THE ADAMS SPECTRAL SEQUENCE FOR $THH(MU)^{S^1}$

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1. A Bockstein spectral sequence

Write $T$ for the circle group of complex numbers of modulus 1, and let $C_m \subset T$ be the cyclic group of order $m$. We also write $C = C_p$. Let $X$ be a naive $T$-spectrum. We have maps

$$X^T \xrightarrow{F} X^{C_p^n} \xrightarrow{V} \Sigma^{-1}X^T$$

where $F$ is the inclusion of fixed points and $V$ comes from a transfer map. Using the identifications

$$X^T = F(\Sigma^\infty_+ T/T, X)^T$$
$$X^{C_p^n} = F(\Sigma^\infty_+ T/C_p^n, X)^T$$

they can be described as follows. Consider the projection $T/C_p^n \to T/T$. It induces a stable map $\pi_n: \Sigma^\infty_+ T/C_p^n \to \Sigma^\infty_+ T/T$, and there is also a $T$-transfer map $t_n: \Sigma \Sigma^\infty_+ T/T \to \Sigma^\infty_+ T/C_p^n$, where the extra suspension is suspension by the (trivial) adjoint representation of $T$. Applying $F(\cdot, X)^T$ to $\pi_n$ and $t_n$ gives us $F$ and $V$, respectively. Replacing $X$ by $\tilde{E}T \wedge F(ET_+, X)$ we get the maps

$$X^{tT} \xrightarrow{F} X^{tC_p^n} \xrightarrow{V} \Sigma^{-1}X^{tT},$$

and by taking continuous homology with respect to the corresponding Tate towers we get a sequence of completed $\mathcal{A}_*$-comodules

$$0 \xrightarrow{} H_*(X^{tT}) \xrightarrow{F_*} H_*(X^{tC_p^n}) \xrightarrow{V_*} \Sigma^{-1}H_*(X^{tT}) \xrightarrow{} 0.$$

Let $\alpha: T_+ \wedge X \to X$ be the $T$-action map, and let $\rho: T \to T/C$ be the $p$'th root isomorphism. The composition

$$\Sigma X \xrightarrow{\rho \wedge 1} T_+ \wedge X \xrightarrow{\alpha} X$$

defines a map $\sigma: \Sigma X \to X$. We can pull back the $T/C$-action on $X^{tC}$ to get a $T$-spectrum $\rho^*X^{tC}$. As above we get a map

$$0 \xrightarrow{} H_*(X^{tC}) \xrightarrow{F_*} H_*(X^{tC}) \xrightarrow{V_*} \Sigma^{-1}H_*(X^{tC}) \xrightarrow{} 0.$$

Lemma 1.3. The composition

$$\pi_n \circ t_n: \Sigma \Sigma^\infty_+ T/T \to \Sigma^\infty_+ T/T$$

is null-homotopic. The composition

$$FV: X^{tC} \to \Sigma^{-1}X^{tC}$$

is the desuspension of $\bar{\sigma}$.  

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Proof. We will use an explicit geometric model for the transfer map \( t_n : \Sigma \Sigma T \rightarrow T \Sigma T / \Sigma C \). Write \( S^1(n) \simeq T / C^{p^n} \) for the circle with \( T \)-action given by \( t \cdot s = t^n s \). Take a tubular neighborhood of \( S^1(n) \subset \mathbb{C} \), which is equivalent to the normal bundle \( \eta \) of \( S^1(n) \subset \mathbb{C} \). Dividing out by the complement of this neighborhood and one-point compactifying the source results in a map

\[
p : S^2 \rightarrow \text{Th}(\eta) \cong S^2 / \{N, S\},
\]

where \( N \) and \( S \) denotes the north and south pole of \( S^2 \), respectively. The map \( p \) just pinches \( N \) and \( S \) together. The point \( N \) is taken as the basepoint. \( T \)-equivariantly, each circle of latitude is a copy of \( S^1(n) \), and this specifies the \( T \)-action in both the source and the target. Now \( t_n \) is the \( (\text{desuspended}) \) stabilization of \( p \).

