Nef cycles on some hyperkähler fourfolds

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Abstract

We study the cones of surfaces on varieties of lines on cubic fourfolds and Hilbert schemes of points on K3 surfaces. From this we obtain new examples of nef cycles which fail to be pseudoeffective.

Introduction

The cones of curves and effective divisors are essential tools in the study of algebraic varieties. Here the intersection pairing between curves and divisors allows one to interpret these cones geometrically in terms of their duals; the cones of nef divisors and ‘movable’ curves respectively. In intermediate dimensions however, very little is known about the behaviour of the cones of effective cycles. Here a surprising feature is that nef cycles may fail to be pseudoeffective, showing that the usual geometric intuition for ‘positivity’ does not extend more generally. In the paper [7], Debarre–Ein–Lazarsfeld–Voisin presented examples of such cycles of codimension two on abelian fourfolds.

In this paper, we show that similar examples can also be found on certain holomorphic symplectic varieties (or ‘hyperkähler manifolds’). For these varieties, the cones of curves and divisors are already well-understood thanks to several recent advances in hyperkähler geometry [2–5,14]. Our main result is the following:

**Theorem 0.1.** Let $X$ be the variety of lines of a very general cubic fourfold. Then the cone of pseudoeffective 2-cycles on $X$ is strictly contained in the cone of nef 2-cycles.

On such a hyperkähler fourfold, the most interesting cohomology class is the second Chern class $c_2(X)$, which represents a positive rational multiple of the Beauville–Bogomolov form on $H^2(X,\mathbb{Z})$. This means that the class carries a certain amount of positivity; for example it intersects products of effective divisors non-negatively. It is therefore natural to ask when this class is represented by an algebraic subvariety of $X$. As we shall see in Section 4, this turns out not to be the case, at least for $X$ generic.

**Proposition 0.2.** Let $X = S^{[2]}$ be the Hilbert square of a very general K3 surface $S$. Then no multiple of $c_2(X)$ is numerically equivalent to an effective cycle.

The main idea of the proof is to deform $X$ to the Hilbert square of a special K3 surface containing many curves of self-intersection $-2$ and $0$. On the special fiber it is relatively easy to see that the second Chern class $c_2(X)$ is at least not in the interior of the effective cone, because it has intersection number $0$ with the fibers of a Lagrangian fibration. (On the other hand, after going to such a deformation $c_2(X)$ may no longer
be nef). Then a more careful argument using how $c_2(X)$ intersects the various the Lagrangian planes in $X$ shows that in fact no multiple is effective.

In the last section we consider fourfolds of generalized Kummer type. Using results of Hassett–Tschinkel, we find that $3c_2(X)$ is in fact effective on every such fourfold.

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1 Preliminaries

We work over the complex numbers. For a projective variety $X$, let $N_k(X)$ (resp. $N^k(X)$) denote the $\mathbb{R}$-vector space of dimension (resp. codimension) $k$-cycles on $X$ modulo numerical equivalence. In $N_k(X)$ we define the pseudoeffective cone $\text{Eff}_k(X)$ to be the closure of the cone spanned by classes of $k$-dimensional subvarieties. A class $\alpha \in N_k(X)$ is said to be big if it lies in the interior of $\text{Eff}_k(X)$. This is equivalent to having $\alpha \equiv \epsilon h^{\dim X - k} + e$ for $h$ the class of a very ample divisor; $\epsilon > 0$; and $e$ an effective 2-cycle with $\mathbb{R}$-coefficients.

A codimension $k$-cycle is said to be nef if it has non-negative pairing with any $k$-dimensional subvariety. We let $\text{Nef}_k(X) \subset N_k(X)$ denote cone spanned by nef cycles; this is the dual cone of $\text{Eff}_k(X)$. For most of the varieties in this paper, it is known that numerical and homological equivalence coincide, so we may consider these as cones in $H^{2k}(X, \mathbb{R})$ and $H^{2k}(X, \mathbb{R})$ respectively. If $Y \subset X$ is a subvariety, we let $[Y] \in H^*(X, \mathbb{R})$ denote its corresponding cohomology class.

