Decoherence of a qubit by non-Gaussian noise at an arbitrary working point

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The decoherence of a qubit due to a classical non-Gaussian noise with correlation time longer than the decoherence time is discussed for arbitrary working points of the qubit. A method is developed that allows an exact formula for the phase-memory functional in the presence of independent random telegraph noise sources to be derived.

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I. INTRODUCTION

Reducing the decoherence induced by interaction with the environment is one of the major challenges in the practical implementation of quantum computing. In particular, for solid state qubits this seems to be the most important issue. In this paper we discuss the effect of noise with a correlation time that is long compared to the decoherence time of the qubit. The work is directly motivated by experiments1,2 on Josephson charge qubits (JCQ), but due to their general nature they can also be relevant to many other systems. The JCQ is built around a small superconducting grain connected to a superconducting reservoir by a Josephson junction. By means of a capacitively coupled gate voltage one can control the number of Cooper pairs on the small grain. Because of the Coulomb blockade there will be a preferred number of Cooper pairs except at special degeneracy points where the energies of states with \( n \) and \( n+1 \) coincide. The degeneracy is lifted by the Josephson coupling, giving the usual level anticrossing picture sketched in Fig. 1. If the gate voltage is never too far from the degeneracy point we can ignore transitions to other levels and the device can be regarded as a two level system or qubit.

Consider noise in this system originating from fluctuations in the electric potential of the grain. This could be either due to fluctuations in the gate voltage source, or to fluctuating charges in the environment. Both would correspond to fluctuations of the working point position along the horizontal axis of the figure, with a corresponding change in energy. The experiments1 were conducted with the gate voltage away from the degeneracy point, where to a good approximation the energy is a linear function of the potential. This case is covered by the theory3,4 for the case of free induction and narrow distribution of coupling constants between the qubit and the noise source and for both free induction and echo at arbitrary distribution of coupling constants, respectively. Low-frequency fluctuations as a source of decoherence in qubits were addressed in several papers,5 see Ref. 6 for a review of some results. Quantum aspects of non-Markovian dynamics of a qubit were addressed in Ref. 7.

Using a modified circuit Devoret et al.2 were able to work at the degeneracy point where to first order there is no change in energy when the potential is changed. The idea is that this would make the device less sensitive to electrostatic noise. This is then called the optimal working point. At this point the expectation value of the charge is the same for both states. The purpose of this paper is to extend the theory3,4 to this situation and to see how dephasing is changed as the optimal point is approached. The initial stage of decoherence near the optimal point was numerically studied in Ref. 8.

The paper is organized as follows. In Sec. II the model for the qubit interacting with the noise is presented. In Sec. III we discuss a short time expansion that illustrates in a simple way the interplay of the effect of several uncorrelated fluctuators at the optimal point. The main purpose of this section is providing a qualitative understanding, as the results are contained within the full solution presented in Sec. IV.

II. MODEL

Consider the Hamiltonian of the Josephson qubit, cf. with Ref. 9,

\[ H = \frac{1}{2} \Delta \sigma_z - \frac{1}{2} E_c \sigma_x. \] (1)

Here \( \Delta = E_c C_g (V_{g} - V_{g}^{\text{opt}})/e \), where \( E_c \) is the charging energy, \( V_g - V_{g}^{\text{opt}} \) is the deviation in gate voltage from the optimal

![FIG. 1. Energy levels of Josephson qubit as function of the gate potential.](image-url)
working point, $C_j$ is the gate capacitance, and $e$ the electron charge. The energy levels as a function of the gate potential are shown in Fig. 1.

In principle, both $\Delta$ and $E_J$ can fluctuate in time, but for clarity we will only consider the noise in $\Delta$. Noise in $E_J$ was considered in Ref. 10 and can be treated in a similar way.

We will describe fluctuations in $\Delta$ by a random time-dependent quantity $\nu(t)$. The Hamiltonian of the qubit is then

$$H = \frac{1}{2}[\Delta + \nu(t)]\sigma_z - \frac{1}{2}E_J\sigma_x. \quad (2)$$

One can express $\nu(t)$ through fluctuations of the effective gate voltage, $\delta V_g(t)$ as $\nu(t)\equiv E_J C_g \delta V_g(t)/e$. One of the sources of such fluctuations is hopping background charges.\textsuperscript{3,4,11} We briefly recall the model of Refs. 3 and 4. It is assumed that there are dynamic defects characterized by two metastable states. They can be, for example, traps near the electrodes able to capture electrons from the electrodes and then re-emit them, or pairs of traps with one electron bouncing between them. Each dynamic defect is represented as a classical fluctuator producing random telegraph noise. That is, the state of each fluctuator is represented by a random function $\chi(t)$ which switches between the two values, $\pm 1/2$. We adopt the simplest model where it is assumed that the probabilities to jump between the states in two directions are the same. As a result, the equilibrium populations of the states are equal. This assumption can be relaxed\textsuperscript{4} without change of the physical conclusions. Indeed, since only fluctuators with energy splittings $\pm kT$ contribute to the decoherence, the equilibrium populations of the efficient states do not essentially differ. The probability to switch $n$ times between these states during the time $t$ is assumed to be given by the Poisson distribution

$$P_n(t) = \frac{\gamma^n}{n!} e^{-\gamma}. \quad (3)$$

Here $\gamma$ is the characteristic switching frequency of the given fluctuator. This means that the switching events occur independently of each other. The fluctuation $\nu(t)$ is a sum of the contributions of different fluctuators,

$$\nu(t) = \sum_i \nu_i \chi_i(t). \quad (4)$$

Different fluctuators are assumed to be statistically independent,

$$\langle \chi_i(t_1) \chi_j(t_2) \rangle = \frac{1}{4} e^{-2\gamma|t_1-t_2|} \delta_{ij}. \quad (5)$$

Several papers\textsuperscript{3,4,11} have addressed the dephasing in the presence of a large number of fluctuators with a broad range of switching rates, $\gamma$. This situation is relevant for systems showing $1/f$ noise. However, recent experiments\textsuperscript{1,12} indicate the presence of one or a small number of dominant fluctuators, and we believe that this is the typical situation in these devices. We will therefore here focus on the effect of a small number of fluctuators.

