

The Donsker Delta Function, a Representation Formula for Functionals of a Lévy Process and Application to Hedging in Incomplete Markets

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Abstract

We use white noise theory and Wick calculus, together with the Donsker delta function, to find an explicit expression for φ in the representation formula

$$g(\eta_T) = E[g(\eta_T)] + \int_0^T \int_{\mathbb{R}} \varphi(t, z) \tilde{N}(dt, dz)$$

where

$$\eta_t = \int_0^t \int_{\mathbb{R}} z \tilde{N}(dt, dz), \quad t \geq 0,$$

is a pure jump Lévy process with centered Poisson stochastic measure \tilde{N} .

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1 Introduction.

Let

$$\eta_t = \int_0^t \int_{\mathbb{R}} z \tilde{N}(dt, dz), \quad t \geq 0,$$

be a pure jump Lévy process on the probability space (Ω, \mathcal{F}, P) with centered Poisson stochastic measure \tilde{N} . And let \mathcal{F}_t , $t \geq 0$, be the completed filtration generated by this process. For any \mathcal{F}_T -measurable $\xi \in L_2(P)$ (where $T > 0$ is constant) there exists a unique \mathcal{F}_t -adapted process $\varphi \in L_2(P \times \lambda \times \nu)$, i.e. $E \int_0^T \int_{\mathbb{R}} \varphi^2(t, z) \nu(dz) dt < \infty$, such that

$$\xi = E[\xi] + \int_0^T \int_{\mathbb{R}} \varphi(t, z) \tilde{N}(dt, dz).$$

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Because of applications in mathematical finance, it is of interest to find φ explicitly. This can be done, for instance, via Malliavin calculus and a Lévy version of the Clark-Haussmann-Ocone theorem (see e.g. [BDLØP], [LSUV], [DØP], [NS], [ØP]); otherwise the non-anticipating differentiation can be applied (see [D1], [D2]).

The purpose of this paper is to show that white noise theory and the Donsker delta function can be used to find an explicit formula for φ (Theorem 2.6). This method does not need the Malliavin calculus for Lévy processes nor does it involve conditional expectations, as the previous methods do.

As an application of our representation theorem we give conditions under which a claim of the form $\xi = g(\eta_T)$ is not hedgeable in a (incomplete) financial market with stock prices driven by $S(t)$, $t \geq 0$, according to the dynamics $dS(t) = S(t^-)d\eta_t$, ($S(0) > 0$) (Theorem 3.1 and Theorem 3.2).

2 Mathematical setting and tools.

2.1 White noise framework.

In this section we set our framework and we give a survey of the most relevant results that will be used in the sequel.

In the probability space (Ω, \mathcal{F}, P) we consider a real *pure jump Lévy process* η_t , $t \geq 0$, which is the stochastic process characterized by the Kolmogorov-deFinetti law

$$(2.1) \quad \log E[e^{iu\eta_t}] = t \cdot \int_{\mathbb{R}} (e^{iuz} - 1 - iuz)\nu(dz), \quad t \geq 0.$$

Here $\nu(dz)$, $z \in \mathbb{R}$, is the so-called Lévy measure; namely, it is the σ -finite Borel measure on $\mathbb{R} \setminus \{0\}$ that describes the distribution of the jump size. Note that we assume that

$$(2.2) \quad \mu := \int_{\mathbb{R}} z^2 \nu(dz) < \infty.$$

In the sequel we will always deal with the cadlag modification of this Lévy process. We refer to [B] and [Sa] as detailed monographs on Lévy processes. In particular we recall that the pure jump Lévy process admits the following stochastic integral representation

$$(2.3) \quad \eta_t = \int_0^t \int_{\mathbb{R}} z \tilde{N}(ds, dz), \quad t \geq 0,$$

via the centered Poisson stochastic measure

$$\tilde{N}(dt, dz) := N(dt, dz) - \lambda(dt)\nu(dz), \quad t \geq 0, z \in \mathbb{R}.$$

Here $N(dt, dz)$ is the Poisson stochastic measure with $EN(dt, dz) = \lambda(dt)\nu(dz)$ and $\lambda(dt) = dt$ denotes the Lebesgue measure. Cf. [I].