For a variety $X$, we denote by $X^{(n)}$ denote the $n$-th symmetric product and $X^{[n]}$ the Hilbert scheme parameterizing length $n$ subschemes of $X$.

1.1 Holomorphic symplectic fourfolds

We will study effective 2-cycles on certain holomorphic symplectic varieties (or hyperkähler manifolds). By definition, such a variety is a smooth, simply connected algebraic variety admitting a non-degenerate holomorphic two-form $\omega$ spanning $H^{2,0}(X)$. In dimension 4 there are currently two known examples of such hyperkähler manifolds up to deformation: Hilbert schemes of points on a K3 surface and generalized Kummer varieties.

A hyperkähler manifold $X$ carries an integral, primitive quadratic form $q$ on the cohomology group $H^2(X, \mathbb{Z})$ called the Beauville–Bogomolov form. The signature of this form is $(3, b_2(X) - 3)$, and $(1, b_2(X) - 3)$ when restricted to the Picard lattice $\text{Pic}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. For the hyperkähler fourfolds considered in this paper it is known that the second Chern class $c_2(X) = c_2(T_X)$ represents a positive rational multiple of the Beauville–Bogomolov form [20]. It follows from this and standard properties of the Beauville–Bogomolov form that

$$c_2(X) \cdot D_1 \cdot D_2 \geq 0$$

for any two distinct prime divisors $D_1, D_2$ on $X$.

Recall that a subvariety $Y \subset X$ of dimension $\frac{1}{2} \dim X$ is said to be Lagrangian if the restriction of $\omega$ to the smooth part of $Y$ is trivial. In this case $\omega$ induces an
isomorphism between the normal bundle $N_{Y|X}$ and the cotangent bundle $\Omega_1^1$. Using the normal bundle sequence, we find that $c_2(X) \cdot [Y] = 2c_2(Y) - c_1(Y)^2$.

1.2 Specialization of effective cycles

In the proof of Theorem 1 and Proposition 2, we will need a certain semi-continuity result for effective cycles. This result is likely known to experts, but we include it here for the convenience of the reader and for future reference.

**Proposition 1.1.** Let $f : \mathcal{X} \to T$ be a smooth family of projective varieties over a smooth variety $T$ and suppose that $\alpha \in H^{k,k}(\mathcal{X}, \mathbb{Z})$ is a class such that $\alpha|_{\mathcal{X}_t}$ is effective on the very general fiber. Then $\alpha|_{\mathcal{X}_t}$ is effective for every $t \in T$.

**Proof.** This follows basically from the theory of relative Hilbert schemes. For a given point $t_0 \in T$, we will show that effective algebraic cycles on nearby fibers extend to $\mathcal{X}_{t_0}$.

Choose a differential trivialization of the family $\sigma : \mathcal{X}_U \simeq X_{t_0} \times U$ in a neighbourhood around $t_0 \in T$. This induces a specialization map of cohomology groups

$$H^{2k}(\mathcal{X}_U, \mathbb{R}) \cong H^{2k}(X_{t_0}, \mathbb{R}) \tag{2}$$

which is compatible with the restriction mapping. The main point is that there are at most countably many components $\rho_i : H_i \to T$ of the Hilbert scheme parameterizing subschemes supported in the fibers of $f$. For a component $H_i$, let $\pi_i : U_i \to H_i$ denote the universal family of $H_i$. This fits into the diagram

$$\begin{array}{ccc}
U_i & \longrightarrow & X \\
\downarrow & & \downarrow f \\
H_i & \longrightarrow & T
\end{array}$$

where $\pi_i$ is flat. Let $T' = \bigcup_i \rho_i(H_i)$, where the union is taken over all indices $i$ such that $\rho_i$ is not surjective. This is a countable union of proper closed subsets of $T$.

