Advanced computational models for stratified flows

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The study of water waves and internal waves in a stratified fluid has a long and colourful history, partly due to the mathematical elegance of the equations (and the challenges of nonlinearity) but also due to practical needs to understand the impact of waves on technology and the environment (see e.g. the review by Helfrich & Melville 2006). Research in this area, and in fluid dynamics generally, continues to expand and intensify — advances in our understanding lead, naturally, to new questions, a situation no doubt the result of the complexity of the underlying mathematical equations. At the core of this problem is *turbulence*, a notoriously difficult feature of all high-Reynolds number fluid flows. Our lack of understanding of turbulence, and the great difficulties in reliably modelling turbulence will likely keep researchers busy for years to come.

In this work, we shall focus on fundamental aspects of internal waves, including their form, stability and (in some cases) breakdown into stratified turbulence. The latter turbulence itself is rich in phenomena and replete with structure, and indeed can be thought of as the natural environment within which internal waves form, propagate, interact and break down.

The novelty of the present work is the use of an exceptionally-high resolution numerical method called the "Contour-Advective Semi-Lagrangian" (CASL) algorithm (Dritschel & Ambaum, 1997) to reveal phenomena in unprecedented detail. This method combines accurate, efficient aspects of standard numerical methods (pseudo-spectral, semi-Lagrangian) with those of Lagrangian contour methods (Dritschel, 1988). Specifically, materially-conserved quantities like density or buoyancy here are represented as a set of contours, which move freely through the flow (they are not determined by data on a grid, but rather can be used to determine these data). Advection of the contours is nearly trivial: just the solution of a simple, first-order ODE, $\mathrm{d}\boldsymbol{x}_i/\mathrm{d}t = \boldsymbol{u}(\boldsymbol{x}_i, t)$, for each node $\boldsymbol{x}_i$ on each contour. This avoids the most serious errors arising from computing the nonlinear advection term in grid-based methods (or from interpolation in semi-Lagrangian methods). The result is a much more efficient and accurate numerical method, as comparative studies have repeatedly demonstrated (Dritschel & Scott 2009; Fontane & Dritschel 2009; Dritschel & Fontane 2010 and references).

Here, we focus on the well-studied system of a two-dimensional fluid in the x-z plane, under gravity, and making use of the simplifying Boussinesq approximation. It is useful to write the equations in terms of vorticity $\zeta = \partial w/\partial x - \partial u/\partial z$ (where u and w are the x and z velocity components) and buoyancy $b = -g(\rho - \rho_0)/\rho_0$ (where g is gravity and ρ_0 is a constant reference density). Then, the system is governed by

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = \frac{\partial b}{\partial x} \qquad ; \qquad \frac{\mathrm{D}b}{\mathrm{D}t} = 0 \qquad ; \qquad \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0 \tag{1}$$

for the inviscid case studied here. Here $\mathrm{D}F/\mathrm{D}t = \partial F/\partial t + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}F$ is the material derivative for any field F. The velocity field is recovered by inversion of $\nabla^2\psi = \zeta$ with $u = -\partial\psi/\partial z$ and $v = \partial\psi/\partial x$.

We consider a rectangular domain of dimensions L_x and L_z, with free-slip boundaries at $z = 0$ and L_z and either open (periodic) boundaries at $x = 0$ and L_x or free-slip ones. The numerical method is closely similar to that described in Dritschel & Ambaum (1997) and Dritschel & Fontane (2010), but with two major extensions. Notably, both b and ζ are represented by contours, though the latter is additionally represented by gridded fields to uptake the vorticity source term $\partial b/\partial x$, as explained in Dritschel & Fontane (2010). The first extension is the direct computation of the

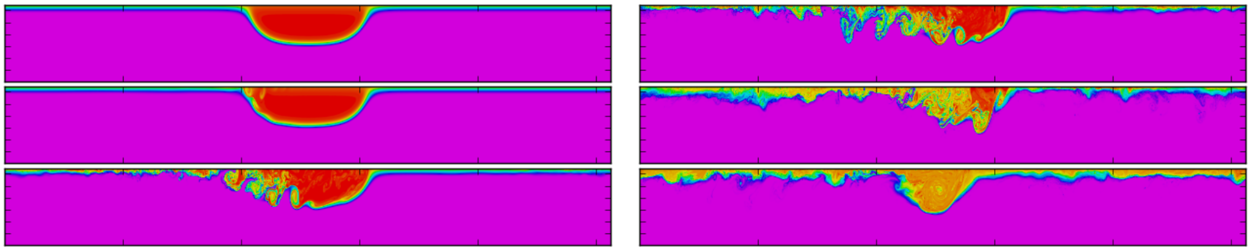

Figure 1: An ISW propagating on a stratification profile with a uniform buoyancy layer beneath a linearly stratified layer. Shown is the buoyancy field at times $t = 0$, 60, 90, 150, 210 and 300 units of time (down left column then down right one) in a periodic domain of width 2π and height $\pi/10$ (stretched in the images) and in a reference frame moving at the speed of the initial wave.

source term from the buoyancy contours, avoiding finite-difference discretisation errors. The second extension is the inclusion of boundaries, requiring a sophisticated inversion method for computing ψ from ζ and new extensions of the contour representation to allow for open contours starting and terminating in boundaries. These extensions have been thoroughly tested and enable one to maintain exceptional accuracy and efficiency.

The method has been used to study a number of fundamental problems including internal solitary waves (ISWs), the lock-exchange problem, downslope gravity currents, and (with additional features) turbidity currents. Here, we illustrate just one. In a series of papers, we have studied the form, properties and stability of large-amplitude ISWs. These (often tidally-generated) waves are commonly observed near the shelf break in coastal waters. They are important to understand not least due to their potentially damaging effects on marine structures. In King, Carr & Dritschel (2011), a new method was devised to compute the steadily-propagating form of large-amplitude ISWs in arbitrary (but stable) background stratifications. The stability of these waves was studied in Carr, King & Dritschel (2011,2012). Here, following this latter study, we illustrate a very-high resolution CASL simulation of an ISW with a trapped recirculating core. The evolution of the flow, in the full (x-periodic) domain, is shown at a few characteristic times in figure 1. The wave develops a shear instabilty after the mid point of the wave. In addition the rear stagnation point destabilises, which leads to the propagation of a disturbance forward through the core to the leading front of the wave. This destabilises the front of the wave leading to violent billowing and mixing throughout the length of the wave. The original irrotational, nearly stagnant core is here seen to acquire significant circulation as a result of turbulent mixing, and the wave appears to move towards a markedly different quasi-steady form.

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