Next we recall the white noise framework that we are going to exploit. Our presentation is based on [DØP] and [ØP]. Further references can be found in [BK], [HKPS], [Ku], [O],

for example, where the general theory is given in full completeness for the Gaussian white noise.

For the probability space (Ω, \mathcal{F}, P) equipped with the completed filtration \mathcal{F}_t , $t \geq 0$ ($\mathcal{F} = \mathcal{F}_\infty$), generated by the pure jump Lévy process, we consider the standard space $L_2(P)$.

In this space we construct the orthogonal basis K_α , $\alpha \in \mathcal{A}$, as follows. Here \mathcal{A} denotes the set of all multi-indices $\alpha = (\alpha_0, \alpha_1, \dots)$ which have only finitely many non-zero values $\alpha_i \in \mathbb{N} \setminus \{0\}$.

First of all we consider the orthonormal basis φ_i , $i \in \mathbb{N}$, in $L_2(\lambda)$ constituted by the Laguerre functions (order 1/2). Moreover we take an orthonormal basis ψ_j , $j \in \mathbb{N}$, in $L_2(\nu)$ of polynomial type. See e.g. [ØP] for further details. We assume that for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$(2.4) \quad \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} e^{\rho|z|} \nu(dz) < \infty.$$

This guarantees that the measure ν integrates all polynomials of degree greater than or equal to 2. Then we can consider the products

$$(2.5) \quad \zeta_k(t, z) = \varphi_i(t) \psi_j(z)$$

for $k = k(i, j)$ as a bijective mapping $k : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ (e.g. the diagonal counting of the Cartesian product $\mathbb{N} \times \mathbb{N}$).

For any $\alpha \in \mathcal{A}$ with $\max\{i : \alpha_i \neq 0\} = j$ and $|\alpha| := \sum_i \alpha_i = m$, we can define

$$\begin{aligned} \zeta^{\otimes \alpha}((t_1, z_1), \dots, (t_m, z_m)) &:= \zeta_1^{\oplus \alpha_1} \otimes \dots \otimes \zeta_j^{\otimes \alpha_j}((t_1, z_1), \dots, (t_m, z_m)) \\ &= \zeta_1(t_1, z_1) \cdots \zeta_1(t_{\alpha_1}, z_{\alpha_1}) \cdots \zeta_j(t_{\alpha_1+\dots+\alpha_{j-1}+1}, z_{\alpha_1+\dots+\alpha_{j-1}+1}) \cdots \zeta_j(t_{\alpha_m}, z_{\alpha_m}) \end{aligned}$$

and $\zeta^{\otimes 0} = 1$. Moreover, we denote the corresponding *symmetrized tensor product* by $\zeta^{\hat{\otimes} \alpha}$. We can now construct an *orthogonal basis* K_α , $\alpha \in \mathcal{A}$, in $L_2(P)$ as follows:

$$(2.6) \quad K_\alpha := I_{|\alpha|}(\zeta^{\hat{\otimes} \alpha}), \quad \alpha \in \mathcal{A},$$

where

$$I_n(f) = n! \int_0^\infty \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} f(t_1, z_1, \dots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, z_n)$$

is the Itô iterated integral with respect to the centered Poisson stochastic measure. See [I]. Here $f \in L_2(\lambda \times \nu)^n$ is symmetric. Note that

$$E[I_m(g) \cdot I_n(f)] = 0, \quad m \neq n$$

and

$$E[I_n(f)^2] = n! \|f\|_{L_2(\lambda \times \nu)^n}^2$$

for all symmetric $g \in L_2(\lambda \times \nu)^m$ and $f \in L_2(\lambda \times \nu)^n$ ($m, n \in \mathbb{N}$).

Hence every $\xi \in L_2(P)$ admits the chaos expansion

$$(2.7) \quad \xi = \sum_{\alpha \in \mathcal{A}} c_\alpha K_\alpha \quad (c_\alpha \in \mathbb{R})$$

and

$$(2.8) \quad \|\xi\|_{L_2(P)}^2 = \sum_{\alpha \in \mathcal{A}} c_\alpha^2 \|K_\alpha\|_{L_2(P)}^2 = \sum_{\alpha \in \mathcal{A}} c_\alpha^2 \alpha!$$

where $\alpha! := \alpha_1! \alpha_2! \dots$ for $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{A}$.