The noise affects the qubit in two ways: (i) it causes shifts in the energy levels of the two states and thereby introduces a random contribution to the relative phase of the two states (dephasing) and (ii) it causes transitions between the two states leading to energy relaxation. Let us for the moment concentrate on the first effect, dephasing. The important quantity that we study is the phase-memory functional

$$\Psi = \exp \left[ \frac{i}{\hbar} \int_0^t dt' \beta(t') \delta E(\{\chi(t')\}) \right] \quad (6)$$

where $\delta E(\{\chi(t')\})$ is the shift in the energy splitting of the two levels caused by the fluctuators, while $\beta(t)$ is some function which depends on the qubit manipulation procedure, as will be explained below. The phase-memory functional describes the relative phase picked up during time evolution by one state of the qubit relative to the other.

Diagonalizing Eq. (2) we get the eigenenergies

$$E_{\pm} = \pm \frac{1}{2} [\Delta + \nu] \pm \frac{1}{2} [\Delta - \nu]. \quad (7)$$

If the working point is far from the optimal one, $\Delta \gg E_J$, we can neglect the Josephson energy, and coupling of the fluctuators to the qubit is linear,

$$E_{\pm} \approx \pm \frac{1}{2} \sqrt{\Delta^2 + E_J^2} \left(1 \pm \frac{\nu \Delta}{\Delta^2 + E_J^2}\right). \quad (8)$$

An exact formula for the phase-memory functional for a linear coupling in the case of a single fluctuator coupled to the qubit was derived in Refs. 3, 4, and 13. In this limit, because of the linearity of $E$ in $\nu$, the phase-memory functional for any number of fluctuators is found by simply multiplying the phase-memory functionals of different fluctuators. In the regime of exponential decay this procedure corresponds to simply adding the decay rates and the resulting decay rate is represented by the average over the distributions of fluctuator parameters.\textsuperscript{4} The optimal point, $\Delta = 0$, was studied in various approximations and numerically in Refs. 11 and 14.

The aim of the present work is to derive an exact formula for the phase-memory functional of a single fluctuator similar to that in Refs. 3 and 4 that is applicable at an arbitrary working point. Averaging over a large number of fluctuators is in the general case not as simple as in the linear regime since the phase-memory functional is not the product of individual factors for each fluctuator. However, we will extend the analysis to a small number of fluctuators, which we believe is relevant to qubit experiments.\textsuperscript{1}

To make the main idea clear let us first study the expansion at short times, $\gamma t \ll 1$. Firstly, this expansion provides some insight; secondly, the short-time situation can be most important for realistic qubits since at long times the phase-memory functional has already decayed to a very low value.

### III. Phase-Memory Functional at Small Times, $\gamma t \ll 1$

We expand the energy in the limit

$$\nu \ll E_0 = \sqrt{\Delta^2 + E_J^2}$$

to obtain
where $E^* = E_0^2/E_f$. When averaging the phases we subtract the initial values so that we only get what comes from fluctuator jumps. Thus $\Psi = \langle e^{i\phi(t)} \rangle$ where,

$$
\phi(t) = \frac{1}{\hbar} \int_0^t \beta(t') dt' \left[ \sum_i \frac{\Delta}{2E_0} v_i [\chi_i(t') - \chi_i^0] \right] + \sum_{ij} \frac{v_j v_i}{4E^*} [\chi_i(t') \chi_j(t') - \chi_i^0 \chi_j^0].
$$

(10)

is the random phase shift. Here $\chi_i^0 = \chi_i(0)$ while the average is calculated over random telegraph processes in the fluctuators. Except in Eq. (20) we have also averaged over the initial state $\chi_i^0$ of each fluctuator.

If there is only linear coupling, each fluctuator appears in only one term in the exponent. Since we assume the fluctuators to be statistically independent the average can be split into a product of averages over each individual fluctuator. With the quadratic term included this is no longer possible. Here we will study the cases of one and two fluctuators.

(a) Free induction signal. Let us assume that $\beta(t) = \theta(t)$ where $\theta(t)$ is the Heaviside unit step function. This assumption corresponds to the free induction signal. With only one fluctuator the quadratic term is identically zero, and we have

$$
\Psi_1 = \left\langle e^{i \int_0^t \beta(t') dt' [\chi(t') - \chi^0]} \right\rangle, \quad \nu \equiv \frac{\Delta}{2E_0} v.
$$

(11)

Let us evaluate the memory functional in the limit of $\gamma t \ll 1$, first for one, and then for two fluctuators. Then the possibility of more than one jump is negligible, and we write for the probabilities of zero and one jump $P_0 = 1 - \gamma t$ and $P_1 = \gamma t$. The memory functional can be calculated by averaging over the time, $t_1$, between the jumps, which at $\gamma t \ll 1$ are equally probable:

$$
\Psi_1 = P_0 + \frac{P_1}{t} \int_0^t dt_1 \cos \nu(t-t_1) = 1 - \frac{\gamma \nu}{\nu - \sin \nu \nu}.
$$

If $\nu = (\Delta/2E_f) t \ll 1$ we can expand the sine get $\Psi_1 = 1 - \gamma \nu^2 t^2 / 6$. At the optimal point $\nu = 0$ the fluctuator is not visible to the qubit. At other working points the dephasing rate is proportional to $\Delta^2$. Note however, the fact that a single fluctuator invisible at the optimal point presupposes that the optimal point already is known, and will thus depend on the protocol that is used to establish this point.

Now take the case of two fluctuators. We have to calculate $\Psi_2 = \langle e^{i\phi(t)} \rangle$ where

$$
\phi(t) = \int_0^t dt' \left[ \nu_1 (\chi_1 - \chi_1^0) + \nu_2 (\chi_2 - \chi_2^0) + 2\lambda_{12} (\chi_1 \chi_2 - \chi_1^0 \chi_2^0) \right]
$$

(12)

and $\lambda_{12} = v_1 v_2 / E^* \hbar$. Again, for $\gamma t \ll 1$ we get

$$
\Psi_2 = 1 - \sum_{i=1,2} \left[ \frac{\gamma_i}{2} \left( \sin \left( \gamma_i \frac{\Delta}{2E_0} t \right) v_i \pm \lambda_{12} \right) \right].
$$

(13)

Assuming $\nu_1, \nu_2, \lambda_{12} \ll t^{-1}$ and expanding sines we get:

$$
\Psi_2 = 1 - \sum_i \frac{\gamma_i}{6} (\nu_i^2 + \lambda_{12}^2) t^3
$$

$$
= \left( 1 - \frac{\gamma_1 + \gamma_2}{6} \lambda_{12}^2 t^3 \right) \prod_i \left( 1 - \frac{\gamma_i v_i^2 t^3}{6} \right).
$$

(14)

We see that in this limit one can split the expression for $\Psi$ in factors corresponding to the individual fluctuators just as in the case of linear coupling, but there appears an additional factor due to the nonlinear coupling. It is easy to see that a similar pattern will also appear for a larger number of fluctuators. At longer times this product structure is lost.

