element represented by the composite of the following maps.

$$S f(V_1 \oplus \cdots \oplus V_j) \xrightarrow{f_*} S^{V_1 \oplus \cdots \oplus V_j} \xrightarrow{g_1 \wedge \cdots \wedge g_j} S^{V_1 \oplus \cdots \oplus V_j} \xrightarrow{f_*} S f(V_1 \oplus \cdots \oplus V_j)$$

By taking $G$–fixed points we get the nonequivariant $E_\infty$ ring spectrum $S^G = (S_G)^G$ with an action

$$\xi_j: \mathcal{L}^G(j) \times \Sigma_j (S^G)^\wedge j \to S^G$$

of the nonequivariant $E_\infty$ operad $\mathcal{L}^G$ of $G$–equivariant isometries. The corresponding infinite loop space $\Omega^\infty (S^G)$ is the space $Q_G(S^0)^G = \operatorname{colim}_{V \subset U} (\Omega^V S^V)^G$, with the inherited $\mathcal{L}^G$–action

$$\eta_j: \mathcal{L}^G(j) \times \Sigma_j (Q_G(S^0)^G)^\wedge j \to Q_G(S^0)^G.$$
Now consider the image of the cycle $e_i \otimes z^p$ under the map

$$C_*(O(p)) \otimes_{\Sigma_p} C_*(E)^{\otimes p} \xrightarrow{\cong} C_*(O(p) \ltimes \Sigma_p \ E^{\wedge p}) \xrightarrow{\xi_p} C_*(E),$$

and denote its image in homology by $Q_i(x)$. Here $\xi_p$ is the $E_\infty$ structure map. Then for $p = 2$ define $Q_i(x) = 0$ when $i < q$, and

$$Q_i(x) = Q_{i-q}(x)$$

when $i \geq q$. For $p > 2$ define $Q_i(x) = 0$ when $2i < q$, and

$$Q_i(x) = (-1)^i v(q) \cdot Q_{(2i-q)(p-1)}(x)$$

when $2i \geq q$, where $v(q) = (-1)^{q-1}(p-1)/((\frac{1}{2}(p-1))!p^q)$. See Bruner, May, McClure and Steinberger [2, Chapter III] for more details.

The spectra $S^G$ and $K(\Phi E^G)$ should be equivalent as $E_\infty$ ring spectra, but we will only need the following weaker result.

**Lemma 2.1** There is an equivalence $S^G \simeq K(\Phi E^G)$ of spectra such that the induced isomorphism $H_*(S^G) \cong H_*(K(\Phi E^G))$ commutes with the Dyer–Lashof operations.

**Proof** Our first goal is to construct a commutative diagram:

$$\begin{array}{ccc}
E \Sigma_j \ltimes \Sigma_j (N^G)^{\wedge j} & \xrightarrow{\lambda_j} & N^G \\
\downarrow^{1 \ltimes \phi^{\wedge j}} & & \downarrow^{\phi} \\
E \Sigma_j \ltimes \Sigma_j (N^G)^{\wedge j} & \xrightarrow{\cong} & (E \Sigma_j \ltimes L^G(j)) \ltimes \Sigma_j D^{\wedge j}_t & \xrightarrow{\mu_j} & D_t & \xrightarrow{\cong} & N^G \\
\downarrow^{1 \ltimes \psi^{\wedge j}} & & \downarrow^{\cong} & & \downarrow^{\psi} \\
L^G(j) \ltimes \Sigma_j C^{\wedge j}_t & \xrightarrow{\nu_j} & C_t & \xrightarrow{\eta_j} & Q_G(S^0)^G
\end{array}$$

(2-3)

We start by describing the space $C_t$. Let $V$ be an indexing space in $U$. For each finite $G$-set $X$, consider the space $E_V(X)$ of $X$-tuples of distance-reducing embeddings of $V$ in $V$, closed under the action of $G$. More precisely, this is the space of $G$-equivariant maps $\coprod_X V \to V$ such that the restriction to each summand $g: V \to V$ is an embedding that satisfies $|g(v) - g(w)| \leq |v - w|$ for all $v, w \in V$. Let $K_V(X)$
be the space of paths $[0, 1] \to E_V(X)$ such that the embeddings at the endpoint 0 are identities, and the embeddings at 1 have disjoint images. Now let

$$K_{\mathcal{U}}(X) = \operatorname{colim}_{V \subset U} K_V(X).$$

These are $G$–equivariant versions of the spaces in the Steiner operad [22]. The group Aut$^G(X)$ acts freely on $K_{\mathcal{U}}(X)$ by permuting the embeddings, and the space $C_{\mathcal{U}}$ is to be the disjoint union

$$C_{\mathcal{U}} = \bigsqcup_{[X]} K_{\mathcal{U}}(X)/ \operatorname{Aut}^G(X)$$

where $X$ ranges over all isomorphism classes of finite $G$–sets.