Let \( q : \text{Th}(\eta) \rightarrow S^1 \) be the map that wraps each fiber around the target circle, with the point at infinity mapped to the basepoint in \( S^1 \). Here \( T \) acts trivially on the target. This map stabilizes to \( \pi_n \), and since the composition \( q \circ p : S^2 \rightarrow S^1 \) is \( T \)-equivariantly contractible (slide the image of the south pole one round back to the basepoint), this proves the first result.

For the second, consider the adjoints ...

\[\square\]

Lemma 1.4. The sequence (1) is short exact, and is split (as \( \mathcal{A}_* \)-comodules) when \( n \geq 2 \).

Proof. The induced maps of Tate spectral sequences form a short exact sequence of \( E^2 \)-terms,

\[
P(t^{k \pm 1}) \otimes H_* (X) \xrightarrow{f \otimes 1} E(u) \otimes P(t^{k \pm 1}) \otimes H_* (X) \xrightarrow{v \otimes 1} \Sigma^{-1} P(t^{k \pm 1}) \otimes H_* (X),
\]

where \( f \) is the inclusion, and \( g \) maps \( t^i \) to zero and \( u t^i \) to \( \Sigma^{-1} t^i \). In other words, \( F_* \) includes the even columns and \( V_* \) projects onto the odd columns. Since the first and last spectral sequence can only have differentials in the even pages, it follows that the same is true for the middle spectral sequence, and that the sequence of maps is short exact for every \( E^r \)-term. Combined with the fact that the composition \( V_* F_* \) is zero on the abutments, since \( \pi \circ t \simeq * \), this implies the first result.

To establish the splitting, consider the commutative diagram

\[
\begin{array}{ccc}
H_* (X^{t C^{p^n}}) & \rightarrow & H_* (X^{t C}) \\
F_* & / & V_* \\
0 & \rightarrow & \Sigma^{-1} H_* (X^{t T}) & \rightarrow & 0.
\end{array}
\]

The composition \( V_* F_* \) is multiplication by \( p^{n-1} \) integrally, hence is zero since we use mod \( p \) coefficients. Now \( V_* F_* = 0 \), so a splitting for the unnamed map is given by \( F_*^{-1} F_* \). \[\square\]

We will use the short exact sequence (1) as a tool to compute Ext-groups. If we can compute \( \text{Ext}_{\mathcal{A}_*} (F_p, H_* (X^{t T})) \), then the splitting immediately gives us \( \text{Ext}_{\mathcal{A}_*} (F_p, H_* (X^{t C^{p^n}})) \) for \( n \geq 2 \). For \( n = 1 \) the sequence (1) is typically not split, but we will construct a spectral sequence to pass from \( X^{t C_1} \) to \( X^{t T} \). In that case the sequence represents a non-trivial element

\[
\delta \in \text{Ext}_{\mathcal{A}_*}^{1, 1} (H_* (X^{t T}), H_* (X^{t T})).
\]

Assume now that \( X \) is a ring spectrum and that \( T \) acts by ring maps. Then \( H_* (X^{t T}) \) is a completed \( \mathcal{A}_* \)-comodule algebra, and we can pull back the element \( \delta \) along the unit map \( \eta : F_p \rightarrow H_* (X^{t T}) \) to an element \( e \in \text{Ext}_{\mathcal{A}_*}^{1, 1} (F_p, H_* (X^{t T})) \). Explicitly \( e \) is represented by the top sequence in the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & H_* (X^{t T}) & \xrightarrow{F_*} & H_* (X^{t C}) & \xrightarrow{V_*} & \Sigma^{-1} H_* (X^{t T}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H_* (X^{t T}) & \rightarrow & E & \rightarrow & \Sigma^{-1} F_p & \rightarrow & 0 \\
\end{array}
\]
where the right-hand square is a pullback.

