Internal solitons and breathers in three-layer flow

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The intensive internal waves in the ocean are studied actively in the framework of various mathematical models. Main attention is paid to the solitary waves (solitons), which are observed very often on the oceanic shelves in all oceans. The dynamics of the weakly and moderate nonlinear internal waves can be described by the extended Korteweg – de Vries equation called the Gardner equation:

\[
\frac{\partial \eta}{\partial t} + (\alpha \eta + \alpha_1 \eta^3) \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0
\]  

(1)

where \( \eta \) is the isopycnic displacement on the mode maximum depth, and the coefficients of quadratic \( \alpha \) and cubic \( \alpha_1 \) nonlinearities and dispersion \( \beta \) are dependent on the stratification features and water depth only [1]. For the practical case of the two-layer flow many rigorous results concerning the properties of the solitary waves are obtained. Here the nonlinear wave dynamics is analyzed for three-layer approximation of the density stratification. We consider a three-layer flow with upper and lower layers of equal thicknesses \( h \) and the same density jump \( \Delta \rho \) across each interface. The total depth is denoted by \( H \), and rigid boundaries at the top and bottom are used. For this symmetric stratification the quadratic nonlinear coefficient is zero (\( \alpha = 0 \)) and the Gardner equation (1) reduces to the modified Korteweg–de Vries (mKdV) equation with coefficients [2]

\[
c = \sqrt{gh}, \quad \beta = \frac{ch}{4} \left( H - \frac{4h}{3} \right), \quad \alpha_1 = -\frac{3c}{4h^2} \left( 13 - \frac{9H}{2h} \right)
\]  

(2)

where \( g = g \Delta \rho / \rho \). In the case of closed location of the both interfaces the cubic nonlinear term is negative as for two-layer flow. Here we pay attention to the case when the cubic nonlinear coefficient \( \alpha_1 \) is positive. It is realized only if \( h/H < 9/26 \) (Fig. 1). When \( \alpha_1 \) is positive two kinds of stable-state solutions exist for this equation: solitary waves (solitons)

\[
\eta(x,t) = a \sech \left[ \sqrt{\frac{\alpha_1}{6\beta}} \left( x - \frac{\alpha_1 a^2 t}{6} \right) \right],
\]  

(3)

where \( a \) is wave amplitude, it can have either sign, and breather (oscillatory wave packets):

\[
\eta(x,t) = -\frac{4qH}{\cosh \theta} \left[ \frac{\cos \phi - (q/p) \sin \phi \cdot \tanh \theta}{1 + (q/p)^2 \sin^2 \phi \cdot \text{sech}^2 \theta} \right]
\]  

(4)
where parameters $\varphi$ and $\theta$ are

$$\theta = 2q \frac{x}{L} + 8q(3p^2 - q^2) \frac{f}{T} + \theta_0, \quad \varphi = 2p \frac{x}{L} + 8p(p^2 - 3q^2) \frac{f}{T} + \varphi_0$$

(5)

and $L = \frac{1}{H} \sqrt{\frac{6\beta}{\alpha_1}} T = \frac{6}{\alpha_1 H} \sqrt{\frac{6\beta}{\alpha_1}}$. Parameter $q$ is responsible for the amplitude of the mKdV breather and “envelope phase” $\theta$ and parameter $p$ is responsible for the “carrier phase” $\varphi$ mainly.

Fig.2a,b presents the characteristic breather shapes for the same $q$ and different $p$.

Fig.2. Breather shapes
It is interesting to note on Fig. 2 a, b that the same parameter $q$ leads to the same envelope shape but with different number of carrier waves due to different $p$ and it seems the number of “carrier waves” into the envelope depends on the ratio $q/p$ (Fig. 2 a, c, blue breathers). Both of parameters $p$ and $q$ describe the “phase” and “group” speeds of breather

$$V_{\text{phase}} = 4(3q^2 - p^2), \quad V_{\text{gr}} = 4(q^2 - 3p^2)$$

As it can be seen from (6) both speed can be zero (in reference system) for various values of $q/p$. So, to compare with the soliton, the breather may move faster as well as slower the linear wave.

The Gardner equation (as well as for mKdV) is completely integrated and various rigorous results are obtained allowing to obtain the solution of the Cauchy problem [3]. One of the interesting results here is the modulational instability of the wave packets resulting to the generation of the large-amplitude short-lived pulses (internal rogue waves). Fig. 3 demonstrates characteristic wave profiles developed from weakly modulated sine train [4].

![Fig.3. Large – amplitude pulse developed from sin-modulated IW group](image)

Nonlinear wave processes presented above are investigated in the framework of asymptotic reductions of the governing Euler equations for stratified fluid. To study strongly nonlinear waves within basic hydrodynamic equations is very difficult task and here numerical approaches are effective. The existence of solitary waves in the framework of the full Euler equations is proved now and very often modeled, but nothing is known about the breather solutions for the Euler equations. Using the numerical model of the Euler equations for three-layer stratification we confirmed the “long-time” existence of breather in frameworks of full nonlinearity and dispersion [5]. Fig. 4 shows the density contours displacement due to breather for various times. Fig. 5 is the Hovmöller plot of horizontal velocity perturbation at the sea surface. Given results illustrate the stability of the breather propagation.

In the coastal oceans, the interaction of currents (such as the barotropic tide) with topography can generate large-amplitude, horizontally propagating internal waves. These waves often occur
in regions where the waveguide properties vary in the direction of propagation. We consider the modeling of these waves by nonlinear evolution equations of the Korteweg–de Vries type (Gardner equation) with variable coefficients, adding the Earth rotation and bottom friction. We use these models to describe the shoaling of intensive internal waves over such continental shelves as the Malin Shelf, North-West Shelf of Australia and some Russian Arctic shelves [6-8]. The theoretical results are compared with various numerical simulations and observations.

Fig. 4. Density contours showing the evolving wave field. (a) $t=106$, (b) $t=317$, (c) $t=388$, (d) $t=458$

Fig. 5. Hovmöller plot of horizontal velocity perturbation at the surface. Contoured values are for magnitudes 0.1–0.9 in steps of 0.1 and between 0.07 and 0.1 in steps of 0.01.

References