The action of the operad $L^G$ on $C_{\mathcal{U}}$ is defined as follows. Let $f: U^j \to U$ be a $G$–linear isometry, and let $[g_i], 1 \leq i \leq j$, be elements in $C_{\mathcal{U}}$, represented by paths of $X_i$–tuples of embeddings $g_i \in K_{\mathcal{U}}(X_i)$. Denote the component paths of embeddings that constitute $g_i$ by $g_{i,x_i}$, where $x_i \in X_i$. The resulting element $v_j(f; [g_1], \ldots, [g_j])$ in $C_{\mathcal{U}}$ is represented by an element in $K_{\mathcal{U}}(X_1 \times \cdots \times X_j)$, which on the summand indexed by $(x_1, \ldots, x_j)$ is given by $f \circ (g_{1,x_1} \times \cdots \times g_{j,x_j}) \circ f^{-1}$.

There is a map $C_{\mathcal{U}} \to Q_G(S^0)^G$, given by evaluating a Steiner path in $E_V(X)$ at 1 to get a $G$–equivariant embedding $e: \bigsqcup X V \to V$, and then applying a folded Pontryagin–Thom construction to obtain a $G$–equivariant map $q: S^V \to S^V$, which is a point in $Q_G(S^0)^G$. Given the distance-reducing embedding $e$, let $S^V \to \bigsqcup_X S^V$ be the $G$–equivariant map that is given by $e^{-1}$ on the image of $e$ in $V \subset S^V$ and maps the remainder of $S^V$ to the base point of $\bigsqcup_X S^V$. Let $\bigsqcup_X S^V \to S^V$ be the fold map that is the identity on each summand. The folded Pontryagin–Thom construction $q$ is the composite of these two $G$–maps. If we permute the embeddings indexed by $X$ we get the same element in $Q_G(S^0)^G$, so our map is well-defined. A comparison of definitions shows that this construction is compatible with the $L^G$–actions on $C_{\mathcal{U}}$ and $Q_G(S^0)^G$, so the lower square in (2-3) commutes.

Let

$$D_{\mathcal{U}} = \bigsqcup_{[X]} (E \operatorname{Aut}^G(X) \times K_{\mathcal{U}}(X))/ \operatorname{Aut}^G(X),$$

where $\operatorname{Aut}^G(X)$ acts diagonally on the product. The nerve $N\mathcal{E}^G$ splits as a sum of components

$$N\mathcal{E}^G \simeq \bigsqcup_{[X]} B\operatorname{Aut}^G(X),$$

where the disjoint union is over the isomorphism classes of $G$–sets $X$. Projection on the first factor in $D_{\mathcal{U}}$ followed by this homotopy equivalence gives the map.
$D_{ij} \to NE_{it}^G$ in (2-3), while the map $D_{ij} \to C_{ij}$ is the projection on the second factor. There is an induced action of the product operad $E \times L^G$ on $D_{ij}$, defined as follows. Let $(e, f) \in E \times L^G(j)$ and let $(e_i, f_i) \in E \times L^G(X) \times C_{ij}(X)$ represent elements in $D_{ij}$, for $1 \leq i \leq j$. The image under $\mu_j$ is the element represented by $(\lambda_j(e_1), \ldots, e_j, v_j(f_1, \ldots, f_j))$. This makes the upper and middle squares in (2-3) commute.

Let

$$NE_{it}^G = \left( \bigsqcup_{(H)} (E \times L^G(G/H) \times K_{ij}(G/H))/\text{Aut}^G(G/H) \right)_+,$$

where the coproduct is taken over the conjugacy classes of subgroups $H$ of $G$. The map $NE_{it}^G \to D_{ij}$ in (2-3) is the inclusion of the components indexed by the isomorphism classes of transitive $G$–sets.