The interplay between the linear and quadratic coupling is now quite clear. If $\nu \gg \lambda$ the linear coupling is dominant. Approaching the optimal point will reduce the dephasing until $\nu$ becomes smaller than $\lambda$ where the term proportional to $\lambda^2$ becomes most important. This contribution results from the interplay between the two fluctuators and cannot be eliminated; thus it represents the minimal dephasing possible at the optimal point. The physical reason for this is quite easy to understand. Switching of fluctuator one shifts the average point that fluctuator two is working around. Both positions of fluctuator one cannot represent the optimal point with respect to fluctuator two, and some dephasing is bound to occur. This is the most important physical insight that distinguishes the quadratic coupling from the linear. In the case of quadratic coupling, even if the different fluctuators in themselves are independent, their effect on the qubit will be influenced by the positions of all the others. With linear coupling one finds that slow fluctuators, with $\gamma t \ll 1$ do not contribute to the dephasing. This is no longer true for quadratic coupling, as they play a role in determining the effect of the fast fluctuators even if they do not have time to switch during the experiment. Thus very slow fluctuators may be of great importance.

(b) Two-pulse echo. Perhaps more directly related to the experiments are the echo signals. These are found using Eq. (10) where for two-pulse echo

$$
\beta(t) = \begin{cases} 
+1 & \text{for } t < \tau, \\
-1 & \text{for } \tau < t < 2\tau.
\end{cases}
$$

(15)

Here $\tau$ is the delay between the initial pulse and the echo pulse, and the echo signal is centered around $2\tau$. For short times ($\gamma \tau < 1$) this gives for one fluctuator

$$
\Psi_1 = 1 - \frac{2\gamma}{\nu} \nu \tau - \sin \nu \tau,
$$

(16)

and for two fluctuators

$$
\Psi_2 = 1 - \sum_{i=1,2} \left[ 2\gamma_i \tau - \gamma_i \left( \frac{\sin(\nu_i \pm \lambda_{12}) \tau}{\nu_i \pm \lambda_{12}} \right) \right].
$$

(17)
IV. EXACT SOLUTION FOR SMALL NUMBER OF FLUCTUATORS

We now turn to the calculation of the phase-memory functional for arbitrary times. The method described here is in principle applicable to an arbitrary number of fluctuators, but the resulting formulas quickly get impractically large when the number of fluctuators increase. The main idea is to consider the phase \( \phi \) as a random variable with some probability distribution \( p(\phi, t) \) that will depend on time. Once this is known the phase-memory function is given by \( \Psi = \int d\phi e^{i\phi} p(\phi, t) \). By mapping to a correlated random walk problem we derive a master equation for the probabilities \( p(\phi, t) \). The details of how to calculate \( p(\phi, t) \) are given in the Appendix.

A. Distribution function for one fluctuator far from the optimal point

Free-induction signal. Let us first discuss the results for the distribution function in the simplest case, that of one fluctuator far from the optimal point. The phase-memory functional for this problem was derived in Refs. 3 and 4, and in the end we will rederive the same expression. However, it is the simplest example for illustrating the general method, and it gives insight to the relation between the Gaussian approximation, and the fluctuator model. To better understand the meaning of the results it is useful to recall the standard picture of dephasing by a Gaussian noise that is well known from NMR-physics (see, e.g., Ref. 15). If the time \( t \) entering the phase-memory functional

\[
\Psi = \langle e^{i\phi(t)} \rangle, \quad \phi(t) = \int_0^t dt' \nu(t'), \quad \nu(t) = \nu \chi(t)
\]

is much longer than the correlation time \( \approx \gamma^{-1} \) of the fluctuating function \( \nu(t) \), the integral can be considered as the sum of a large number of uncorrelated contributions. Consequently, by the central limit theorem, the phase will be distributed according to a Gaussian

\[
p(\phi) = \frac{1}{\sqrt{2\pi\gamma \nu^2}} e^{-\phi^2/(2\gamma \nu^2)}, \quad \Psi = e^{-\langle \phi^2 \rangle /2}.
\]

From Eq. (5) for the correlation function we get

\[
\langle \phi^2 \rangle = \left( \frac{\nu}{2\gamma} \right)^2 (2\gamma t + e^{-2\gamma t} - 1) \approx \frac{\nu^2 t^2}{4\gamma} \quad \text{at} \quad \gamma t \gg 1.
\]

Thus, in the Gaussian approximation, the phase-memory functional decays exponentially at \( \gamma t \gg 1 \) with the rate

\[
\Gamma_{\phi}^{(G)} = \nu^2/8\gamma.
\]

Using the method explained in the Appendix we can find an exact solution for the distribution function of \( \phi \)

\[
p(\phi, t) = e^{-\nu^2 t^2} \left[ \delta(\phi - \nu t/2) + \frac{\gamma (t + 2\phi \nu) I_1(\gamma \sqrt{t^2 - (2\phi \nu)^2})}{\nu \sqrt{t^2 - (2\phi \nu)^2}} \right] \times \left[ \theta(\frac{2\phi}{\nu} - t) - \theta(\frac{2\phi}{\nu} + t) \right],
\]

where the different signs correspond to different initial states of the fluctuator and \( I_1(z) \) is the modified Bessel function. Without jumps of the fluctuator, the result would be only the moving \( \delta \) pulse of a constant amplitude, \( \delta \phi \sim \nu t/2 \). The value \( \nu t/2 \) is the maximal possible value of \( \phi \) acquired for the time \( t \), while the jumps of the fluctuators account for the smooth part. Averaging over the initial state of the fluctuator we get

\[
p(\phi, t) = e^{-\nu^2 t^2} \left[ \delta(\phi - \nu t/2) + \delta(\phi + \nu t/2) \right] + \frac{\gamma}{\nu} I_1(\nu \sqrt{1 - (2\nu t/\sqrt{\gamma})^2}) \frac{1}{\sqrt{1 - (2\nu t/\sqrt{\gamma})^2}}.
\]

Here \( \theta \) functions are omitted. This is plotted in Fig. 2 for the times \( t = 1, 5, 10 \) for \( \gamma = 1 \) and \( \nu = 1 \). Arrows represent \( \delta \)-function peaks.

