Constrained Delaunay Triangulations (CDT)

Recall:
“Definition 1” (Constrained triangulation). A triangulation $\Delta$ with prespecified edges or breaklines between nodes.

Motivation:
- Geological faults in oil recovery
- Rivers, roads, lake boundaries in GIT
- Repr. non-convex boundaries and holes
- Linear features in CAD models
- Meshing for FEM (boundaries, interior & exterior)
Algorithm III from Preliminaries

Given a predefined constraint $E_{i,j}$. Suppose that the endpoints $p_i$ and $p_j$ are in $\Delta$.

1. For all $T_i$ in $\Delta$,
   If $(Int(T_i) \cap E_{i,j} \neq \phi)$,
   remove $T_i$ from $\Delta$

We get one or more regions $R_i$ on each side of $E_{i,j}$ inside simple closed polygond. (More than one if $E_{i,j}$ intersects nodes.)

2. Triangulate each region $R_i$ with Algorithm I (Protruding point removal).

- May include holes and arbitrary exterior boundary
Delaunay Triangulation of a PSLG

We will generalize the theory of Delaunay triangulation from $\Delta(P)$ to $\Delta(G)$ where $G = G(P,E_c)$ is a PSLG (Planar Straight Line Graph).

$E_c$ are constrained edges. The endpoints of $E_c$ are in $P$.

Recall from conventional Delaunay triangulations:

- MaxMin angle criterion
- Lexicographical ordering of indicator vectors
- Definition of local- and global optimum
- The circle criterion (will be redefined)
- The dual (Voronoi diagram) will not be considered here! (involved and not so useful)

**DEFINITION (Visibility).** $p_i$ and $p_j$ are visible to each other if $\overline{p_ip_j}$ does not intersect the interior of any edge in $E_c$. 
**DEFINISJON (CDT).** A CDT $\Delta(G)$ of a PSLG $G(P, E_c)$ is a triangulation containing the edges $E_c$ such that $C(t)$ of any triangle $t$ in $\Delta(G)$ contains no point of $P$ in its interior which is visible from all the three nodes of $t$.

**DEFINISJON: Modified circle criterion (relaxed)**

**DEFINISJON:** Edges in $\Delta(G)$ that are not in $E_c$ are called Delaunay edges; triangles in $\Delta(G)$ are called Delaunay triangles.

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![Diagram](attachment:diagram.png)

a) $G(P, E_c)$  b) $DT(P)$  c) $CDT(G)$
if $E_c = \phi \Rightarrow G = P$ and $\Delta(G) = \Delta(P, \phi) = \Delta(P)$; a conventional Delaunay triangulation.
Generalization of the theory to CDTs
(Brief, consult cited papers for details)

In (b), \(e'\) is constrained

**Lemma.** The modified circle criterion and the MaxMin angle criterion are equivalent for strictly convex quadrilaterals.

Recall: *indicator vector, lexicographical measure, ...*

**Lemma(\(^*\)).** The *indicator vector* (of a constrained triangulation) becomes *lexicographically larger each time* an edge of a strictly convex quadrilateral is swapped according to the Delaunay swapping criteria.

Basis for Lawson’s LOP Algorithm
LOP applied to a $\Delta(G)$, where $G$ is a PSLG:

1. Start with an arbitrary $\Delta(G)$

2. Repeat swapping of edges according to modified circle criterion

- Does the LOP converge when applied to $\Delta(G)$?
- What does it converge to?
- Do we reach a global optimum?

Lemma (*) and the fact that the number of possible triangulations of $G$ is finite guarantees that the algorithm terminates after a finite number of edge swaps.

\[ \Downarrow \]

**Definition.** Locally optimal edge:

1. When the decision is not to swap it in the LOP.
2. Edges in $E_c$ and boundary edges are locally optimal by default.

**Definition.** Locally optimal triangulation; accordingly.
**THEOREM.** All interior edges of a triangulation $\Delta(G)$ ($G \equiv G(P, E_c)$) are locally optimal

$$\uparrow$$ (if and only if)

the modified circle criterion holds for all triangles.