**Theorem 1.6.** Let $X$ be a naive commutative $T$-ring spectrum that is bounded below and of finite type over $\mathbb{F}_p$, and assume the top short exact sequence in (5) is non-split. Then there is a (strongly?) convergent spectral sequence of algebras

$$E_1^{*,*} = P(e) \otimes \text{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, H^*_c(X^{TC})) \Rightarrow \text{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, H^*_c(X^{TC})),$$

where $e$ in degree $(1, -1)$ is a permanent cycle, as are all its powers. The differentials $d_*$ have degree $(1, -1)$ and are of the form $d_* (e^k \otimes x) = e^{k+1} \otimes d_*(x).$ The first differential $d_1$ is given by $d_1 (e^k \otimes x) = e^{k+1} \otimes \sigma_*(x),$ where

$$\bar{\sigma}_*: \text{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, H^*_c(X^{TC})) \to \text{Ext}^{*,*+1}_{\mathcal{A}_*}(\mathbb{F}_p, H^*_c(X^{TC}))$$

is induced by the map $\bar{\sigma}$ in (2).

**Proof.** We use the spectral sequence terminology of Boardman. Write $\text{Ext}(M)$ for $\text{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, M).$ The short exact sequence (1) induces a long exact sequence of Ext-groups

$$\cdots \to \text{Ext}^*(H^*_c(X^{IT})) \to \text{Ext}^*(H^*_c(X^{IT})) \to \text{Ext}^*(H^*_c(X^{IT})) \xrightarrow{\delta} \text{Ext}^{*+1}(H^*_c(X^{IT})) \to \cdots.$$

Since the homology groups are commutative algebras, these Ext-groups have induced algebra structures, and the boundary map $\delta$ is just multiplication by $e$. We construct the unrolled exact couple (truncated at $s = 0$)

$$\cdots \xrightarrow{e} \text{Ext}^*(H^*_c(X^{IT})) \xrightarrow{e} \text{Ext}^*(H^*_c(X^{IT})) \xrightarrow{e} \text{Ext}^*(H^*_c(X^{IT})) \xrightarrow{e} \cdots$$

$$\begin{array}{ccc}
\text{Ext}^*(H^*_c(X^{TC})) & \xrightarrow{F_*} & \text{Ext}^*(H^*_c(X^{TC})) \\
\downarrow_{V_*} & & \downarrow_{V_*} \\
\text{Ext}^*(H^*_c(X^{IT})) & \xrightarrow{F_*} & \text{Ext}^*(H^*_c(X^{IT})) \\
\downarrow_{V_*} & & \downarrow_{V_*} \\
\end{array}$$

where each triangle is an exact couple formed by coiling up (7). Since the inverse limit of the horizontal sequence vanishes, this results in a spectral sequence converging to the (attained) colimit $\text{Ext}^*(H^*_c(X^{IT}))$ with differential $d_1 = F_* \circ V_*.$ The identification of the first differential follows immediately from Lemma 1.3. Note that $e$ has degree $(1, -1), F_*$ has degree $(0, 0),$ and $V_*$ has degree $(0, 1).$

This spectral sequence has a natural tri-grading, but we want to make one of the gradings (the “Bockstein degree”) implicit, by instead recording the power of the element $e.$

((more details about multiplicative structure etc))

2. The Adams $E_2$-term

The starting point of our computation is the homology of $\text{THH}(MU)$ and $\text{THH}(BP).$ First we recall the homology of $MU$ and $BP$ as comodule algebras over the dual Steenrod algebra $\mathcal{A}_*.$ As an algebra, $\mathcal{A}_* = E(\tau_0, \tau_1, \ldots) \otimes P(\xi_1, \xi_2, \ldots)$ with $\tau_k$ in degree $2p^k - 1$ and $\xi_k$ in degree $2^k - 1,$ assuming $p$ is odd. When $p = 2$, $\mathcal{A}_* = P(\xi_1, \xi_2, \ldots)$ with $\xi_k$ in degree $2^k - 1.$ Here we are actually using the conjugates of the usual generators without changing the notation. Let $H_*(-) = H_*(-; \mathbb{F}_p)$.

