Recent Development of Multi-layer Models for Fully Nonlinear Internal Wave Propagation

Philip L.-F. Liu and Xiaoming Wang
School of Civil and Environmental Engineering
Cornell University

Large amplitude internal waves have been frequently observed in the ocean environment, especially in straits, on continental shelves and coastal zones through either field survey or remote sensing. One of the main mechanisms of internal wave generation is the interaction between tides and the submarine topography, such as the shelf-edges, ridges or sills. Soliton-like internal waves often emerge. Once generated, they are generally capable of propagating for a long distance before encountering any further significant bathymetric variation. In the South China Sea, Synthetic Aperture Radar (SAR) images from satellites clearly show surface signatures of huge internal packets (with wavelengths from 5 km to 500m), which are believed to be generated by diurnal tides over the shallow ridge (about 200 - 300m deep) in the Luzon Strait and propagate with a speed up to 2.0 m/s over a distance more than 400km in the South China Sea, and finally reach the continental shelf of Southern China.

Internal waves, due to their large amplitude vertical motion, are thought to have contributions to nutrient mixing, especially when they break in coastal regions. Internal waves in the coastal ocean will also affect the offshore structures, mooring systems and the operation of submarine vehicles.

Numerous experimental and theoretical efforts have been made to study the characteristics of internal waves including their generation, evolution and interactions with other waves and bathymetric features. Korteweg-de-Vries-type (KdV) equations have played a primary role in understanding the essential features of the observations. However, KdV equations’ application is limited to uni-directional weakly nonlinear waves. An approach that does include weakly horizontal 2-dimensional (H2D) effects is the Kadomtsev and Petviashvili (K-P) type equations and has been applied to situations such as wave propagation through a narrow channel. K-P equations require that the spatial variation of the wave form in the transverse direction is small compared with that along the direction of wave propagation.

Satellite observations in the South China Sea clearly indicate strong two-dimensional wave-wave interactions and the reflection and refraction of internal waves by bathymetry and islands. Using the ray theory, Small (2001a, b) extended the application of KdV equations into a two-dimensional ocean basin, studying the shoaling and refraction of internal solitary waves. To account for the H2D effects, Choi and Camassa (1996) derived a set of weakly nonlinear equations for H2D internal waves in a 2-layer fluid system with constant density in each layer. However, only one-dimensional stationary solutions were demonstrated. Using a similar depth-integrating approach, a weakly nonlinear H2D model in a 2-layer system was also derived by Lynett and Liu (2002) for internal waves over a slowly varying bathymetry. The model has been successfully implemented to study the evolution, refraction and reflection of internal waves in the South China Sea and the Strait of Gibraltar. The results, qualitatively, produce a fairly good match with the SAR observations.

Due to the generally small density stratification in the ocean, internal waves may attain relatively large amplitudes, which have been widely observed in the ocean. For example, in South China Sea, wave amplitude up to 165m was observed in 340m water depth with a background upper layer of about 40 m. Thus, weakly nonlinear theories are not adequate in describing the evolution of large amplitude internal waves. Recently, more advanced models have been
developed to include the strongly nonlinear effects often observed in the ocean for a 2-layer fluid system.

In a 2-layer system, the pycnocline is ignored and is replaced by a density discontinuity at the interface. This configuration cannot accurately describe the smooth transition of density stratification in pycnocline, especially when the thickness of pycnocline is not negligible. In addition, the 2-layer setup also suppresses the occurrence of higher-order internal wave modes. In laboratory experiments and field measurements, mode-2 internal waves have been frequently observed. The higher-order internal wave modes travel at much slower speed than the first-mode wave and is usually generated by the interaction of the first-mode internal waves with bottom topography, breaking of the first-mode or intrusion of fluid into mixing layer.

Euler equations, or full Navier-Stokes equations when viscous effect must be included, are most suitable to study large amplitude internal waves, especially, in a continuous density stratification. However, these equations are difficult to solve and can be very costly to simulate even in relatively simply situations. These approaches are very time-consuming when the horizontal two-dimensional effects are considered and extremely difficult to apply to large-scale simulations. Consequently, those studies are mostly focusing on horizontally one-dimensional problems.

Thus, there is a need for developing models that are sufficiently accurate to capture the dynamics occurring in physically realistic situations, yet simple enough to be applied efficiently. Practically, the fluid continuous stratification in water depth can be approximated by a multiple layered fluid system with constant density in each layer. Within each layer, if the wave is long compared to the layer thickness, the depth-integration approach can be employed to include the vertical dependence and to reduce the dimensionality in the resulting equations. The depth-integrated 2-layer model is the simplest setup for this multiple layered system. Using this concept, Choi (2000) derived a set of depth-integrated equations for one-dimensional large amplitude internal waves. However, only the stationary solution in a 2-layer system was demonstrated.

In this paper, by assuming that the ratio of the layer thickness to the wave length is small \( \mu^2 = 1 \) and using a similar depth-integrated approach by Lynett and Liu (2002), a H2D multi-layer internal wave model is developed without making any assumption on wave amplitude. Thus, the new model is capable of studying large-amplitude internal waves in the ocean. Since the continuously stratified fluid is approximated by a stack of \( N \) homogeneous layers, theoretically, the model can be used to simulate higher-order modes up to \( N-1 \).