If $t_1 \in T - T'$ and $\Gamma \subset \mathcal{X}_{t_1}$ is any subvariety of codimension $k$, then there exists a component $\pi : U \to H$ of the Hilbert scheme such that $\Gamma$ is a fiber of $\pi$, and $\rho(H) = T$. Let $\phi : H' \to H$ be a desingularization of $H$ and define $U' = H' \times_H U$ and $\mathcal{X}' = H' \times_T \mathcal{X}$, with the induced (smooth) morphism $\pi' : \mathcal{X}' \to H'$. We have $\mathcal{U}' \hookrightarrow \mathcal{X}'$ and a corresponding element in the local system $[U'] \in H^0(H', R^{2n-2k}\pi'_*\mathbb{Z})$.

Pick two points $t'_0, t'_1$ in the same connected component of $\pi'^{-1}(U)$ mapping to $t_0, t_1$ respectively. By construction, $[\Gamma] \in H^{2k}(\mathcal{X}'_{t'_1}, \mathbb{Z})$ is the restriction of $[\mathcal{U}']$ to the fiber over $t'_1$. Similarly, the restriction of $\mathcal{U}'$ to $\mathcal{X}_{t'_0}$ is effective, and this is the image of $[\Gamma]$ via the specialization homomorphism (2). By linearity, it follows that if $\alpha|_{\mathcal{X}_{t_1}}$ represents an effective cycle on $\mathcal{X}_{t_1}$, then also $\alpha|_{\mathcal{X}_{t_0}}$ is effective on $\mathcal{X}_{t_0}$.

In other words, just like in the the case of divisors, the effective cones can only become larger after specialization. Note that the proposition also implies that a class $\alpha$ which is big on a very general fiber is also big on every fiber.

**Remark 1.2.** Essentially the same proof works in the Kähler setting by replacing the Hilbert scheme with the relative Douady space, by results of Fujiki.
2 The variety of lines of a cubic fourfold

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. The variety of lines $X = F(Y)$ on $Y$ is a smooth 4-dimensional subvariety of the Grassmannian $Gr(2, 6)$. A fundamental result due to Beauville and Donagi [6] says that $X$ is a holomorphic symplectic variety. Moreover, $X$ is deformation-equivalent to the Hilbert square $S^2$ of a K3 surface.

From the Grassmannian embedding we obtain natural cycle classes on $X$. In particular, if $U$ denotes the tautological rank 2 bundle on $Gr(2, 6)$, we consider the Chern classes $g = c_1(U^\vee)|_X$ and $c = c_2(U^\vee)|_X$. Note that $g$ defines a polarization on $X$, corresponding to the Plücker embedding of $X$ in $\mathbb{P}^{14}$. When $Y$ is very general, the vector space of codimension 2 Hodge classes $H^{2,2}(X) \cap H^4(X, \mathbb{Q})$ is two-dimensional, generated by $g^2$ and $c$. We have the following intersection numbers:

$$g^4 = 108, \quad g^2 \cdot c = 45, \quad c^2 = 27. \quad (3)$$

The cubic polynomial defining $Y$ shows that $X$ is the zero-set of a section of the vector bundle $S^3 U^\vee$ on $Gr(2, 6)$. Using this description, a standard Chern class computation shows that

$$c_2(X) = 5g^2 - 8c. \quad \text{See for example [1] or [20] for detailed proofs of these statements.}$$

2.1 Surfaces in the variety of lines

There are several interesting surfaces on the fourfold $X$. For example, we may consider restrictions of codimension two Schubert cycles from the ambient Grassmannian $Gr(2, 6)$. In terms of $g^2$ and $c$, these cycles are given by $g^2 - c$ and $c$. Moreover, since on $Gr(2, 6)$ every effective cycle is nef (this is true on any homogeneous variety), also their classes remain nef and effective when restricted to $X$.

There are two natural surfaces on $X$ with class proportional to $g^2 - c$. For example, fixing a general line $l \subset Y$, the surface parameterizing lines meeting $l$ represents the class $\frac{1}{2}(g^2 - c)$. Also, the variety of lines of ‘second type’ (that is, lines with normal bundle $O(1)^2 \oplus O(-1)$ in $Y$) is a surface with corresponding class $5(g^2 - c)$ (see [6]).