**Proposition 2.1.** We have the following identifications of $\mathcal{A}_*$-comodule algebras. For $p = 2,$

$$H_*(MU) = P(\xi_k^2 \mid k \geq 1) \otimes P(b_i \mid i \geq 2, i \neq 2^k - 1)$$

$$H_*(BP) = P(\xi_k^2 \mid k \geq 1),$$

where $P(\xi_k^2) \subset \mathcal{A}_*$ and the $b_i$ are comodule primitives in degree $2i$. For $p$ odd,

$$H_*(MU) = P(\xi_k \mid k \geq 1) \otimes P(b_i \mid i \geq 1, i \neq p^k - 1)$$

$$H_*(BP) = P(\xi_k \mid k \geq 1),$$
where \( P(\xi_k) \subset A^* \) and the \( b_i \) are comodule primitives in degree \( 2i \).

Proposition 2.1 serves as input to a collapsing Bökstedt spectral sequence, with output the following result. For convenience, let \( b_{p^{k-1}} = \xi_k \) for odd primes \( p \) and \( b_{p^{k-1}} = \xi_k^2 \) for \( p = 2 \).

**Proposition 2.2.** There are isomorphisms of \( \mathcal{A}_* \)-comodule algebras

\[
H_*(THH(MU)) \cong H_*(MU) \otimes E(\sigma b_i \mid i \geq 1) \\
H_*(THH(BP)) \cong H_*(BP) \otimes E(\sigma b_{p^{k-1}} \mid k \geq 1),
\]

where the classes \( \sigma b_i \) are \( \mathcal{A}_* \)-comodule primitives.

To describe the comodule \( H_*(THH(MU)^{IC}) \), we first recall the (homological) Singer construction. Let \( M \) be an \( \mathcal{A}_* \)-comodule, and define the completed \( \mathcal{A}_* \)-comodule \( R_+(M) = L \hat{\otimes} M \), where

\[
L = \begin{cases} 
P(u^{\pm 1}) & \text{if } p = 2 \\
E(u) \otimes P(t^{\pm 1}) & \text{if } p \neq 2
\end{cases}
\]

with \( u \) in degree \(-1\) and \( t \) in degree \(-2\). The \( \mathcal{A}_* \)-comodule structure on \( R_+(M) \) is given by

\[
Sq_*^i (u^r \otimes x) = \sum_j \begin{pmatrix} -r - s - 1 \\ s - 2j \end{pmatrix} u^{r+s-j} \otimes Sq_*^j(x)
\]

for \( p = 2 \), and there are corresponding but more complicated formulas for odd primes. In particular there is always a non-trivial Bockstein going from \( u \otimes x \) to \( u^2 \otimes x \). The map \( \epsilon : M \to R_+(M) \) given by

\[
\epsilon(x) = \sum_{r=0}^{\infty} u^{-r} \otimes Sq_*^r(x)
\]

for \( p = 2 \), and similarly for odd \( p \), is a comodule map (its image is the largest actual sub-comodule inside \( R_+(M) \)). The map \( \epsilon \) induces an isomorphism on Ext-groups.

Let \( \gamma : THH(MU) \to THH(MU)^{IC} \) be the right hand map in the Tate diagram for \( THH(MU) \), and similarly for \( THH(BP) \). Lunøe-Nielsen and Rognes prove the following.