Thanks to these chaos expansions we can characterize the following spaces and chain of embeddings. Cf. [ØP], for example.

By $(S)_\rho$ ($0 \leq \rho \leq 1$) we denote the space of all random variables $f = \sum_{\alpha \in \mathcal{A}} c_\alpha K_\alpha \in L_2(P)$ such that

$$\|f\|_{\rho, k}^2 = \sum_{\alpha \in \mathcal{A}} (\alpha!)^{1+\rho} c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \quad \text{for all } k \in \mathbb{N}$$

where $(2\mathbb{N})^{k\alpha} := (2 \cdot 1)^{k\alpha_1} (2 \cdot 2)^{k\alpha_2} \dots (2 \cdot j)^{k\alpha_j}$ if $j = \max\{i : \alpha_i \neq 0\}$. And by $(S)_{-\rho}$ we denote the space of all random variables $F = \sum_{\alpha \in \mathcal{A}} c_\alpha K_\alpha \in L_2(P)$ such that

$$\|F\|_{-\rho, -k}^2 = \sum_{\alpha \in \mathcal{A}} (\alpha!)^{1-\rho} c_\alpha^2 (2\mathbb{N})^{-k\alpha} < \infty \quad \text{for some } k \in \mathbb{N}.$$

The subspaces $(S)_\rho$ and $(S)_{-\rho}$ are respectively equipped with the projective topology and the inductive topology induced by the above seminorms. Note that for any $F = \sum_{\alpha \in \mathcal{A}} a_\alpha K_\alpha \in (S)_{-\rho}$ and $f = \sum_{\alpha \in \mathcal{A}} b_\alpha K_\alpha \in (S)_\rho$ the action

$$\langle F, f \rangle := \sum_{\alpha \in \mathcal{A}} a_\alpha b_\alpha \alpha!$$

is well defined and thus the space $(S)_{-\rho}$ is the dual of $(S)_\rho$, i.e. $(S)_{-\rho} = (S)_\rho^*$. We remark that, for $\rho = 0$, the spaces $(S) := (S)_0$ and $(S)^* = (S)_0^* = (S)_{-0}$ appear respectively as a Lévy version for the *Hida test function space* and *Hida distribution space* for pure jump Lévy processes. See e.g. [HKPS], [HoØUZ], [Ku], [O]. For $\rho = 1$, the spaces $(S)_1$ and $(S)_{-1}$ are the Lévy version of the *Kondratiev test function space* and the *Kondratiev distribution space* respectively. See [K] and also [HoØUZ], for example.

The following relationships hold true

$$(2.9) \quad (S)_1 \subset (S)_\rho \subset (S) \subset L_2(P) \subset (S)^* \subset (S)_{-\rho} \subset (S)_{-1}.$$

The relevance of these spaces will be clarified in the sequel. For instance $(S)^*$ is rich enough to contain the white noise of the centered Poisson stochastic measure and of the pure jump Lévy process as its elements. In fact, let us consider the random variable $\xi = \tilde{N}(t, B) \in L_2(P)$, for any Borel set $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, then the following chaos expansion can be written:

$$\tilde{N}(t, B) = \sum_{i, j \geq 1} \int_0^t \int_{\mathbb{R}} \varphi_i(s) \psi_j(z) \nu(dz) ds \cdot K_{e^{k(i, j)}}$$

via the application of the orthogonal basis in $L_2(\lambda)$ and $L_2(\nu)$ and the the diagonal counting - cf. (2.5). Moreover, for any $k \in \mathbb{N}$, the multi-index $\epsilon^k = (\epsilon_1^k, \epsilon_2^k, \dots)$ is defined by

$$\epsilon_i^k := \begin{cases} 1, & i = k \\ 0, & \text{otherwise.} \end{cases}$$

Here we refer to [ØP] for all the details. Then we can define the *white noise for the centered Poisson stochastic measure* as the element

$$(2.10) \quad \dot{\tilde{N}}(t, z) := \sum_{i, j \geq 1} \varphi_i(t) \psi_j(z) \cdot K_{\epsilon^{k(i, j)}}$$

in $(S)^*$, for almost all $t \geq 0$, $z \in \mathbb{R}$. Naturally it appears as the Radon-Nikodym derivative

$$\dot{\tilde{N}}(t, z) = \frac{\tilde{N}(dt, dz)}{dt \times \nu(dz)} \quad \text{in } (S)^*.$$