We observe that the central region is similar to a Gaussian, but at short times this is cut off by the \( \delta \) functions represented by the vertical lines. At \( \gamma t \ll 1 \) the function is indeed close to a Gaussian, as can be seen from the asymptotic expansion of the Bessel function

\[
e^{-\nu^2 t^2} I_1(\nu \sqrt{1 - (2\nu t)^2}) \approx \frac{1}{\sqrt{2\pi \nu \gamma}} e^{-\nu^2 t^2/2 \gamma}.
\]

Comparing to Eq. (18) we see that we recover the result for the dephasing rate in Eq. (19).

However, we know from Ref. 4 that if \( \nu > 2\gamma \) we have pronounced non-Gaussian behavior. We can now understand this from the point of view of the distribution function. The smooth central part of this indeed approaches a Gaussian for \( \gamma t \gg 1 \) and this gives the dephasing rate (19), but the \( \delta \) functions at the ends only decay at the rate \( \gamma \). As long as the \( \Gamma_{\phi}^{(G)} \) of (19) is smaller than \( \gamma \) the decay will be controlled by the central part and the Gaussian approximation is valid. If \( \Gamma_{\phi}^{(G)} > \gamma \) the decay is limited by the \( \delta \) functions, and is set by the rate \( \gamma \).

Using the distribution function (21) one can calculate the phase-memory functional (see the Appendix) and one finds for \( \gamma t \ll 1 \) an exponential decay with rate

\[
\Gamma_{\phi} = \gamma - \text{Re}(\gamma^2 - \nu^2/4).
\]

Figure 3 shows \( \Gamma_{\phi} \) as function of \( \nu \) at \( \gamma = 1 \) for the Gaussian and the fluctuator models.

Two-pulse echo. The same method as outlined in the Appendix allows one to calculate the phase distribution func-
tion, \( p_e(\phi, \tau) \) for the two-pulse echo signal. The result can be expressed in the form
\[
p_e(\phi, \tau) = e^{-2\tau} \left[ \delta(\phi) + \frac{\gamma}{\pi} \int_0^\infty dk \cos k\phi \sin w_k \tau + w_k \cos w_k \tau \right].
\]
Here \( w_k = \sqrt{(k\nu/2)^2 - \gamma^2} \), while \( \tau \) is the delay time between the first and second pulse. The smooth part of the distribution given by the second item in the above formula is plotted in Fig. 4. It is qualitatively similar to the phase distribution for the free induction.

B. Dephasing rate as function of working point

Let us then consider the decay of the phase-memory functional for different working points. Again we recall the situation in NMR physics where the loss of the signal after the spins are set precessing by a \( \pi/2 \)-pulse is caused by two independent processes. The phase-memory functional considered above measures the random contributions to the phase caused by a fluctuating energy difference between the two states (which in an NMR experiment is caused by fluctuations in the magnetic field parallel to the external main field). In addition there are processes which flip the spin from one state to another, so called \( T_1 \) processes. These are caused by fluctuations in the magnetic field normal to the external field. If we denote the decay rate of the excited state into the ground state \( \Gamma_1 \) and add the two contributions we have the total decay of the spin precession signal
\[
\Gamma_2 = \frac{1}{2} \Gamma_1 + \Gamma_\phi,
\]
see Ref. 15, or, for qubits, Ref. 14.

Let us now discuss how the relative strength of the two terms of Eq. (23) changes as we change the working point of the qubit. Looking back at the Hamiltonian (2) we remember that it is equivalent to a spin 1/2 particle in a static magnetic field \( B = E_j e_x + \Delta e_z \), while the noise is always along the \( z \) axis, \( \nu = \nu e_z \). We denote the angle between \( B \) and the \( z \) axis by \( \theta = \arctan(E_j/\Delta) \). In particular, \( \theta = 0 \) corresponds to working far from the degeneracy point where \( \delta \gg E_j \) while \( \theta = \pi/2 \) is the degeneracy (optimal) point \( \Delta = 0 \). The time evolution of the qubit is then a precession on the Bloch sphere around the total field (Fig. 5).

Far from the degeneracy point the main field is directed along the \( z \) axis (pointing at the north pole of the sphere), whereas at degeneracy it is along the \( x \) axis (on the equator). All the time the noise vector \( \nu \) is in the \( z \) direction. We see that the noise component parallel to the external field, which gives the \( \Gamma_\phi \), is largest far from degeneracy, which agrees with our previous discussion. However, the noise normal to the field, giving \( T_1 \) processes, is maximal at degeneracy. In the Gaussian approximation this is given by (see Ref. 15)
\[
\Gamma_1 = \frac{1}{T_1} = \sin^2 \theta \int_0^\infty d\tau \nu(t) \nu(t+\tau) \cos \frac{\nu_0 \tau}{\hbar}.
\]
Using the correlator (5) one can rewrite this expression for the case of \( N \) identical fluctuators with parameters \( \nu \) and \( \gamma \) as follows:
\[
\Gamma_1 = \frac{N}{2} \frac{\hbar^2 \nu^2 \gamma}{E_0^2 + 4 \gamma^2} \left( \frac{E_j}{E_0} \right)^2.
\]
At this point we should warn the reader that although we will use this expression for \( \Gamma_1 \) in Eq. (23) it is not likely to be relevant to typical qubit experiments were the energy splitting of the qubit \( E_0 \gg T \). This means that our high temperature approximation is not very good. In reality, the main contribution to the relaxation is expected to come from quan-
decoherence rates obtained by a numerical simulation of the
cess. All curves are calculated for the case of two fluctuators
resulting decay time, 2, respectively. The points represent the
average in Eq. (20).

The Gaussian approximation for the \( \Gamma_{\phi} \) is more difficult
to obtain because the square root in the energy (7) makes the
average in Eq. (18) not treatable analytically. However we
can expand in the lowest order in \( v/E_0 \), which is a good
approximation except at a close vicinity to the degeneracy
point, where the coefficient in front goes to zero and higher
order terms need to be calculated. This gives the same result
as the \( \Gamma_2^{0d} = 1/T_2^{0d} \) of Ref. 15,

\[
\Gamma_{\phi}^{(G)} \approx \frac{1}{2} \cos^2 \theta \int_0^\infty dt \langle \sin^2 v(t) \rangle,
\]

which for \( N \) identical uncorrelated fluctuators yields

\[
\Gamma_{\phi}^{(G)} = \frac{N v^2}{8 \gamma} \left( \frac{\Delta}{E_0} \right)^2.
\] (25)

Now we want to compare these Gaussian results with the
exact expressions found using the method in the Appendix.