**Theorem 2.3.** There is a commutative diagram

\[
\begin{array}{ccc}
H_*(THH(MU)) & \xrightarrow{\gamma_*} & H_*(THH(MU)^{IC}) \\
\downarrow{\epsilon} & \Phi \cong & \\
R_+(H_*(THH(MU))) & & \\
\end{array}
\]

of complete \( \mathcal{A}_* \)-comodules, and the map \( \Phi \) is an isomorphism. In particular the map \( \gamma_* \) is an Ext-equivalence. An identical statement holds for \( BP \) instead of \( MU \).

From this we can easily describe the Ext-groups of \( H_*(THH(MU)^{IC}) \) and \( H_*(THH(BP)^{IC}) \).

**Corollary 2.4.** There are algebra isomorphisms

\[
\text{Ext}^*_{\mathcal{A}_*}(\mathbb{F}_p, H_*(THH(MU)^{IC})) = P(a_k \mid k \geq 0) \otimes P(b_i \mid i \geq 1, i \neq p^k - 1) \otimes E(\sigma b_i \mid i \geq 1) \\
\text{Ext}^*_{\mathcal{A}_*}(\mathbb{F}_p, H_*(THH(BP)^{IC})) = P(a_k \mid k \geq 0) \otimes E(\sigma b_{p^{k-1}} \mid k \geq 1),
\]

with \( a_k = (2p^{k-1}, b_i = (0, 2i) \) and \( \sigma b_i = (0, 2i + 1) \).

**Proof.** A standard calculation gives

\[
\text{Ext}^*_{\mathcal{A}_*}(\mathbb{F}_p, H_*(MU)) = P(a_k \mid k \geq 0) \otimes P(b_i \mid i \geq 1, i \neq p^k - 1)
\]

with \( a_k \) in bi-degree \((1, 2p^k - 1)\) and \( b_i \) in bi-degree \((0, 2i)\). Proposition 2.2 and Theorem 2.3 then combine to prove the corollary for \( MU \). The case \( BP \) is similar. \( \square \)
We now want to use the spectral sequence of the previous section to pass from Ext of the cyclic Tate construction to Ext of the circle Tate construction. First we have to show that the extension (5) is non-trivial.

**Lemma 2.5.** The short exact sequence

\[ 0 \longrightarrow H^*_c(THH(MU)^T) \longrightarrow E \longrightarrow \Sigma^{-1}F_p \longrightarrow 0 \]

representing \( e \in \text{Ext}^{1,1}_{\mathcal{A}_c}(F_p, H^*_c(THH(MU)^T)) \) does not split as \( \mathcal{A}_c \)-comodules. Similarly for \( BP \).

**Proof.** First consider the corresponding extension for the sphere spectrum

\[ (6) \quad 0 \to P(t^\pm 1) \to E_S \to \Sigma^{-1}F_p \to 0 \]

pulled back from

\[ (7) \quad 0 \to H^*_c(S^T) \to H^*_c(SIC) \to \Sigma^{-1}H^*_c(S^T) \to 0. \]

Identifying \( H^*_c(S^T) \) with \( R_+(F_p) = E(u) \otimes P(t^{\pm 1}) \), the right hand map in (7) sends \( u \otimes 1 \) to \( \Sigma^{-1}1 \). This can be seen by looking at the induced map on homological Tate spectral sequences. There is a Bockstein \( \beta(u \otimes 1) = t \otimes 1 \) in \( R_+(F_p) \), so the extension (7) is non-trivial. Since both \( u \otimes 1 \) and \( t \otimes 1 \) are in \( E_S \), the extension (6) is also non-trivial.