Analogously, let us consider the Lévy process (2.3). Then for any t , $\eta_t \in L_2(P)$ has the following chaos expansion:

$$(2.11) \quad \eta_t = \sum_{i \geq 1} \mu \int_0^t \varphi_i(s) ds \cdot K_{\epsilon^{k(i, 1)}}.$$

And we can then define

$$(2.12) \quad \dot{\eta}_t := \mu \sum_{i \geq 1} \varphi_i(t) K_{\epsilon^{k(i, 1)}} \quad (t \geq 0)$$

(with μ as in (2.2)) which is an element of $(S)^*$. Here it acquires the meaning of derivative of η_t with respect to time

$$\dot{\eta}_t = \frac{d\eta_t}{dt} \quad \text{in } (S)^*.$$

We call $\dot{\eta}_t$, $t \geq 0$, the *Lévy white noise*. Moreover the following relationship

$$\dot{\eta}_t = \int_{\mathbb{R}} z \dot{\tilde{N}}(t, z) \nu(dz), \quad t \geq 0,$$

holds true.

The *Lévy-Wick product* $F \diamond G$ of two elements $F = \sum_{\alpha \in \mathcal{A}} a_\alpha K_\alpha$ and $G = \sum_{\beta \in \mathcal{A}} b_\beta K_\beta$ in $(S)_{-1}$ is defined by

$$(2.13) \quad F \diamond G = \sum_{\alpha, \beta \in \mathcal{A}} a_\alpha b_\beta K_{\alpha+\beta}.$$

It can be shown that the spaces $(S)_1, (S), (S)^*$ and $(S)_{-1}$ are closed under Wick products.

One of the useful features of the Wick product is the following relationship within Itô stochastic integration and Bochner integration:

$$(2.14) \quad \int_0^t \int_{\mathbb{R}} Y(s, z) \tilde{N}(ds, dz) = \int_0^t \int_{\mathbb{R}} Y(s, z) \diamond \tilde{N}(s, z) \nu(dz) ds.$$

We also mention that for all $F \in (S)_{-1}$ one can define the *Wick exponential* $\exp^\diamond F \in (S)_{-1}$ by

$$(2.15) \quad \exp^\diamond F := \sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n}.$$

Example 2.1. For any $u \in \mathbb{R}$ and $t \geq 0$ we have that

$$(2.16) \quad \exp^\diamond \left(\int_0^t \int_{\mathbb{R}} (e^{iuz} - 1) \tilde{N}(ds, dz) + t \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right) = \exp(iu\eta_t).$$

Proof. This is Lemma 3.1.3 in [MØP]. ■

If $Y : [0, T] \rightarrow (S)_{-1}$ is differentiable in $(S)_{-1}$, then the *Wick chain rule* holds. In particular, we have

$$(2.17) \quad \frac{d}{dt} \exp^\diamond Y(t) = \exp^\diamond(Y(t)) \diamond \frac{d}{dt} Y(t).$$

We refer to [ØP] for a more general discussion about these properties of the Wick product.

2.2 The Donsker delta function.

In this white noise framework we can define the Donsker delta function as follows. See [MØP], for example.

Definition 2.2. For a given real random variable $X \in (S)_{-1}$, the *Donsker delta function* of X is a continuous function $\delta.(X) : \mathbb{R} \rightarrow (S)_{-1}$ such that

$$(2.18) \quad \int_{\mathbb{R}} h(x) \delta_x(X) dx = h(X)$$

for all measurable functions $h : \mathbb{R} \rightarrow \mathbb{R}$ for which the integral is well-defined in $(S)_{-1}$.

Following [MØP] we consider a pure jump Lévy process η_t , $t \geq 0$, which satisfies the condition: there exists $\varepsilon \in (0, 1)$ such that

$$(2.19) \quad \lim_{|u| \rightarrow \infty} |u|^{-(1+\varepsilon)} \operatorname{Re} \log E[e^{iu\eta_t}] = \infty.$$

Remark 2.3. Note that this property (2.19) implies that the probability law of η_t , $t \geq 0$, is absolutely continuous with respect to the Lebesgue measure.