In Fig. 6 the relaxation times (inverses of the relaxation
rates) \( T_1 \) (curve 2) and \( 1/\Gamma_{\phi}^{(G)} \) (curve 1) are shown. They are
calculated according to Eqs. (24) and (25), respectively. The
decay time, \( T_2 = \Gamma_{\phi}^{-1} \) (curve 4), of the phase-memory functional
calculated using the method of the Appendix and the
resulting decay time, \( T_2 \) (curve 3), of the spin signal according
to Eq. (23) are also shown. The points represent the
decoherence rates obtained by a numerical simulation of the
time evolution according to the Hamiltonian (2), which
performs averaging over many realizations of the random
process. All curves are calculated for the case of two fluctuators
with coupling strength \( v/E_j = 0.1 \) and switching rate
\( \gamma/E_j = 0.1 \). Thus \( v/\gamma = 1 \) and this case belongs to the so-
called weak coupling regime.9 We see that for all working
points the decay is well described by the Gaussian approxi-
mation. This is because the decoherence rates always are slower than the limiting rate \( \gamma \) set by the correlation time of the
fluctuators, similar to what was described above for the
case far from the degeneracy point. For most working points
the rate \( \Gamma_{\phi} \gamma \) and this dominates the \( \Gamma_2 \). Close to the
degeneracy point we see that \( \Gamma_1 \) becomes more important and at
degeneracy it dominates completely giving \( \Gamma_2 \approx \Gamma_1/2 \).

The same quantities for the case of strong coupling
\( v/E_j = 0.1, \gamma/E_j = 0.01, v/\gamma = 10 \) are shown in Fig. 7. Far
from the degeneracy the situation is similar to the one
described earlier, with the rate, \( \gamma \), determined by the decay of the
\( \delta \)-function peaks in the distribution function. Closer to
the degeneracy the decoherence rate is slower and the Gaussian
approximation gives good results. For a single fluctuator it is easy to give a criterion for when the Gaussian approxi-
mation is valid or not. From Eq. (9) we get that the effective
coupling strength is \( \frac{\Delta}{E_0} \) and inserting this into Eq. (22) we
find that the Gaussian approximation can be used when
\( \gamma < (\nu/4E_0)^2 \). For a larger number of fluctuator it is less
simple to write a similar condition, as one has to use method
in the Appendix to derive an expression corresponding to Eq.
(22) which quickly becomes impractical for larger numbers
except numerically.

Note that in the case of strong coupling the difference
between the approximate expression \( \Gamma_{\phi}^{(G)} \) and the exact \( \Gamma_{\phi} \)
becomes more noticeable. At large \( \Delta \) this is because of the
essentially non-Gaussian character of the noise, as discussed before. At small \( \Delta \) the difference comes from the fact that the
\( \Gamma_{\phi}^{(G)} \) only gives the Gaussian character of the noise, as discussed before. At small \( \Delta \) the difference comes from the fact that the
(22) (which is hard to find analytically) the result would agree
completely with the \( \Gamma_{\phi} \) calculated by the method in the Ap-
pendix since the decay time is much longer than the corre-
lation time \( 1/\gamma \) of the noise and the central limit theorem
should work. Note also that at degeneracy the decay is still
dominated by the phase relaxation processes, \( \Gamma_{\phi} \), while the
\( T_1 \) processes only give a small correction.

Figures 6 and 7 should be compared to the experimental
results of Astafiev et al.18 There is clear qualitative agree-
ment, but it is hard to try to make a quantitative fit, especially for the $T_2$ where there is very little data.

Let us rather look back at the Hamiltonian (2) and Fig. 5. So far we only considered noise in the $\sigma_z$ component of the Hamiltonian, as appropriate for noise sources coupled to the charge of the qubit. As discussed in Ref. 10 there is also the possibility of noise in the Josephson coupling (the $\sigma_x$ part). In Fig. 5 this would correspond to a noise vector parallel to the $x$ axis. This noise would be transversal far from degeneracy, giving large $T_2$ and small $T_1$ and it would be longitudinal at the degeneracy point, with $T_1$ large and $T_2$ small. The fact that the experimental results are similar to our Figs. 6 and 7 rather that the opposite shows that in these experiments the noise in the $\sigma_z$ part is dominant.

V. DISCUSSION

A. Measuring the distribution function?

The method that we have used to find the phase-memory functional gave as an intermediate result an expression for the distribution function of the phase $p(\phi,t)$. The phase-memory functional is the average of the quantity $e^{i\phi}$ with this distribution. But can the full distribution function be compared to the experiment? In the first set of experiments only averages could be measured, but recently there has been the possibility that one can experimentally distinguish these in an experiment, but it is hard to try to make a quantitative fit, especially for the $T_2$ where there is very little data.

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B. Non-Gaussian $T_1$?

Notice that it seems from the Figs. 6 and 7 that the energy relaxation time $T_1$ is always well described by the Gaussian approximation. Can we understand this in a better way? The Gaussian approximation is good provided the correlation time of the noise is much shorter than the decay time, $T_1$ or $T_2$. In that case separation of time scales enables one to deduce an exponential decay with rates (Ref. 15)

$$1/T_1 \approx S(E_0), \quad 1/T_2 = 1/2T_1 + \Gamma_\phi, \quad \Gamma_\phi \approx S(0),$$

where $S(\omega)$ is the noise power spectrum. For the telegraph process with switching rate $\gamma$ and coupling strength $v$ we find that the time correlation is given by

$$\langle \chi(t)\chi(0) \rangle = e^{-2\gamma|t|}$$

so the correlation time is $1/2\gamma$ and the noise power spectrum

$$S(\omega) \approx \frac{\gamma v^2}{\gamma^2 + \omega^2}.$$

This gives the relative rates

$$\frac{1}{T_1} \approx \frac{v^2}{\gamma^2 + E_J^2} \leq \left(\frac{v}{E_J}\right)^2, \quad \frac{\Gamma_\phi}{\gamma} \approx \left(\frac{v}{\gamma}\right)^2.$$

We see that as long as the noise is weak compared to the qubit splitting, $v < E_J$, the condition $\gamma T_1 \gg 1$ for the validity of the Gaussian approximation, is always satisfied. For the $T_2$ the situation is different, and the condition $\Gamma_2 \ll \gamma$ is violated. So in the case of dephasing, the Gaussian approximation is not consistent as long as the noise is weak compared to the level spacing.