After identifying \( H^*_c(THH(MU)^{IC}) \) with \( R_+(H_*(THH(MU))) \), the map induced from \( S^T \to THH(MU)^{IC} \) sends \( u \otimes 1 \) to \( u \otimes 1 \). We have a map of extensions

\[ \begin{array}{ccc}
0 & \longrightarrow & H^*_c(S^T) \\
& \downarrow & \downarrow \\
0 & \longrightarrow & H^*_c(THH(MU)^T) \\
& \downarrow & \downarrow \\
0 & \longrightarrow & E \\
& \downarrow & \downarrow \\
0 & \longrightarrow & \Sigma^{-1}F_p \\
& \downarrow & \downarrow \\
0 & \longrightarrow & 0,
\end{array} \]

where the middle map sends \( u \otimes 1 \) to \( u \otimes 1 \). Hence \( v(u \otimes 1) = \Sigma^{-1}1 \) and the extension \( E \) is non-trivial as well. The same argument works for \( BP \). \( \square \)

We will start with the calculation for \( X = THH(BP) \). Using Corollary 2.4, the spectral sequence in Theorem 1.6 is

\[ P(e) \otimes P(a_k \mid k \geq 0) \otimes E(\sigma b_p k - 1 \mid k \geq 1) \Rightarrow \text{Ext}^*_c(F_p, H^*_c(THH(BP)^T)) \]

The map \( \gamma : THH(BP) \to \rho^*TTHH(BP)^{IC} \) is \( T \)-equivariant, which implies that the \( d_1 \)-differential is induced from \( \sigma : \Sigma THH(BP) \to THH(BP) \) via the Ext-isomorphism \( \gamma_* \).

We have the following picture of a part of the \( E_1 \)-page, where we use the Adams grading convention with \( t - s \) as the horizontal axis and \( s \) as the vertical axis. For this picture we choose \( p = 3 \). We have only drawn the algebra generators, in addition to the target of the \( d_1 \)-differential.
We already know that $e$ is a permanent cycle, and so is $a_0$ since there is nothing in (Adams) bi-degree $(-1, 2)$. In homology, $\sigma$ vanishes on the $\sigma \xi_k$ classes, hence $d_1 = \sigma$ is zero on the $\sigma \xi_k$ in the corresponding Ext. What remains are the $a_k$ classes, and we will show that $d_1(a_k) = -e a_0 \sigma \xi_k$, as indicated in the picture for $k = 1$. The element $a_k$ is represented in the (completed) cobar complex by

$$- \sum_{i+j=k} \tau_i \xi_j^{p^i} \in \mathcal{A} \otimes H^*(T \text{HH}(BP)^{TC}),$$

which maps to $-\tau_0 \sigma \xi_k$ under $1 \otimes \sigma$. Since $\tau_0 \sigma \xi_k$ represents $a_0 \sigma \xi_k$, this shows that $d_1(a_k) = e \sigma(a_k) = -e a_0 \sigma \xi_k$.

This results in the $E_2$-term

$$E_2 = P(e) \otimes P(a_0, a_1^p, a_2^p, \ldots) \otimes E(\sigma \xi_k)/(ea_0 \sigma \xi_k \mid k \geq 1).$$

Since the $a_k$ classes with $k \geq 1$ are now gone, there is nothing for $d_1(\sigma \xi_k)$ to hit, so all the $\sigma \xi_k$ have to be permanent cycles. (This also follows from the fact that the image of $d_1$ is divisible by $e^r$.) The only class $d_1(a_k^p)$ could hit is $e^r a_0 \sigma \xi_k$, but this class was already killed by $d_1$. Hence the spectral sequence collapses from the $E_2$-term.

Since the $\sigma \xi_k$ has odd total degree, they are automatically exterior classes in the abutment, at least for odd primes. (Use Steenrod operations in Ext for $p = 2$?) Since $V_e(a_k) = a_0 \sigma \xi_k$ in $\text{Ext}^*_\mathcal{A}_e(\mathbb{F}_p, H^*_e(T \text{HH}(BP)^{TC}))$, which lies in the kernel of multiplication by $e$, the relation $ea_0 \sigma \xi_k = 0$ still holds in the abutment.