The class $c$ is also represented by an irreducible surface. Indeed, for a general hyperplane $H = \mathbb{P}^4 \subset \mathbb{P}^5$ the Fano surface

$$\Sigma_H = \{[l] \in X \mid l \subset H\}$$

is a smooth surface with $[\Sigma_H] = c$, corresponding to the lines in the cubic threefold.

We also have the following less obvious example: Suppose that $Y$ is the hyperplane section of some cubic fivefold $V \subset \mathbb{P}^6$. The variety of planes in $V$ is a smooth surface $F_2(V)$. If $Y$ is general, there is an embedding $F_2(V) \to F(Y)$ given by associating a plane to its intersection with the hyperplane section. The class of the image has class $63c$ (see [11]). In both of these cases the surfaces representing multiples of $c$ are Lagrangian subvarieties of $X$, and are of general type.

By the following result of Voisin [23], the class $c = c_2(U^\vee)$ is the boundary in the cone of effective 2-cycles on $X$:
Lemma 2.1 (Voisin). Let $X$ be the variety of lines on a very general cubic fourfold and let $c = c_2(U^\vee)$. Then $c$ is extremal in the effective cone of 2-cycles.

This result is quite surprising because the surface $\Sigma_H \subset X$ behaves in many ways like a complete intersection: varying the hyperplane $H$ one sees that it deforms in a large family covering $X$ and its normal bundle is an ample vector bundle. For this reason, this surface was used as a counterexample to a question of Peternell in [17]. Voisin’s proof is also very interesting: it uses the fact that $\Sigma_H$ is a Lagrangian submanifold of $X$; the fact that $\Sigma_H$ is not big is essentially a consequence of the Hodge–Riemann relations (see also Proposition 2.6).

Even though the class $c$ is the restriction of an extremal Schubert cycle on $Gr(2,6)$, there is a priori no reason to expect that it should remain extremal when restricted to $X$. Too see how subtle this is, we note that Voisin [23] showed that in the case $Y$ is the variety of lines of a cubic fivefold, the corresponding class $c$ of lines in a hyperplane section of $Y$ is in fact big on $X = F(Y)$.

Even though the subvariety $\Sigma_H$ is very positively embedded in $X$, it is not an ‘ample subscheme’ in the sense of [15]. This is because ample subschemes satisfy a Lefschetz hyperplane theorem on rational homology; in our case $H_1(X, \mathbb{Q}) = 0$ whereas both $\Sigma_H$ and $F_2(V)$ have non-zero $H_1$. As ample subschemes also have ample normal bundles, it is interesting to ask whether Peternell’s question has a positive answer when restricted to such subschemes, e.g., whether smooth ample subvarieties have big cycle classes. As usual, the answer is affirmative for curves and divisors [16].

Voisin’s result gives one face of $\text{Eff}^2(X)$. To bound the other half of the effective cone of $X$, we will consider the class $c_2(X) = 5g^2 - 8c$. Note that this class is already quite positive, since it intersects products of divisors $D_1D_2$ non-negatively (cf. equation (1)). In fact, this class will be shown to lie in the interior of the cone of nef 2-cycles, and is strictly nef, in the sense that $c_2(X) \cdot Z > 0$ for every irreducible surface $Z \subset X$. We will however show that it is not big. Using a specialization argument, we will prove something slightly more general:

Proposition 2.2. Let $X$ be a hyperkähler manifold of dimension $2r$, admitting a Lagrangian fibration. Then $c_k(X)$ is not big, for $1 \leq k \leq r$.

Proof. Let $f : X \to B$ denote this fibration. By a theorem of Matsushita, $\dim B = r$ and the general fiber $A$ of is an $r$-dimensional abelian variety. In particular, $\Omega^1_A \simeq \mathcal{O}_A^\vee$. Let $Y = A \cap H_1 \cap \cdots \cap H_{n-k}$, where $H_i \in |h|$, and $h$ is a very ample line bundle on $X$.