The maps $\phi$ and $\psi$ are defined by commutativity of the right hand triangles in the diagram. We claim that the adjoints

$$\Sigma^\infty NE_{it}^G \to K(\mathcal{E}^G) \simeq K(\Phi \mathcal{E}^G)$$

$$\Sigma^\infty NE_{it}^G \to S^G$$

of the maps $\phi$ and $\psi$, respectively, are both equivalences. Here $K(\mathcal{E}^G)$ is the additive $K$–theory spectrum of $\mathcal{E}^G$, with zeroth space $K(\mathcal{E}^G)_0 = NE_{it}^G$, which only depends on the additive symmetric monoidal structure of $\mathcal{E}^G$. There is an equivalence

$$\Sigma^\infty NE_{it}^G \simeq \bigsqcup_{(H)} \Sigma^\infty BW_G H_+,$$

where $BW_G H = NG H/H \cong \text{Aut}^G(G/H)$ is the Weyl group of $H$ and the wedge sum is over the conjugacy classes of subgroups of $G$. By Waldhausen’s additivity theorem [23, 1.3.2] applied to a suitable filtration of $\mathcal{E}^G$, according to stabilizer types, there is a splitting

$$K(\mathcal{E}^G) \simeq \bigsqcup_{(H)} K(\mathcal{E}(BW_G H)),$$

where $\mathcal{E}(BW_G H)$ is the category of finite free $BW_G H$–sets and equivariant bijections. The map (2-5) is equivalent under these identifications to the wedge sum of the maps

$$\Sigma^\infty BW_G H_+ \to K(\mathcal{E}(BW_G H))$$

that are left adjoint to the inclusions $BW_G H_+ \to NE(W_G H) = K(\mathcal{E}(W_G H))_0$.

The Barratt–Priddy–Quillen–Segal theorem [20, 3.6] says that each of the maps (2-7) is an equivalence, hence (2-5) is an equivalence. The map (2-6) is an equivalence by
the Segal–tom Dieck splitting [14, V.11.2]. The composition of these two equivalences is the equivalence \( S^G \simeq K(\Phi e^G) \) referred to in the statement of the lemma.

We apply the suspension spectrum functor \( H \) to the diagram (2-3), combine it with diagram (2-1) and the \( G \)–fixed part of (2-2), take homology, and end up with the following commutative diagram.

\[
\begin{array}{ccc}
H_*(\Sigma j; H_*(K(\Phi e^G)) \otimes j) & \xrightarrow{K_j} & H_*(K(\Phi e^G)) \\
\downarrow \cong & & \downarrow \phi_* \\
H_*(\Sigma j; H_*(\mathcal{E}^G) \otimes j) & \xrightarrow{H_j} & H_*(\mathcal{E}^G)
\end{array}
\]

(2-8) \[
\begin{array}{ccc}
H_*(\Sigma j; H_*(\mathcal{E}^G) \otimes j) & \xrightarrow{H_j} & H_*(\mathcal{E}^G)
\end{array}
\]

We need the fact that \( \epsilon_1 \) and \( \epsilon_2 \) have the same kernel. In fact, all summands in \( \tilde{H}_*(D_{\ell}) \) indexed by \( G \)–sets with more than one orbit map to zero under both \( \epsilon_1 \) and \( \epsilon_2 \). This follows from the fact that Pontryagin products and additive Dyer–Lashof operations vanish after stabilization. More precisely, a decomposition of a \( G \)–set \( X = \bigsqcup_{i=1}^k n_i (G/H_i) \), where the \( H_i \) lie in distinct conjugacy classes, induces a factorization

\[
B \text{Aut}^G(X) \cong \prod_{i=1}^k B(\Sigma_{n_i} \wr W_G H_i).
\]

The homology group \( H_*(B(\Sigma_{n_i} \wr W_G H_i)) \subset H_*(\mathcal{E}^G) \) is generated by \( H_*(BW_G H_i) \) under iterated Pontryagin products and Dyer–Lashof operations (see Cohen, Lada and May [3, I.4.1]), which all map to zero under \( \epsilon_1 \) and \( \epsilon_2 \) unless \( k = 1 \) and \( n_1 = 1 \).