**Proof.** See Exercise in lecture notes.

So, LOP yields a CDT in accordance with

**DEFINITION (CDT).**

**THEOREM.** A triangulation $\Delta(G)$ is a CDT in accordance with **DEFINITION (CDT)**

$$\uparrow$$ (if and only if)

its indicator vector is lexicographically maximum.

**Proof.** See Exercise with guidelines in lecture notes. (Difficult since a dual construction (Voronoi) cannot be used.)
Uniqueness of a CDT can be deduced from the proof (under the usual assumption that no four points of $P$ are cocircular).

Unique characterization of Delaunay edge and Delaunay triangle in A CDT (Recall, edges in $E_c$ are not Delaunay edges):

**THEOREM (Delaunay edge).** Edge $e_{ij}$ between points $p_i$ and $p_j$ of $P$ is Delaunay

\[ \updownarrow \text{(if and only if)} \]

$p_i$ and $p_j$ are visible to each other and there exists a circle passing through $p_i$ and $p_j$ that does not contain any points of $P$ in its interior visible from both $p_i$ and $p_j$.

**Proof.** See Exercise in lecture notes.
THEOREM (Delaunay triangle). Triangle $t$ with nodes $p_i$, $p_j$ and $p_k$ is Delaunay

$\triangleright$ (if and only if)

$C(t)$ contains no point of $P$ in its interior which is visible from both $p_i$, $p_j$ and $p_k$.

Proof. See Exercise in lecture notes.
Algorithms for Constrained Delaunay Triangulation

Overview:
Similar schemes as for conventional Delaunay, but
1. predefined constrained edges, and
2. modified circle criterion.

Only incremental algorithms are considered here: basic operations:

- inserting a constrained edge into a CDT, and
- inserting a node into a CDT.
Given a PSLG $G(P, E_c)$ where endpoints of $E_c$ are in $P$. 

Algorithm (Compute $\Delta(G)$):

1. Compute $\Delta(P, \phi)$ (conventional Delaunay)

2. for each $e$ in $E_c$
   insert $e$ into $\Delta(P, E'_c)$ and update to a CDT.
   \[ E'_c \leftarrow E'_c \cup e \]

3. (add additional points into $\Delta(G)$)

Recall from conventional Delaunay: two schemes for inserting a point $p$:

- remove triangles in (star shaped) influence region $R^p$ and retriangulate

- split $t$ which contains the insertion point into three new triangles and apply \texttt{recSwap} procedure three times.
Inserting an Edge into a CDT

“Insert” constrained edge $e_c$ between $p_a$ and $p_b$ in $\Delta(P,E_c) \rightarrow \Delta(P,E_c \cup e_c)$.

Influence region $R^{e_c}$ of $e_c$ in $\Delta(G)$ are triangles intersected by $e_c$

Note that no node is inserted or moved $\Rightarrow$ only $R^{e_c}$ is affected! (Recall Theorem)

Influence polygons, $Q^{e_c,L}$ and $Q^{e_c,R}$ on each side of $e_c$. 

Obtain $\Delta(P, E_c \cup e_c)$ by triangulating $Q^{e_c,L}$ and $Q^{e_c,R}$ on each side of the constrained edge $e_c$ separately using modified circle criterion:

“Step-by-step” approach with base line and growing circles:

Retriangulate one $Q$:

1. Let $e_c$ be the first baseline

2. Start with a growing circle with $r = \infty$ and grow it into $Q$.

3. Make a triangle $t$ with the first point $p$ reached that is not separated from $e_c$ with a constraint.

4. Chose a new base line $e_c$ (an edge of $t$) and use the same circle.

5. GOTO 3
The point $p$ is uniquely defined as the point in $P$ that,

i) makes the largest angle at $p$ spanned by the baseline $e_b$, and

ii) $p$ is visible from the endpoints of $e_b$.