The restriction of the normal bundle sequence

$$0 \to T_A|_Y = \mathcal{O}_Y^\oplus \to T_X|_Y \to N_A|_Y = \mathcal{O}_Y^n \to 0$$

to $Y$ shows that $c_k(X) \cdot A \cdot h^{n-k} = 0$. Now if $c_k$ is a big cycle, then it is numerically equivalent to $\epsilon h^k + e$, for some $\epsilon > 0$ and $e$ an effective cycle with $\mathbb{R}$–coefficients. However, the cycle $A \cdot h^{n-k}$ is nef, and so it has strictly positive intersection number with $h^k$. Hence $c_k(X)$ cannot be big. \qed

Lemma 2.3. Let $X = F(Y)$ for a very general cubic fourfold $Y$. Then the second Chern class $c_2(X) = 5g^2 - 8c$ is not big.
Proof. By Proposition 1.1, it suffices to prove the statement for a special cubic fourfold \( Y \) so that \( F(Y) \) admits an abelian surface fibration. To exhibit such a \( Y \) we specialize to the case where \( Y \) is a Pfaffian cubic fourfold, in which case \( F(Y) = S^{[2]} \) for a degree 14 K3 surface \( S \) (provided that \( Y \) does not contain any planes [6]). After specializing \( Y \) even further, so that \( S \) contains an elliptic fibration, we obtain an abelian surface fibration \( S^{[2]} \to (\mathbb{P}^1)^{[2]} = \mathbb{P}^2 \). Then the theorem follows from Proposition 2.2.

Combining this with Voisin’s result we obtain a bound for the effective cone \( \text{Eff}_2(X) \):

**Corollary 2.4.** Let \( X \) be the variety of lines on a very general cubic fourfold. Then

\[
\mathbb{R}_{\geq 0}c + \mathbb{R}_{\geq 0}(g^2 - c) \subseteq \text{Eff}_2(X) \subseteq \mathbb{R}_{\geq 0}c + \mathbb{R}_{\geq 0}(g^2 - \frac{8}{5}c)
\]

**Proof of Theorem 1.** By the previous corollary, we have that \( \text{Eff}_2(X) \subseteq \mathbb{R}_{\geq 0}c + \mathbb{R}_{\geq 0}c_2(X) \). Using the intersection numbers (3), we find by duality:

\[
\text{Nef}^2(X) \supseteq \left( \mathbb{R}_{\geq 0}c + \mathbb{R}_{\geq 0}(5g^2 - 8c) \right)^\vee
= \mathbb{R}_{\geq 0}(20c - g^2) + \mathbb{R}_{\geq 0}(3g^2 - 5c)
\supseteq \text{Eff}_2(X)
\]

The above theorem implies that the class \( c_2(X) \) is in the interior of the nef cone, but not big. We will show in Section 3 that in fact \( c_2(X) \) has no effective multiple.

Note that although \( c_2(X) \) is nef for very general cubics, there might be other cubics for which it is not. Indeed, taking \( Y \) to contain a plane, we obtain a surface \( P = \mathbb{P}^2 \subset X \), on which \( c_2(X) \cdot P = -3 \).

**Remark 2.5.** In his thesis [18], M. Rempel also considered examples of pseudoeffective cycles on hyperkähler fourfolds, in particular on the variety of lines of a cubic fourfold. Among his many results, he shows that the class \( 3g^2 - 5c \) (which is proportional to \( c_2(X) - \frac{1}{5}g^2 \)) has no effective multiple. The proof involves a nice geometric argument using Voisin’s rational self-map \( \varphi : X \to X \).

### 2.2 Lagrangian submanifolds

Using the same method as in Voisin’s proof of [23, Proposition 2.4], one obtains the following

**Proposition 2.6.** Let \( X \) be an holomorphic symplectic variety of dimension four and let \( Y \subset X \) be a Lagrangian surface. Then \([Y]\) is in the boundary of \( \text{Eff}_2(X) \).

The classes of Lagrangian subvarieties of \( X \) thus span a face of the effective cone \( \text{Eff}_2(X) \) (which may have codimension \( \geq 2 \)). This face can be described as the set of classes orthogonal to \( \omega \wedge \varpi \). The main point is that if \( Y \) is not Lagrangian, then \( \omega \wedge \varpi \) restricts to a volume form on \( Y \) and so \([Y] \cdot \omega \wedge \varpi > 0 \).