In developing the model equations, the water column is divided into \( N \) layers; within each layer, the density is a constant. The fluid is also assumed to be inviscid and incompressible. The coordinate system is chosen so that the plane formed by \( x \)-axis and \( y \)-axis coincides with the free surface at rest, with \( z \)-axis pointing upwards. The \( n \)-th layer is bounded by \( \eta_n \) and \( \eta_{n+1} \) and, therefore, \( \eta_{N+1} \) describes the bottom floor boundary. The physical variables are scaled by the characteristic length of the wave motion \( l_0 \) as the horizontal length scale, the characteristic water depth \( h_0 \) as the vertical length scale and \( U_0 \) as the characteristic velocity. Assuming that \( \mu^2 = (h_0 / l_0)^2 = 1 \), the governing equations for the interfacial surface displacement \( \eta_n \), the layer-averaged velocity \( \mathbf{u}_n \), can be written in the following dimensionless depth-integrated forms.
\[
\frac{\partial \eta_n}{\partial t} - \frac{\partial \eta_{n+1}}{\partial t} + \nabla \cdot [(\eta_n - \eta_{n+1}) \overline{u}_n] = 0,
\]

(1)

\[
\frac{\partial \overline{u}_n}{\partial t} + \overline{u}_n \cdot \nabla \eta_n + \nabla p_n(z = \eta_n) + \frac{1}{\rho_n} \nabla p_n(z = \eta_n) + \mu^2 \frac{\partial}{\partial t} \{A_n \nabla S_n + B_n \nabla T_n\}
\]

\[
+ \mu^2 \nabla \cdot \left[\overline{u}_n \cdot (A_n \nabla S_n + B_n \nabla T_n) + \frac{1}{2} T_n^2 - \frac{\eta_n^2}{2} P_n - \eta_n R_n\right] = O(\mu^4).
\]

(2)

where \( n = 1, N \) and

\[
p_n(z = \eta_n) = p_{n-1}(z = \eta_{n-1}) + \rho_{n-1}(\eta_{n-1} - \eta_n)
\]

\[
+ \mu^2 \rho_{n-1}\left[\frac{\eta_n^2 - \eta_{n-1}^2}{2} P_{n-1} + (\eta_n - \eta_{n-1})R_{n-1}\right] + O(\mu^4),
\]

(3)

\[
A_n = \frac{1}{6}(\eta_n^2 + \eta_n \eta_{n+1} + \eta_{n+1}^2), \quad B_n = \frac{1}{6}(\eta_n + \eta_{n+1}), \quad S_n = \nabla \cdot \overline{u}_n, \quad T_n = -\nabla \cdot \left[\overline{u}_n \eta_{n+1}\right] - \frac{\partial \eta_{n+1}}{\partial t},
\]

(4)

\[
P_n = \frac{\partial S_n}{\partial t} + \overline{u}_n \cdot \nabla S_n - S_n^2, \quad R_n = \frac{\partial T_n}{\partial t} + \overline{u}_n \cdot \nabla T_n - S_n T_n.
\]

(5)

Thus, for this \( N \)-layer fluid system, there are \( 3N \) unknowns, \( \eta_n, \overline{u}_n, \) and \( \overline{v}_n \) \((n = 1, \ldots, N)\), and \( 3N \) equations. It should be noted that on the top surface, \( z = \eta_1, \) \( p_1 = p_a \). If the top surface is a free surface, \( p_a \) denotes the atmospheric pressure, which is usually taken as a constant \((p_a = 0)\). If the top surface is a rigid lid \((\eta_1 = 0)\), \( p_a \) is unknown pressure at the top boundary. Since no assumption on wave amplitude is made, this set of the depth-averaged governing equations describes fully nonlinear, weakly dispersive wave evolution in a fluid with \( N \) density layers.

A high-order numerical algorithm solving the model equations have been implemented. Here, numerical results for a strongly nonlinear internal solitary wave are compared with the experimental data by Grue et al. (1999). In their experiments a 2-layer fluid system, with fresh water on top of brine water was carefully created. Salt water with density of \( \rho_2 = 1,022 \, \text{kg} / \text{m}^3 \) was used for the lower layer with thickness of \( h_2 = 0.62 \, \text{m} \). Fresh water of density \( \rho_1 = 999.0 \, \text{kg} / \text{m}^3 \) was gently filled on top of the salt water layer; the top layer thickness was \( h_1 = 0.15 \, \text{m} \). Solitary waves with different amplitudes were generated by removing a gate with an initial volume of fresh water behind the gate.

In the following figures, numerical results for the interfacial profile (left panel) for \( a_0 / h_1 = 0.62 \) and the horizontal velocity (right panel) for \( a_0 / h_1 = 0.78 \) are compared with the experimental data. In the plot for the interfacial surface, solutions based on the KdV equation is also shown. It is quite clear that the present numerical solutions agree with experimental data very well. More examples will be discussed in the workshop.
References