Remark 2.4. The condition (2.19) is sufficient to guarantee the existence of the non-zero local time of the pure jump Lévy process.

From now on we shall assume that (2.19) holds.

We now recall the following result from [MØP].

Theorem 2.5. The Donsker delta function $\delta_x(\eta_t)$ exists in $(S)_{-1}$ and it has the explicit form

$$(2.20) \quad \delta_x(\eta_t) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp^{\diamond} \left\{ \int_0^t \int_{\mathbb{R}} (e^{iuz} - 1) \tilde{N}(ds, dz) + t \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) - iux \right\} du.$$

2.3 A representation formula for functionals of a Lévy process.

We now turn to the main result of this paper. It is well known that any \mathcal{F}_T -measurable random variable $\xi \in L_2(P)$ can be written as

$$(2.21) \quad \xi = E[\xi] + \int_0^T \int_{\mathbb{R}} \varphi(t, z) \tilde{N}(dt, dz)$$

for some (unique) \mathcal{F}_t -adapted process φ such that

$$(2.22) \quad E \left[\int_0^T \int_{\mathbb{R}} \varphi^2(t, z) \nu(dz) dt \right] < \infty.$$

The problem of finding φ explicitly is related to stochastic differentiation. It can be approached by using Malliavin calculus and the (generalized) Clark-Haussmann-Ocone formula - see [BDLØP], [DØP] and [ØP], and also [LSUV], [NS], for example. Otherwise the non-anticipating stochastic differentiation can be applied - see [D1], [D2].

Here we present a different method based on the Donsker delta function. The method, which is related to the one in [AØU], does not require any Malliavin calculus and no conditional expectations have to be computed. But nevertheless it gives an explicit result.

Theorem 2.6. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a given function in $L_1(\lambda)$ such that its Fourier transform

$$\hat{g}(u) := \frac{1}{2\pi} \int_{\mathbb{R}} g(x) e^{-iux} dx$$

belongs to $L_1(\lambda)$. Then, for $T \geq 0$, we have

$$(2.23) \quad g(\eta_T) = \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} du + \int_0^T \int_{\mathbb{R}} \varphi(t, z) \tilde{N}(dt, dz),$$

where

$$(2.24) \quad \varphi(t, z) = \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ (T - t) \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} \exp(iu\eta_t) (e^{iuz} - 1) du.$$

Proof. Define

$$Y_u(t) := \exp^{\diamond} \left\{ \int_0^t \int_{\mathbb{R}} (e^{iuz} - 1) \tilde{N}(ds, dz) \right\}, \quad u \in \mathbb{R}, t \in [0, T].$$

First let us assume that g is continuous with compact support. Then, by (2.18), (2.20), (2.17) and (2.14) we have

$$\begin{aligned}
g(\eta_T) &= \int_{\mathbb{R}} g(y) \delta_y(\eta_T) dy \\
&= \int_{\mathbb{R}} \frac{1}{2\pi} g(y) \left(\int_{\mathbb{R}} Y_u(T) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) - iuy \right\} du \right) dy \\
&= \int_{\mathbb{R}} Y_u(T) \left(\int_{\mathbb{R}} \frac{1}{2\pi} g(y) e^{-iuy} dy \right) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} du \\
&= \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} \left(Y_u(0) + \int_0^T \frac{\partial}{\partial t} Y_u(t) dt \right) du \\
&= \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} du \\
&+ \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} \int_0^T Y_u(t) \diamond \int_{\mathbb{R}} (e^{iuz} - 1) \tilde{N}(t, z) \nu(dz) dt du \\
&= \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} du \\
&+ \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} \int_0^T \int_{\mathbb{R}} Y_u(t) (e^{iuz} - 1) \tilde{N}(dt, dz) du \\
&= \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} du + \int_0^T \int_{\mathbb{R}} \varphi(t, z) \tilde{N}(dt, dz),
\end{aligned}$$

where

$$\varphi(t, z) = \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ T \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} Y_u(t) (e^{iuz} - 1) du.$$