This proposition can be used to prove extremality of other cycle classes:

(i) If \( S \) is a K3 surface and \( C \subset S \) is a smooth curve, then \( C^{[2]} \subset S^{[2]} \) is Lagrangian.

\[\]
(ii) Any surface $S$ with $H^{2,0}(S) = 0$ (e.g., a rational surface) in a hyperkähler fourfold $X$ is Lagrangian, hence extremal. For a concrete example, let $Y \subset \mathbb{P}^5$ be a cubic fourfold containing a plane $P$, then the dual plane $P^\vee \subset X = F(Y)$ parameterizing lines in $P$ is extremal. (See also [18, §3.2]). Of course here, extremality of $P^\vee$ also follows since it can be contracted by a birational map.

3 The second Chern class of the Hilbert square of a K3

In this section we will prove Proposition 2, that there is no effective cycle representing a multiple of $c_2(S^{[2]})$ for a very general K3 surface $S$. We will prove this by a specialization argument by deforming to the Hilbert scheme of points of a K3 surface containing many $(-2)$-curves. Here the presence of infinitely many Lagrangian planes is essentially what forces $c_2(X)$ to be non-effective. This argument was inspired by Piatetski-Shapiro and Shafarevich’s proof of the Torelli theorem for K3 surfaces [19], where the $(-2)$-curves are used for a similar purpose.

Let $S$ be a K3 surface and let $X = S^{[2]}$ be the Hilbert scheme of length 2 subschemes on $S$. Let $\pi : X \to S^{(2)}$ be the Hilbert–Chow morphism, that is, the blow-up of the diagonal in the symmetric product. This morphism induces decompositions $H^2(X, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z} \delta$ and $\text{Pic}(S^{[2]}) = \text{Pic}(S) \oplus \mathbb{Z} \delta$ which are orthogonal with respect to the Beauville–Bogomolov form. Here $2\delta$ is the divisor corresponding to non-reduced subschemes. Moreover, for two divisor classes $\alpha, \beta \in \text{Pic}(X)$, we have

$$c_2(X) \cdot \alpha \cdot \beta = 30q(\alpha, \beta).$$

In the proof we will make frequent use of the fact that $c_2(X) \cdot P = -c_1(\mathbb{P}^2)^2 + 2c_2(\mathbb{P}^2) = -3$ for any Lagrangian plane $P \subset X$.

Proposition 3.1. Let $S$ be a generic K3 surface as above and let $X = S^{[2]}$. Then no multiple of $c_2(X)$ is represented by an effective cycle.

Proof. As before, it suffices to prove the corresponding statement on a special K3 surface. We will choose $S$ to contain infinitely many $(-2)$-curves subject to the genericity condition that they don’t all pass through a finite set of points. Such K3 surfaces aren’t too hard to construct explicitly (e.g. using elliptic fibrations). We may also assume that $S$ is embedded in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Fix such a K3 surface and suppose to the contrary that some multiple of $c_2(X)$ is represented by an effective cycle $Z$ on $X$. Let $i : E \to X$ be the inclusion of the exceptional divisor. We may write

$$Z = i_* D + \Gamma$$

for $\Gamma$ an effective 2-cycle with no components contained in $E$, and $D$ an effective divisor in $E$. By construction $i^* \Gamma$ is an effective 1-cycle on $E$.

First we claim that $D \neq 0$. Note that $i^* c_2(X) = p^* c_2(S)$, so if $Z$ has no components in $E$, then $i^* Z$ is a cycle which is supported on a finite sum of fibers of $p$. However, for every $(-2)$-curve $C \subset S$ we obtain a Lagrangian plane $P = C^{[2]}$ with $c_2(X) \cdot P = -3$, and so $Z \cap P$ has a 1-dimensional component for every $P$. On $P \cong \mathbb{P}^2$ any two effective divisors intersect, so this component meets $E$. It follows that $i^* Z \cap i^* P \neq \emptyset$ and
We claim that if i was the case, then one can find a deformation of $H^0$. By a result of Kobayashi [12], $S$ has no global symmetric differentials, so $H^0(E,\mathcal{O}_E(m)) = H^0(S, S^m\Omega^2_S) = 0$ for all $m \geq 1$. It follows that the line bundle $M \in \text{Pic}(S)$ is non-trivial.