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APPENDIX: CALCULATION OF DISTRIBUTION FUNCTION

1. Single fluctuator

To explain the method we rederive the result for one fluctuator with linear coupling since this is the simplest case. Let us for simplicity also assume that $\beta(t)=1$ at $t>0$ and start with the calculation of $\Psi_1(t)$.

Let us discretize the integral (12) for $\lambda_{12}=0$ introducing small time steps $\tau = t/N$, where $N \gg 1$. Then the random phase shift $\phi(t)$ can be expressed as follows:

$$\phi(t) = \nu \tau \sum_{n=1}^{N} X_n, \quad X_n = \chi(n \tau).$$

Hence, the integration over time can then be thought of as a random walk process, where at each time step the random walker moves a step $\sigma = \nu \tau / 2$ in the direction depending on the current position of the fluctuator. The steps are correlated, but only with the previous step. The probability that a
step is in the same direction as the previous one is 
\( \alpha = 1 - \gamma r \) and the probability for a step to be in the opposite
direction is \( \beta = \gamma r \). Let \( m \) be the number of steps from the
origin (so that the position is \( x = \sigma m \)). We want to find the
probability \( P_n(m) \) to be in position \( m \) at time step \( n \)
(dimensional time \( t_n = n \gamma \)). This is found by the following method.
We split the probability in two parts: the probability to reach
point \( m \) coming from the right, \( A_n(m) \) and from the left \( B_n(m), \) so that \( P_n(m) = A_n(m) + B_n(m). \) We then have the
equations
\[
A_{m+1}(m) = \alpha A_n(m - 1) + \beta B_n(m - 1),
\]
\[
B_{m+1}(m) = \beta A_n(m + 1) + \alpha B_n(m + 1). \tag{A1}
\]
We need the continuum limit, letting \( N \to \infty \), and \( \tau \to 0 \) with
\( N \tau = t \) fixed. Writing
\[
A(\phi,t) = a(\sigma \sigma, \sigma \sigma) = A_n(m),
\]
\[
b(\phi,t) = b(\sigma \sigma, \sigma \sigma) = B_n(m) \tag{A2}
\]
and expanding to first order in \( \tau \) we get
\[
a + \tau a_i = \alpha(a - \sigma \sigma a) + \beta(b - \sigma \sigma b),
\]
\[
b + \tau b_i = \beta(a - \sigma \sigma a) + \alpha(b - \sigma \sigma b). \tag{A3}
\]
Here subscripts \( \phi \) and \( t \) denote partial derivatives with re-
spect to \( \phi \) and \( t \), respectively. Adding and subtracting these we get (with \( p = a + b \) and \( q = a - b)\)
\[
\tau p_i = (\beta - \alpha) \sigma q \phi,
\]
\[
q + \tau q_i = (\alpha - \beta) q - \alpha \sigma p \phi. \tag{A4}
\]
Differentiating the second of these and inserting \( q \phi \) from the
first we obtain the final equation for \( p \)
\[
p_n + 2 \gamma p_\phi = \left( \frac{\gamma}{2} \right) p_{\phi \phi} \tag{A5}
\]
which is called the telegraph equation. We guess the solution
\( p = e^{i(k \phi - \omega t)} \) and get the dispersion relation
\[
\omega = -i \gamma \pm \sqrt{\frac{\nu}{2}} \kappa^2 - \gamma^2. \tag{A6}
\]
The general solution is then
\[
p(\phi,t) = \frac{d \kappa}{2\pi} \left[ a_\kappa e^{-i \kappa \phi} + b_\kappa e^{-i \kappa \phi} \right] e^{i \kappa \phi}. \tag{A7}
\]
The coefficients \( \{a_\kappa, b_\kappa\} \) can be obtained from the initial
conditions,
\[
\delta(\phi) = p(\phi,0) = \int_{-\infty}^{\infty} d \kappa [a_\kappa + b_\kappa] e^{i \kappa \phi} \Rightarrow a_\kappa + b_\kappa = 1, \tag{A8}
\]
\[
\nu \chi_0 \delta'(\phi) = p_r(\phi,t) = -i \int_{-\infty}^{\infty} d \kappa [a_\kappa + \omega \cdot b_\kappa] e^{i \kappa \phi}
\]
\[
\Rightarrow a_\kappa + \omega \cdot b_\kappa = -\kappa \nu \chi_0, \tag{A9}
\]
which yield
\[
\{a_\kappa, b_\kappa\} = \frac{1}{2} \left( 1 \pm \frac{\kappa \nu \chi_0 - i \gamma}{\sqrt{\kappa^2 \nu^2/4 - \gamma^2}} \right). \tag{A10}
\]
This is to be inserted into Eq. (A7). The result is given by
Eq. (20).

The memory functional can be expressed as the expectation value
\( \Psi_1 = \int \mathcal{D} \phi d \phi \langle \phi(t) e^{i \phi} \rangle. \) Thus
\[
\Psi_1 = \int d \kappa [a_\kappa e^{-i \omega \tau} + b_\kappa e^{-i \kappa \phi}] \delta(\kappa + 1)
\]
\[
= \frac{1}{2 \mu} e^{-\gamma \sum \mu \pm \frac{i \nu \chi_0}{\gamma}} \mu^{\mu \gamma}, \tag{A11}
\]
where \( \mu = \sqrt{1 - \nu^2/4 \gamma^2}. \) This agrees with the result of Ref. 4.

2. Two fluctuators

Now we turn to the case of two fluctuators. Again we
discretize (12) to get
\[
\phi_\mu = \gamma \sum_{n=1}^{N} (\nu \chi_1 + \nu \chi_2 + 2 \lambda \chi_1 \chi_2).
\]

There are now three kinds of steps, depending on the settings
of the fluctuators. If both have the value +\( \frac{\gamma}{2} \) there is a step
\( \alpha = \nu + \frac{\gamma}{2}, \) if both are -\( \frac{\gamma}{2} \) there is a step
\( \beta = -\nu + \frac{\gamma}{2}, \) and for one of each the step is
\( \gamma = \frac{\gamma}{2}. \) We have the following probabilities
for each jump, depending on the previous state:

\[
\begin{array}{ccc}
\text{Previous} & \text{\( \alpha \)} & \text{\( \gamma \)} \\
\alpha & \alpha^2 & 2 \alpha \beta \\
\gamma & \alpha \beta & \alpha^2 + \beta^2 \\
\beta & 2 \alpha \beta & \beta^2
\end{array}
\]