By (2.16) we have

$$Y_u(t) \exp \left\{ t \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} = \exp(iu\eta_t).$$

Hence we get

$$\varphi(t, z) = \int_{\mathbb{R}} \hat{g}(u) \exp \left\{ (T-t) \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} \exp(iu\eta_t) (e^{iuz} - 1) du.$$

For the general case, let g be as in Theorem 2.6. Then there exists a sequence $g_n \leq g$, $n = 1, 2, \dots$, of continuous functions with compact support such that $g_n \rightarrow g$ in $L_1(\lambda)$. Then $\hat{g}_n(u) \rightarrow \hat{g}(u)$ pointwise dominatedly. Since (2.23) holds for all g_n , it follows by dominated convergence that it holds for g , as required. ■

3 Application to mathematical finance.

Consider a mathematical market where there are two investment possibilities:

- a risk free asset (e.g. bond), where the price $S_0(t)$, $t \geq 0$, is constantly equal to 1
- a risky asset (e.g. stock), where the price $S(t)$ at time t is given by

$$dS(t) = S(t^-)d\eta_t, \quad S(0) > 0,$$

where $\eta_t = \int_0^t \int_{\mathbb{R}} z \tilde{N}(ds, dz)$ as before. Assume that $z > -1$ for a.a. z with respect to ν . This guarantees that $S(t) > 0$ for all $t \geq 0$.

We say that an \mathcal{F}_T -measurable random variable $\xi \in L_2(P)$ (called a *claim*) is *hedgeable* or *replicable* in this market if there exists an \mathcal{F}_t -adapted process $\psi(t)$, $t \in [0, T]$, in $L_2(P \times \lambda)$ such that

$$(3.1) \quad \xi = E[\xi] + \int_0^T \psi(t) dS(t) = E[\xi] + \int_0^T \psi(t) S(t^-) d\eta_t.$$

The market is called *complete* if all claims are replicable and *incomplete* otherwise. Since (2.3) holds, (3.1) can be written as

$$(3.2) \quad \xi = E[\xi] + \int_0^T \int_{\mathbb{R}} \psi(t) S(t^-) z \tilde{N}(dt, dz).$$

Thus, by comparing (3.2) with (2.21), we see that ξ is hedgeable if and only if the process φ in (2.21) has the product form

$$(3.3) \quad \varphi(t, z) = \psi(t) S(t^-) z \quad \text{for almost all } t, z \text{ with respect to } \lambda \times \nu.$$

It is well known that, unless η is the centered Poisson process, the market is *incomplete*. Hence there exist claims ξ which are not hedgeable. It is of interest to find out which claims are replicable and which claims are not. Our next result gives information in this direction.

Theorem 3.1. Let g be as in Theorem 2.6 and define

$$(3.4) \quad f(t, u) = \hat{g}(u) \exp \left\{ (T-t) \int_{\mathbb{R}} (e^{iuz} - 1 - iuz) \nu(dz) \right\} \exp(iu\eta_t).$$

Suppose that $\xi := g(\eta_T)$ is replicable and let $\psi(t)$ be as in (3.1). Then for almost all $t \in [0, T]$, we have

$$(3.5) \quad \hat{f}(t, \cdot)(z) := \frac{1}{2\pi} \int_{\mathbb{R}} f(t, u) e^{-iuz} du = \frac{1}{2\pi} \int_{\mathbb{R}} f(t, u) du - \frac{1}{2\pi} \psi(t) S(t^-) z$$

for all $z \in -\text{supp } \nu$, where $\text{supp } \nu$ is the smallest closed set in \mathbb{R} in which ν has all its mass.

Proof. By Theorem 2.6 and (3.3) we have

$$\varphi(t, z) = \int_{\mathbb{R}} f(t, u) (e^{iuz} - 1) du = \psi(t) S(t^-) z,$$

for all $z \in \text{supp } \nu$ and almost all $t \in [0, T]$. Hence

$$\frac{1}{2\pi} \int_{\mathbb{R}} f(t, u) e^{-iuz} du = \frac{1}{2\pi} \int_{\mathbb{R}} f(t, u) du - \frac{1}{2\pi} \psi(t) S(t^-) z \quad \text{for all } z \in -\text{supp } \nu.$$

By this we end the proof. ■

Theorem 3.2. Let g be as in Theorem 2.6 and suppose $\text{supp } \nu$ is unbounded. Then $g(\eta_T)$ is not replicable unless $g = 0$.