Let \( x \in H_*(K(\Phi e^G)) \), and let \( y \in H_*(S^G) \) be the element corresponding to \( x \) under \( \psi_* \circ \phi_*^{-1} \), via an element \( z \in H_*(\mathcal{E}^G) \). We need to show that the image \( Q_i(x) \) of \( \epsilon_i \otimes z \otimes p \) under the top map corresponds, via the isomorphism, to the image \( Q_i(y) \) of \( \epsilon_i \otimes y \otimes p \) under the bottom map. The element \( z \) maps to an element \( Q_i(z) \in \tilde{H}_*(D_{\ell}) \), which further maps to \( Q_i(x) \) and \( Q_i(y) \) under \( \epsilon_1 \) and \( \epsilon_2 \), respectively. Let \( w \in H_*(\mathcal{E}^G) \) map to \( Q_i(x) \) under \( \phi_* \). Since the maps \( \epsilon_1 \) and \( \epsilon_2 \) have the same kernel, the elements \( Q_i(z) \) and \( w \) have the same image in \( H_*(S^G) \), which implies the result. \( \square \)

Remark 2.2 The additive equivalence \( S^G \simeq K(\mathcal{E}^G) \simeq K(\Phi e^G) \) of spectra can be realized as the \( G \)–fixed part of a \( G \)–equivalence \( \mathcal{S}_G \simeq K_G(\mathcal{E}) \) of \( G \)–spectra, for example using Shimakawa’s construction [21] of \( G \)–equivariant \( K \)–theory spectra. Presumably this is a \( G \)–equivalence of \( E_\infty \) ring \( G \)–spectra.
3 Proof of the main theorem

Recall the $E_{\infty}$ structure maps $\lambda_j: E_j \vee \Sigma_j (N \mathcal{E}^G)^{\wedge j} \to N \mathcal{E}^G$. We have inclusions $B\Aut^G(G) \to N \mathcal{E}^G$ and $\delta: B\Aut^G(G^j) \to N \mathcal{E}^G$, corresponding to the summands indexed by $X = G$ and $X = G^j = G \times \cdots \times G$, respectively, in the decomposition (2-4) of $N \mathcal{E}^G$. Restricting $\lambda_j$ to these summands, we have a commutative diagram

\[
\begin{array}{ccc}
E_j \vee \Sigma_j (N \mathcal{E}^G)^{\wedge j} & \xrightarrow{\lambda_j} & N \mathcal{E}^G \\
\downarrow \quad \quad \quad \downarrow \delta \quad \quad \quad \downarrow \\
E_j \vee \Sigma_j B\Aut^G(G^j) & \xrightarrow{\cong} & B(\Sigma_j \vee \Aut^G(G^j)) \\
\end{array}
\]

where the homomorphism $\phi$ sends an element $(\sigma; f_1, \ldots, f_j)$ in $\Sigma_j \vee \Aut^G(G^j)$ to the $G$–automorphism $f_{\sigma^{-1}(1)} \times \cdots \times f_{\sigma^{-1}(j)}$ of $G^j$.

We write $\Sigma_j \vee \Aut^G(G) \cong \Sigma_j \vee G$ for the wreath product $\Sigma_j \vee \Aut^G(G^j)$. The free $G$–set $G^j$ splits into $k = |G^j|^{-1}$ orbits, and we fix a $G$–equivariant bijection $G^j \cong \bigsqcup_k G$. This induces an isomorphism $\Aut^G(G^j) \cong \Aut^G(\bigsqcup_k G)$, and we also have $\Aut^G(\bigsqcup_k G) \cong \Sigma_k \vee G$. Thus we get a commutative diagram

\[
\begin{array}{ccc}
B(\Sigma_j \vee \Aut^G(G^j)) & \xrightarrow{\cong} & B\Aut^G(G^j) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
B(\Sigma_j \vee G) & \xrightarrow{\cong} & B(\Sigma_k \vee G) \\
\end{array}
\]

where we also write $\phi$ for the induced homomorphism $\Sigma_j \vee G \to \Sigma_k \vee G$.

Now we specialize to the case $p = 2$. First we study the Dyer–Lashof operation $Q^2: H_1(S^{C_2}) \to H_3(S^{C_2})$.