The total probability must now be split in three parts
\( P_n(\phi) = A_n(\phi) + B_n(\phi) + C_n(\phi), \) where \( A_n(\phi) \) is the probability
to reach point \( \phi \) at time step \( n \) with a \( \alpha \) jump, \( B_n(\phi) \) with a
\( \beta \) jump, and \( C_n(\phi) \) with a \( \gamma \) jump. We have then the set of
equations
\[
A_{n+1}(\phi) = \alpha^2 A_n(\phi - \alpha \tau) + \beta^2 B_n(\phi - \beta \tau) + \alpha \beta C_n(\phi - \alpha \tau), \tag{A12}
\]
\[
B_{n+1}(\phi) = \beta^2 A_n(\phi - \beta \tau) + \alpha^2 B_n(\phi - \beta \tau) + \alpha \beta C_n(\phi - \beta \tau), \tag{A12}
\]
\[
C_{n+1}(\phi) = 2 \beta A_n(\phi - \beta \tau) + 2 \alpha \beta B_n(\phi - \beta \tau) + \alpha^2 + \beta^2 \\
\times C_n(\phi - \beta \tau). \tag{A12}
\]
Again we introduce continuous variables \( a,b,c \) and expand
to first order in \( \tau \) to get
we get the dispersion equation for
We then guess the solution in the form
This can be written in matrix form
where
\[
M = \begin{pmatrix}
-2\gamma - \bar{\alpha}\partial\phi & 0 & \gamma \\
0 & -2\gamma - \bar{\beta}\partial\phi & \gamma \\
2\gamma & 2\gamma & -2\gamma + \bar{\gamma}\partial\phi
\end{pmatrix}.
\]
(A13)
We then guess the solution in the form
\[
a = Ae^{i(\kappa\phi - \omega t)}
\]
(A16)
which gives the eigenvalue equation
\[
-i\omega a = \tilde{M}A,
\]
(A17)
where \(\tilde{M}\) is the matrix \(M\) with \(\partial\phi\) replaced by \(i\kappa\). From this we get the dispersion equation for \(\omega\) in terms of \(\kappa\):
\[
\omega^3 + \left(6\gamma i - \frac{\lambda}{4}\right)\omega^2
+ \left(-\kappa^2\nu^2 - 8\gamma^2 - 4\gamma i\kappa - \frac{\lambda^2}{2} + \kappa^2\frac{\lambda^2}{4}\right)\omega
- 2\gamma i\kappa^2\left(\nu^2 + \frac{\lambda^2}{4}\right) - \kappa^2\lambda\left(\nu^2 - \frac{\lambda^2}{4}\right) = 0.
\]
This equation has the three solutions: \(\omega_0\) and \(\omega_\kappa\) where \(\omega_0\) is the solution that goes continuously to \(-2i\gamma\) when \(\lambda \to 0\). The general solution is then
\[
a(t) = \int \frac{d\kappa}{2\pi} A_\kappa \begin{pmatrix}
\delta^a & 0 & 0 \\
0 & \delta^b & 0 \\
0 & 0 & \delta^c
\end{pmatrix} e^{i\omega t},
\]
(A18)
where
\[
A_\kappa = \begin{pmatrix}
a^a_\kappa & b^a_\kappa & c^a_\kappa \\
a^b_\kappa & b^b_\kappa & c^b_\kappa \\
a^c_\kappa & b^c_\kappa & c^c_\kappa
\end{pmatrix}
\]
(A19)
is a matrix of coefficients that has to be determined by the initial conditions.

Since we have three coefficients to determine for each of the \(a\), \(b\), and \(c\) it appears that we need to specify both the functions \(a(\phi,0)\) \cdots and the first two derivatives. However, because of the special form of the equations we can calculate all derivatives at \(t=0\) from the functions \(a(\phi,0)\) \cdots
\[
a,(\phi,0) = \tilde{M}a(\phi,0), \quad a_n(\phi,0) = \tilde{M}^2a(\phi,0).
\]
(A20)
The typical initial conditions would then correspond to specifying the initial type of jump in the random walk. For example if this was of type \(\alpha\) we would have \(a(x,0) = \delta(x)\) and \(b(x,0) = c(x,0) = 0\). Note that in this simple case where only one of the \(a\), \(b\), and \(c\) are nonzero at \(t=0\) the procedure could be simplified by writing the general solution for \(p=a+b+c\) and initial conditions for this. The more general case would be that all of \(a\), \(b\), and \(c\) are nonzero and the complete matrix \(A_{\kappa}\) is needed. This would be the case, for example, when calculating echo signals, where the equations after the echo pulse has to be solved with initial conditions of this type corresponding to the solution of the equations before the echo pulse is applied.

Introducing the Fourier transformed functions (in \(\phi\))
\[
\tilde{a}_\kappa = \begin{pmatrix}
\bar{a}_\kappa \\
\bar{b}_\kappa \\
\bar{c}_\kappa
\end{pmatrix} = A_\kappa \begin{pmatrix}
e^{-i\omega_0 t} \\
e^{-i\omega_\kappa t} \\
e^{-i\omega_\kappa t}
\end{pmatrix}
\]
(A21)
we can write the initial conditions as
\[
\tilde{a}_\kappa(t=0) = \tilde{A}_\kappa \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix},
\tilde{M}\tilde{a}_\kappa(t=0) = -iA_\kappa \begin{pmatrix}
\omega_0 \\
\omega_\kappa \\
\omega_\kappa
\end{pmatrix},
\tilde{M}^2\tilde{a}_\kappa(t=0) = -\tilde{A}_\kappa \begin{pmatrix}
\omega_0^2 \\
\omega_\kappa^2 \\
\omega_\kappa^2
\end{pmatrix}.
\]
(A22)
These can be written more compactly if we introduce the matrix \(\tilde{M}_{\kappa}\) with the left-hand sides of the above equations as columns
\[
\begin{pmatrix}
\tilde{a}_\kappa(t=0) \\
\tilde{M}\tilde{a}_\kappa(t=0) \\
\tilde{M}^2\tilde{a}_\kappa(t=0)
\end{pmatrix}
\]
\[
= A_\kappa \begin{pmatrix}
1 - i\omega_0 & -i\omega_0^2 \\
1 - i\omega_\kappa & -i\omega_\kappa^2 \\
1 - i\omega_\kappa & -i\omega_\kappa^2
\end{pmatrix}
\]
(A23)
from which the coefficients are found as \(A_{\kappa} = \tilde{M}_{\kappa}\Omega^{-1}\).