Note: No non-procedural approach as for point insertion.
Growing circles
Edge Insertion and Swapping

Recall:

\[ T = 2V_I + V_B - 2 \quad (I) \]
\[ E = 3V_I + 2V_B - 3 \quad (II) \]
\[ E_I = 3V_I + V_B - 3 \quad (III) \]
\[ (T = E - V + 1) \quad (IV) \]

that is, a new constraint edge between two existing nodes does not change the cardinality of edges or triangles.

This suggests that a new constrained edge can be “swapped in place”.
1. Swapping procedure for “inserting” a constraint $e_c$ into $\Delta(P, E_c)$ (operating inside $R^{e_c}$ only).

2. LOP applied to edges inside each of $Q^{e_c,L}$ and $Q^{e_c,R}$ to obtain $\Delta(P, E_c \cup e_c)$

The swapping procedure:

Principle: Swap edges away from the constrained edge $e_c$ such that eventually $e_c$ is included as an edge in the triangulation.

- Let $(p_a, u_1, \ldots, u_n, p_b)$ define the closed influence polygon $Q^{e_c,L}$ to the left of the directed line from $p_a$ to $p_b$.

- Note how the vertices of the polygon are enumerated when it is multiply connected; see figure.

Consider a point $u_m$ each time where the angle $\alpha_m < \pi$ and swap away edges radiating from $u_m$ and intersecting $e_c$.

1. is it always possible to find a point $u_m$ where $\alpha_m < \pi$?
2. is there always a swappable edge at $u_m$?
**Lemma (⋆).** A closed and simply connected polygon $P$ has at least three interior angles smaller than $\pi$.

**Proof.** Let $\alpha_1, \ldots, \alpha_N$, $N \geq 3$, be the interior angles of $P$ and suppose that $k$ of them are smaller than $\pi$. From elementary geometry we know that the sum of the interior angles is $(N - 2)\pi$ and thus,

$$(N - 2)\pi = \sum_{i=1}^{N} \alpha_i = \sum_{\alpha_i < \pi} \alpha_i + \sum_{\alpha_i \geq \pi} \alpha_i.$$

We have $\sum_{\alpha_i \geq \pi} \alpha_i \geq (N - k)\pi$, which inserted above gives,

$$(k - 2)\pi \geq \sum_{\alpha_i < \pi} \alpha_i.$$

The right hand side is non-negative, so we have $k \geq 2$. But since $k > 0$ the right hand side is in fact positive and this gives $k \geq 3$. $\blacksquare$

Simply connected influence polygon:
Lemma; at least one vertex different from $p_a$ and $p_b$ where $\alpha_m < \pi$.

multiply connected: see exercise.
Illustration for Lemma.

- Suppose that there are $r$ edges radiating from $u_m$ and intersecting $e_c$; see figure.
- The endpoints of the $r$ edges on the opposite side of $e_c$ from $u_m$ are denoted $w_1, \ldots, w_r$, numbered counterclockwise. Conventions $w_0 = u_{m-1}$ and $w_{r+1} = u_{m+1}$.

Lemma (★★). There is at least one edge radiating from $u_m$ and intersecting $e_c$, where $\alpha_m < \pi$, that is a diagonal in a convex quadrilateral (and thus swappable).

Proof. The closed polygon defined by the sequence $(w_0, w_1, \ldots, w_r, w_{r+1})$ (not including $u_m$) has at least three interior angles smaller than $\pi$ by Lemma. Thus, there is at least one point $w_s$, $1 \leq s \leq r$ such that the angle $\angle w_{s-1}w_s w_{s+1}$ is smaller than $\pi$. Then the quadrilateral with $(u_m, w_s)$ as a diagonal must be convex since $\angle w_{s+1}, u_m, w_{s-1}$ is also smaller than $\pi$. $\blacksquare$
• Thus, all edges \((u_m, w_s), 1 \leq s \leq r\), where \(\alpha_m < \pi\), can be swapped away from \(u_m\) such that there are no edges left radiating from \(u_m\) and intersecting \(e_c\).
• Result: \(u_m\) is eliminated from the influence polygon.