Consider an elliptic fibration $f : S \to \mathbb{P}^1$ with fiber $e$. This induces an abelian surface fibration $F : S^{[2]} \to \mathbb{P}^2$. Note that the exceptional divisor $E \subset S^{[2]}$ is contracted onto the diagonal in $\mathbb{P}^2$. The map $F|_E : E \to \mathbb{P}^1$ coincides with the map $(f \circ p) : E \to S \to \mathbb{P}^1$. We claim that $i^*\Gamma$ is not contracted by $F$. It suffices to show that $i^*\Gamma \cdot p^*H \neq 0$ where $H$ is ample on $S$. This follows from the following computation:

$$i^*\Gamma \cdot p^*H = (Z-i_D) \cdot E \cdot \pi^*H = D \cdot (\mathcal{O}(1) \cdot p^*H = (\mathcal{O}(a) + p^*M) \cdot \mathcal{O}(1) \cdot p^*H = M \cdot H \neq 0$$

(We may choose $H$ so that this intersection number is non-zero, since $M$ is not numerically trivial). However, since $c_2 \cdot \pi^*e \cdot \pi^*e = 0$ all components of $Z$ and $\Gamma$ are contracted to lower-dimensional subvarieties of $\mathbb{P}^2$. Consider an irreducible component $\Gamma_0$ of $\Gamma$ so that $i^*\Gamma_0$ is not contracted by $F|_E$. We have $F(\Gamma_0) \supseteq (\Gamma_0 \cap E) = \Delta_{p^1}$, and by irreducibility $F(\Gamma_0) = \Delta_{p^1}$. This holds for every elliptic fibration $f : S \to \mathbb{P}^1$. However, if we take three elliptic fibrations $f_1, f_2, f_3$ embedding $S$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we see that

$$\Gamma_0 \subseteq F_1^{-1}(\Delta_{p^1}) \cap F_2^{-1}(\Delta_{p^1}) \cap F_3^{-1}(\Delta_{p^1})$$

However, no reduced length two scheme $\{x, y\} \subset S$ lies on the same fibers of all three elliptic fibrations. The map $S \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is an embedding. It follows that the intersection on the right is supported entirely in $E$, contradicting the assumption that $\Gamma$ has no component in $E$.

Note that it could still be the case that $c_2(S^{[2]})$ is pseudoeffective, e.g., a limit of effective cycles, even for a general $S$. We do not know if the above argument can be used to show that this is not the case.

Remark 3.2. It is well-known that the naïve extension of the Hodge conjecture to non-projective Kähler manifolds fails in general. Such examples were first constructed by Zucker [25] on complex tori (see also Voisin [22]). Here we remark that one can obtain similar examples on hyperkähler manifolds of $K3^{[2]}$-type.

Corollary 3.3. Let $X$ be a very general hyperkähler manifold of $K3^{[2]}$-type. Then $X$ has no analytic 2-dimensional subvarieties.

To see this, recall that for a hyperkähler manifold of $K3^{[2]}$-type we have an isomorphism

$$S^2(H^2(X, \mathbb{Q})) = H^4(X, \mathbb{Q}).$$

Hence for a general $X$, when $\text{Pic}(X) = 0$, the vector space of degree 4 Hodge classes $H^{2,2}(X, \mathbb{C}) \cap H^4(X, \mathbb{Q})$ is generated by a single element, namely $c_2(X)$. However, neither a multiple of $c_2$ or $-c_2$ represents an analytic subvariety on $X$. Indeed, if this was the case, then one can find a deformation of $X$ to $S^{[2]}$ for a projective Kummer K3 surface, and the class would stay effective (cf. Remark 5), contradicting Proposition 2.