The final solution is then
\[
p(\phi,t) = \int_{-\pi}^{\pi} d\kappa \frac{d\kappa}{2\pi} \sum_{\kappa} [a_\kappa e^{-i\omega_0 t} + b_\kappa e^{-i\omega_\kappa t} + c_\kappa e^{-i\omega_\kappa t}] e^{i\kappa\phi},
\]
where \(a_\kappa = \Sigma a_\kappa\) and similarly for \(b_\kappa\) and \(c_\kappa\). Again the average is calculated from
\[
\langle e^{i\phi}\rangle = \int d\phi e^{i\phi} p(\phi,t) = a_{-1} e^{-i\omega_0 t} + b_{-1} e^{-i\omega_\kappa t} + c_{-1} e^{-i\omega_\kappa t},
\]
where \(\kappa = -1\) in the \(\omega\) in the last expression because of the \(\delta\) function from the \(\phi\) integral.

Let us find the explicit expressions for the coefficients \(a_\kappa\), \(b_\kappa\), and \(c_\kappa\) in this case. Adding the lines in Eq. (A23) we get
number of fluctuators. The general solution is

\[ a_\kappa + b_\kappa + c_\kappa = 1, \]
\[ a_\kappa \omega_0 + b_\kappa \omega_+ + c_\kappa \omega_- = \kappa [\nu(\chi_1^0 + \chi_2^0) + 2\lambda \chi_1^0 \chi_2^0] = A_\kappa \]
\[ a_\kappa \omega_0^2 + b_\kappa \omega_+^2 + c_\kappa \omega_-^2 = \kappa^2 [\nu(\chi_1^0 + \chi_2^0) + 2\lambda \chi_1^0 \chi_2^0]^2 \]
\[ -2i \gamma \kappa \nu (\chi_1^0 + \chi_2^0) - 8i \gamma \kappa \lambda \chi_1^0 \chi_2^0 = B_\kappa. \]

Here \( \chi_{1,2} \) represent the initial state of the fluctuators. Also of interest are the values of these averaged over the initial states of the fluctuators. Assuming all four settings are equally probable we have

\[ A_{\kappa}^{\text{avg}} = 0, \quad B_{\kappa}^{\text{avg}} = \frac{1}{2} \kappa^2 \left( \nu^2 + \frac{1}{2} \lambda^2 \right). \]

In terms of these the coefficients are expressed as follows:

\[ c_\kappa = \frac{(\omega_0 - A)(\omega_0 + \omega_+) - (\omega_0^2 - B)}{(\omega_0 - \omega_+)(\omega_0 - \omega_-)}, \]
\[ b_\kappa = \frac{-(\omega_0 - A)(\omega_0 + \omega_-) + (\omega_0^2 - B)}{(\omega_0 - \omega_-)(\omega_0 - \omega_+)} , \]
\[ a_\kappa = 1 - b_\kappa - c_\kappa. \]

(A24)

3. General case

The above method is in principle simple to generalize to any number of fluctuators, but the number of equations increases exponentially in the number of fluctuators.

The general equation is

\[ a_i = M a, \quad a = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \]

Guessing the solution \( a = A e^{i(\omega_0 \tau - \omega t)} \) we get the dispersion equation \(-i\omega a = \tilde{M} A\), which determines the \( n \) eigenvalues \( \omega_i(\kappa) \) \((i = 1 \cdots n)\) as functions of \( \kappa \). Here \( n = 2^N \) with \( N \) the number of fluctuators. The general solution is

\[ a = \int \frac{d\kappa}{2\pi} e^{i\kappa \phi} A \kappa e^{i\omega t}, \quad e^{i\omega t} = \begin{pmatrix} e^{-i\omega_1 t} \\ \vdots \\ e^{-i\omega_n t} \end{pmatrix}. \]

Defining

\[ \vec{a}_\kappa = \begin{pmatrix} \vec{a}_\kappa \\ \vec{b}_\kappa \\ \vec{c}_\kappa \end{pmatrix} = A \kappa e^{i\omega_0 t}, \quad \tilde{M} = M(\partial_\phi \rightarrow i\kappa) \]

we determine the coefficient matrix \( A_\kappa \) from

\[ (\vec{a}_\kappa(t=0) | \tilde{M} \vec{a}_\kappa(t=0) | \cdots | \tilde{M}^{n-1} \vec{a}_\kappa(t=0)) \]

\[ = A_\kappa \begin{pmatrix} 1 - i\omega_1 & \cdots & (-i\omega_1)^{n-1} \\ \vdots & \ddots & \vdots \\ 1 - i\omega_n & \cdots & (-i\omega_n)^{n-1} \end{pmatrix}. \]

The matrix \( \tilde{M} \) can be written as the contraction of a third order tensor \( \tilde{M}_T \) with \( \vec{a}_\kappa \).

Writing the tensor indices we have

\[ [\tilde{M}_\kappa]_{ij} = [\tilde{M}_T]_{ijk} (\vec{a}_\kappa(0))_k, \quad [\tilde{M}_\kappa]_{ijk} = [\tilde{M}_T]_{ijk}^{-1} \]

We then get

\[ [A_\kappa]_{ij} = [\tilde{M}_T]_{lmk} [\Omega^{-1}]_{mj} (\vec{a}_\kappa(0))_k \]

\[ [T]_{ik} = [M_\Theta]_{ijk} [e^{i\omega_0 t}]_j, \]

One can also write equations for echo experiments. Consider the situation where we initially prepare a state, then apply the echo pulse at time \( t_e \), and then measure the state at the time \( 2t_e \) when the echo signal appears (two pulse echo). The state just before the application of the echo pulse has to be calculated as above and then this is used as the initial state for the evolution after the echo (it is assumed that the duration of the echo pulse is short). After the echo pulse the matrix \( \tilde{M} \) is changed because all jumps of the random walk change sign. Then also the \( \omega_i \) change. Let \( \tilde{M}^* \), \( \omega_i^* \) represent quantities before the echo pulse and \( \tilde{M}^* \), \( \omega_i^* \) after the pulse. Then

\[ [\tilde{a}_\kappa(t_e)]_k = [T]_{ik} (\tilde{a}_\kappa(0))_k, \]

\[ [\tilde{a}_\kappa(2t_e)]_k = [T]_{ik} (\tilde{a}_\kappa(t_e))_k \]

with

\[ [T]_{ik} = [M]_{ijk} [e^{i\omega^* t}]_j, \quad [T]_{ik} = [M]_{ijk} [e^{i\omega^* t}]_j. \]
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