Algorithm (\(\star\)) (eliminate \(u_m\), with \(\alpha_m < \pi\)):

1. \(\text{while } (r \geq 1)\)
2. Let \((u_m, w_s)\) be a diagonal in a convex quadrilateral \((u_m, w_{s-1}, w_s, w_{s+1})\).
3. Swap \((u_m, w_s)\) to \((w_{s-1}, w_{s+1})\)
4. \(r \leftarrow r - 1\)

Eventually, \(r = 1\) and and only one edge \((u_m, w_1)\) radiates from \(u_m\) and intersects \(e_c\).

\((u_m, w_1)\) is a diagonal in the quadrilateral \((u_m, u_{m-1}, w_1, u_{m+1})\) that is convex by Lemma.

When \((u_m, w_1)\) is swapped to \((u_{m-1}, u_{m+1})\) in the \(r\)'th cycle of the algorithm, \(u_m\) is isolated from \(e_c\) and eliminated from \(Q\); see \(u_1\) in figure.
Repeat Algorithm on
\[ Q \setminus u_m = (p_a, u_1, \ldots, u_{m-1}, u_{m+1}, \ldots, u_n, p_b) \]
etc. and eventually on \( Q = (p_a, u_1, p_b) \).

Algorithm (Insert constrained edge \( e_c \))

1. while \( n \geq 1 \)
2. Find a point \( u_m, 1 \leq m \leq n \) where \( \alpha_m < \pi \)
3. Apply Algorithm (\( \star \)) to \( u_m \)
4. \( n \leftarrow n - 1 \)
5. \( Q_{e_c,L} \leftarrow (p_a, u_1, \ldots, u_{m-1}, u_{m+1}, \ldots, u_n, p_b) \)

When \( n = 1 \), \( Q = (p_a, u_1, p_b) \) and the interior angle \( \alpha_1 \) at \( u_1 \) is smaller than \( \pi \) by Lemma; see figure.

When \( (u_1, w_1) \) is swapped to \( (p_a, p_b) \) it takes the role as the constrained edge \( e_c \) that has \( p_a \) and \( p_b \) as endpoints.
Finally, apply LOP to edges inside influence polygons $Q^{e_c,L}$ and $Q^{e_c,R}$.

LOP terminates as CDT with $e_c$ as a constraint!

Existence follows from Lemma (⋆), extended to multiply connected polygons, and Lemma (⋆ ⋆).
Inserting a Point into a CDT

Let $\Delta(P \cup p, E_c)$ be the CDT obtained by inserting a point $p$ into a CDT $\Delta(P, E_c)$.

Exact limitation of $R^p$:

**Lemma.** A triangle $t$ in $\Delta(P, E_c)$ will be modified when inserting a point $p$ to obtain $\Delta(P \cup p, E_c)$ if and only if the circumcircle of $t$ contains $p$ in its interior and $p$ is visible from all the three nodes of $t$.

**Proof.** The proof follows directly from Theorem (Delaunay triangle).
How can $\Delta(P \cup p, E_c)$ be obtained?

Alt. 1:

**THEOREM.** All new triangles of $\Delta(P \cup p, E_c)$ have $p$ as a common node.

**Proof.** See Exercise in lecture notes.

$\Delta(P \cup p, E_c)$ can be obtained from $\Delta(P, E_c)$ by removing all triangles of $R^p$ and connecting $p$ to all points of $Q^p$; see Figure (b).

Alt 2, swapping procedure:

As for conventional Delaunay with this modification only: `recSwapDelaunay` must not swap constrained edges.