For $MU$, there are extra classes $b_i$ and $\sigma \xi_k$ on the $s = 0$ line, and differentials $d_1(b_i) = e \sigma b_i$. So the $b_i$ classes die, while the $\sigma \xi_k$ remain but are now $e$-torsion. Again the spectral sequence collapses from the $E_2$-term and the multiplicative extensions are similarly resolved. Summing up, we have the following result.

**Theorem 2.8.** There are algebra isomorphisms

$$\text{Ext}^*_\mathcal{A}_e(\mathbb{F}_p, H^*_e(T \text{HH}(MU)^{TC})) = P(e) \otimes P(a_0, a_k^p \mid k \geq 1) \otimes E(\sigma b_i)/(ea_0 \sigma \xi_k, e \sigma b_i)$$

$$\text{Ext}^*_\mathcal{A}_e(\mathbb{F}_p, H^*_e(T \text{HH}(BP)^{TC})) = P(e) \otimes P(a_0, a_k^p \mid k \geq 1) \otimes E(\sigma \xi_k)/(ea_0 \sigma \xi_k).$$

Let

$$A = P(e) \otimes P(a_0, a_k^p \mid k \geq 1) \otimes P(b_i^p) \otimes E(\sigma b_i)/(ea_0 \sigma \xi_k, e \sigma b_i)$$

$$B = P(e) \otimes P(a_0, a_k^p \mid k \geq 1) \otimes E(\sigma \xi_k)/(ea_0 \sigma \xi_k).$$

From the splitting of (1) we immediately get the following consequence.

**Corollary 2.9.** Let $n \geq 2$. There are identifications

$$\text{Ext}^*_\mathcal{A}_e(\mathbb{F}_p, H^*_e(T \text{HH}(MU)^{TC^n})) = A \oplus \Sigma^{-1} A$$

$$\text{Ext}^*_\mathcal{A}_e(\mathbb{F}_p, H^*_e(T \text{HH}(BP)^{TC^n})) = B \oplus \Sigma^{-1} B,$$

with $A$ and $B$ corresponding to subalgebras.

3. Adams Differentials

We focus on the case $BP$. The Adams spectral sequence we consider is

$$\text{Ext}^*_\mathcal{A}_e(\mathbb{F}_p, H^*_e(T \text{HH}(BP)^{TC})) \Rightarrow \pi_*(T \text{HH}(BP)^{TC}),$$

with $E_2$-term given explicitly by Theorem 2.8. Here is a zoomed out picture of the $E_2$-term. (Again we only draw the algebra generators.)
The element $a_0$ is a permanent cycle, by comparing with the corresponding spectral sequence for $S^{TT}$, and probably for other reasons as well. The map

$$V_* : \text{Ext}^{s,t}_{S_*}(\mathbb{F}_p, H_*(THH(BP))) \rightarrow \text{Ext}^{s,t+1}_{S_*}(\mathbb{F}_p, H_*'^{(THH(BP)^{IT})})$$

sends $a_k$ to $a_0\sigma_\xi_k$, for $k \geq 1$. This is clear from the computation of $d_1 = F_*V_*$ above. Since the Adams spectral sequence for $THH(BP)$ collapses (McClure-Staffeldt), the image $a_0\sigma_\xi_k = V_*(a_k)$ has to be a permanent cycle. What about $\sigma_\xi_k$? And especially $a^k_0$?

For $e$, we can say the following. Since $THH(BP)^{IT} \simeq TF(BP)$, which is connective, we know that all the $e^k$ classes has to support non-trivial differentials. A differential on $e$ will leave $e^p$, so a later differential will have to kill $e^p$, which will leave $e^{p^2}$, and so on. So the Adams spectral sequence will not collapse after finitely many pages. It seems hard to analyze these differentials directly; maybe it’s possible to map to or from this Adams spectral sequence to a modified Adams spectral sequence of some sort, as is done by Ravenel in the case $S^{IT}$. What makes this easier in Ravenel’s case is the fact that the norm-restriction cofiber sequence splits for $S$, which I don’t think is true for $THH(BP)$. 

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