Proof. Suppose $\xi = g(\eta_T)$ is replicable. Since the Fourier transform vanishes at $\pm\infty$ we get, from (3.5) that

$$0 = \lim_{\substack{|z| \rightarrow \infty \\ z \in -\text{supp } \nu}} \hat{f}(t, \cdot)(z) = \lim_{\substack{|z| \rightarrow \infty \\ z \in -\text{supp } \nu}} \left(\frac{1}{2\pi} \int_{\mathbb{R}} f(t, u) du - \frac{1}{2\pi} \psi(t) S(t^-) z \right),$$

for almost all $t \in [0, T]$. This is only possible if $\psi(t) = 0$ almost everywhere. Hence $\xi = E[\xi]$, so g is constant and hence $g = 0$ (since $g \in L_1(\lambda)$). ■

References

- [AØU] Aase, K., Øksendal, B., Ubøe, J.: Using the Donsker delta function to compute hedging strategies. *Potential Analysis* 14 (2001), pp. 351–374.
- [BDLØP] Benth, F.E., di Nunno, G., Løkka, A., Øksendal, B., Proske, F.: Explicit representation of the minimal variance portfolio in markets driven by Lévy processes. *Math. Finance* 13 (2003), pp.54–72.
- [BK] Berezansky, Yu.M., Kondratiev, Yu.G.: *Spectral Methods in Infinite Dimensional Analysis*. Kluwer Academic Publishers, 1995.
- [B] Bertoin, J.: *Lévy Processes*. Cambridge University Press, Cambridge 1996.
- [DØP] Di Nunno, G., Øksendal, B., Proske, F.: White noise analysis for Lévy processes. *Journal of Functional Analysis* 206 (2004), pp. 109–148.
- [D1] Di Nunno, G.: Stochastic integral representation, stochastic derivatives and minimal variance hedging. *Stochastics and Stochastics Reports* 73 (2002), pp. 181–198.
- [D2] Di Nunno, G.: Random Fields Evolution: non-anticipating integration and differentiation. *Theory of Probability and Math. Statistics* 66 (2002), pp. 82–94.
- [HKPS] Hida, T., Kuo, H.-H, Potthoff, J., Streit, L.: *White Noise*. Kluwer, Dordrecht, 1993.
- [HoØUZ] Holden, H., Øksendal, B., Ubøe, J., Zhang T.-S.: *Stochastic Partial Differential Equations - A Modeling, White Noise Functional Approach*. Birkhäuser, Boston 1996.
- [I] Itô, K.: Spectral type of the shift transformation of differential processes with stationary increments. *Trans. Am. Math. Soc.* 81 (1956), pp. 253–263.

- [K] Kondratiev, Yu.G.: Generalized Functions in Problems in Infinite Dimensional Analysis. Ph.D. Thesis, University of Kiev, 1978.
- [Ku] Kuo, H.H.: White Noise Distribution Theory. Prob. and Stoch. Series, Boca Raton, FL: CRC Press 1996.
- [LSUV] Léon, J.A., Solé, J.L., Utzet, F., Vives, J.: On Lévy processes, Malliavin calculus and market models with jumps. Finance and Stochastics 6 (2002), pp. 197–225.
- [MØP] Mataramvura, S., Øksendal, B., Proske, F.: The Donsker delta function of a Lévy process with application to chaos expansion of local time. Annales de l’Institut Henri Poincaré 40 (2004), pp. 553–567.
- [NS] Nualart, D., Schouthens, W.: Chaotic and predictable representations for Lévy processes. Stoch. Process. Appl. 90 (2002), pp. 109–122.
- [O] Obata N.: White Noise Calculus and Fock Space. LNM, 1577, Springer-Verlag, Berlin 1994.
- [ØP] Øksendal, B., Proske, F.: White noise for Poisson random measures. Potential Analysis 21 (2004), pp. 375–403.
- [Sa] Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Studies in Advanced Mathematics, Vol. 68, Cambridge University Press, Cambridge 1999.