**Lemma 3.1** The operation $Q^2$ in $H_*(S^{C_2})$ satisfies $Q^2(\alpha_1) = \alpha_3$.

**Proof** Let $C = C_2$. By Lemma 2.1, we may instead compute $Q^2$ in $H_*(K(\Phi \mathcal{E}^C))$. We let $j = 2$, combine diagrams (2-1), (3-1) and (3-2), apply homology, and end up...
with the upper half of the diagram:

\[
\begin{array}{c}
H_*(E \Sigma_2 \times \Sigma_2 K(\Phi \mathcal{E}^C)^{\wedge 2}) \xrightarrow{\kappa_2} H_*(K(\Phi \mathcal{E}^C)) \\
\downarrow \quad \downarrow \\
H_*(B(\Sigma_2 \wedge C)) \xrightarrow{B\phi_*} H_*(B(\Sigma_2 \wedge C)) \\
\downarrow \quad \downarrow \\
H_*(B(\Sigma_2 \times C)) \xrightarrow{B\psi_*} H_*(B(C \times C)) \\
\end{array}
\]

(3-3)

The vertical homomorphisms in the lower square are induced by the homomorphism \( d = 1 \times \Delta \) that sends \((\sigma, x)\) to \((\sigma, x, x)\), and the inclusion \( i \) of the subgroup \( C \times C = C^2 \) in \( \Sigma_2 \wedge C = \Sigma_2 \wedge C^2 \). The homomorphism \( \psi \) is the restriction of \( \phi \) to \( \Sigma_2 \times C \). It is easily checked that \( \psi \) takes values in the subgroup \( C \times C \) (since \( p = 2 \)) and is given by \( \psi(\sigma, x) = (x, \sigma x) \), using the description of \( \phi \) given after diagram (3-1). We have \( B\psi_*(e_i \otimes 1) = 1 \otimes \alpha_i \) and

\[
B\psi_*(1 \otimes \alpha_j) = \Delta_*(\alpha_j) = \sum_{s+t=j} \alpha_s \otimes \alpha_t,
\]

which combine to give

\[
B\psi_*(e_i \otimes \alpha_j) = \sum_{s+t=i} \alpha_s \otimes (\alpha_i \ast \alpha_t),
\]

where \( \ast \) denotes the Pontryagin product in \( H_*(BC) \) induced by the topological group multiplication \( BC \times BC \to BC \). We recall that \( \alpha_i \ast \alpha_t = \binom{i+t}{i} \alpha_{i+t} \).

By May’s paper [15, 9.1] the map \( Bd_* \) is given by

\[
Bd_*(e_i \otimes \alpha_j) = \sum_k e_{i+2k-j} \otimes \text{Sq}^k_*(\alpha_j) \otimes \text{Sq}^k_*(\alpha_j).
\]

Recall that \( \text{Sq}^k_*(\alpha_j) = \binom{i-k}{k} \alpha_j \otimes \alpha_{j-k} \), where \( \text{Sq}^k \) denotes the dual of the Steenrod operation \( \text{Sq}^k \). In particular \( Bd_*(e_1 \otimes \alpha_2) = e_1 \otimes \alpha_1 \otimes \alpha_1 \), which further maps to \( Q_1(\alpha_1) \) in the upper right hand corner of (3-3). But now \( Q_1(\alpha_1) \) is also the image of \( e_1 \otimes \alpha_2 \) under \( \epsilon_* \circ \delta_* \circ Bt_* \circ B\psi_* \). Using the description of \( B\psi_* \) above, we see that \( Q_1(\alpha_1) \) equals

\[
(3-4) \quad \sum_{s+t=2} \epsilon_* \delta_*(1 \otimes \alpha_s \otimes (\alpha_1 \ast \alpha_t))
\]

The map \( \epsilon_* \) vanishes on decomposables with respect to the product in \( H_*(N \mathcal{E}^C) \) which is induced by the additive symmetric monoidal structure on \( \mathcal{E}^C \). The element
δ∗(1 ⊗ α_s ⊗ (α_1 * α_t)) is the image of

α_s ⊗ (α_1 * α_t) ∈ H_* (BAut^C (C) × BAut^C (C))

in H_*(BAut^C (C ∪ C)) ⊂ H_*(NE^C ) under the map induced by disjoint union, thus

the only nonzero term in (3-4) is the one with s = 0 and t = 2, and Q^2(α_1) = Q_1(α_1) = ε_∗(1 ⊗ α_0 ⊗ (α_1 * α_2)) = α_3.