The above result can also be deduced from the paper [21]. In fact, Verbitsky shows that the generic $X$ does not have any analytic subvarieties of positive dimension.
4 Generalized Kummer varieties

Let $X$ denote a generalized Kummer variety of dimension 4 of an abelian surface $A$. Recall that these are defined as the fiber over 0 of the addition map $A^3 \to A$. In this case, the group $H^2(X, \mathbb{Z})$ can be identified with $H^2(A, \mathbb{Z}) \oplus \mathbb{Z}e$, where the class $e$ is one half of the divisor corresponding to non-reduced subschemes in $A^3$. As before, $c_2(X)$ represents a positive multiple of the Beauville–Bogomolov form and the decomposition of $H^2(X, \mathbb{Z})$ is orthogonal with respect to this form. These varieties are however not deformation equivalent to Hilbert schemes of points on K3 surfaces (e.g., as their Betti numbers are different). By [10, §4], we have a decomposition $H^4(X) = S^2H^2(X) \oplus 1_{80}^X$ where $1_n$ denotes a $\mathbb{Z}^n$ with the intersection form represented by the identity matrix.

There are 81 distinguished rational surfaces on $X$, whose classes are linearly independent in $H^4(X, \mathbb{Q})$: For each $\tau \in A$ let $W_\tau$ denote the locus in $A^3$ of subschemes supported at $\tau$. As shown in [10], these are all isomorphic to a weighted projective space $\mathbb{P}(1,1,3)$. Moreover, when $\tau \in A[3]$ is a 3-torsion point on $A$, $W_\tau$ is a subvariety of $X$. Now, for $p \in A$ there is another special surface in $X$ given by the closure of the locus of subschemes $(a_1, a_2, p) \in A^3$ where $a_1 + a_2 + p = 0 \in A$. This is isomorphic to the Kummer surface of $A$ blown-up at a point. Now, Hassett–Tschinkel showed that as $p \to \tau \in A[3]$, the flat limit of $Y_p$ breaks into two pieces, $W_\tau$ and a surface $Z_\tau$. For example, $Z_0$ is the closure of the locus of points $(0, a, -a)$ with $a \in A - 0$ (and the other $Z_\tau$ are translates of this via the group $A[3]$). Moreover, [10, Proposition 5.1] says that

$$c_2(X) = \frac{1}{3} \sum_{\tau \in A[3]} [Z_\tau].$$

In particular, $3c_2(X)$ represents an effective cycle. As before, this class cannot be big, as it has intersection 0 with a Lagrangian fibration on a deformation of $X$. Using Proposition 2.6, we get the following

**Proposition 4.1.** Let $X$ be a very general projective generalized Kummer fourfold. The surfaces $Z_\tau, W_\tau$ are all extremal, and span two 81-dimensional faces of the effective cone $\text{Eff}H^2(X)$. The second Chern class $c_2(X)$ is effective, but is not big.

Interestingly, the surfaces $Z_\tau$ survive even when passing to a non-projective deformation of $X$; they are trianalytic subvarieties in the sense of [21]. In particular, $c_2$ is effective even on any smooth hyperkähler manifold of Kummer type.

5 An alternative notion of nefness

In the Nakai–Moishezon criterion, a line bundle $L$ is ample if and only if for every $r = 1, \ldots, \dim X$ we have $L^r \cdot Z > 0$ for every subvariety of dimension $r$. This suggests the following naive fix for the definition of nefness for any cycle class: $\sigma \in N^k(X)$ could be defined to be nef if the restriction cycle $i^*\sigma \in N^k(Y)$ is pseudoeffective for every subvariety $i : Y \to X$. In particular, taking $i$ to be the identity map, $\sigma$ would itself be pseudoeffective on $X$. This is related to Fulger and Lehmann’s notion of universal
pseudoeffectivity where the \( f^* \sigma \) is required to be pseudoeffective for any morphism \( f : Y \to X \). We do not know whether these notions are equivalent, or whether this gives a useful definition in general.

References


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