Proof of Theorem 0.2

We now turn to the operations in

H_*(TC(1) (*; 2)) = F_2 ⊕ H_*(R P_∞).

The general formula for the Q^i will follow from Lemma 3.1 and the Nishida relations, which say in particular (see Bruner et al [2, III.1.1]) that

(3-5) Sq^i+j+1 Q^i(α_j) = \sum_k \left( \frac{2^N - j - 1}{2^N - i - 2j - 2k} \right) Q^{k-j-1} Sq^k(α_j),

where N is sufficiently large. When k ≥ j + 2 the element Sq^k(α_j) is zero for degree reasons, and when k ≤ j the fact that Q^{k-j-1} vanishes on classes in degree higher than k − j − 1 implies that Q^{k-j-1} Sq^k(α_j) = 0. Hence the sum in (3-5) simplifies to the single term

Sq^i+j+1 Q^i(α_j) = \left( \frac{2^N - j - 1}{2^N - i} \right) Q^0 Sq^{i+1}(α_j)

for k = j + 1, where N is large.

Bob Bruner has observed that

\left( \frac{2^N - j - 1}{2^N - i} \right) ≡ \left( \frac{2^N + i - 1}{2^N + j} \right) \mod 2, for large N. Here is a quick proof. Let x_k denote the k-th bit in the binary expansion of a natural number x. Then

\left( \frac{2^N - j - 1}{2^N - i} \right) ≡ 1

if and only if (2^N - i)_k = 1 implies (2^N - j - 1)_k = 1 for all k, and

\left( \frac{2^N + i - 1}{2^N + j} \right) ≡ 1

if and only if (2^N + j)_k = 1 implies (2^N + i - 1)_k = 1 for all k. But for N large compared to i, j and k the bit (2^N - i)_k is complementary to (2^N + i - 1)_k, and
\((2^N - j - 1)_k\) is complementary to \((2^N + j)_k\), so
\[
\binom{2^N - j - 1}{2^N - i} \equiv 1
\]
if and only if
\[
\binom{2^N + i - 1}{2^N + j} \equiv 1.
\]
The operations \(Sq^k\) in \(H_*(\mathbb{R}P^\infty_m)\) are given by the formula
\[
Sq^k(\alpha_j) = \binom{j - k}{k} \alpha_{j-k}.
\]
This follows by the corresponding formula for \(\mathbb{R}P^n\) and James periodicity. More precisely, a theorem of James [12] says that given \(m \leq n\), there is a positive integer \(M\), depending only on \(n - m\), such that \(\mathbb{R}P^{n+\ell}_m \simeq \Sigma^\ell \mathbb{R}P^n_m\) when \(\ell\) is a positive multiple of \(2^M\). The space \(\mathbb{R}P^n_m\) is the stunted projective space \(\mathbb{R}P^n/\mathbb{R}P^{m-1}\). If we now define the spectrum \(\mathbb{R}P^n_m\) to be \(\Sigma^{-\ell} \mathbb{R}P^{n+\ell}_m\) for such \(\ell\) (depending on \(n\)), we have that \(\mathbb{R}P^\infty_m = \colim_n \mathbb{R}P^n_m\). The Steenrod operations in \(H_*(\mathbb{R}P^\infty_m)\) can now be calculated from the operations in \(H_*(\mathbb{R}P^{n+\ell}_m)\), and the stated formula follows by noting that the relevant binomial coefficients are \(2^M\)-periodic in the numerator.

In particular \(Sq^{j+1}_*(\alpha_j) = \alpha_{-1}\) for all \(j \geq -1\), and we have
\[
Sq^{i+j+1}_* Q^i(\alpha_j) = \binom{2^N + i - 1}{2^N + j} Q^0(\alpha_{-1}).
\]
If \(Q^0(\alpha_{-1})\) were zero, it would follow that \(Q^i(\alpha_j) = 0\) for all \(i\) and \(j\), since \(Sq^{i+j+1}_*\) is an isomorphism to dimension \(-1\). But this contradicts Lemma 3.1. Hence \(Q^0(\alpha_{-1}) = \alpha_{-1}\), and the formula stated in the theorem follows. \(\Box\)

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