Reconstruction of the Pressure Field in Boussinesq Models with Vorticity

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The effect of a background shear flow on the pressure beneath periodic gravity waves at the surface of a shallow fluid is investigated. It is found that strong background shear leads to non-monotonicity of the pressure field in the sense that the maximum pressure is no longer located under the wave crest, and the fluid pressure near the surface can be below atmospheric pressure. This presentation is based on work reported on in [2].

As a starting point, a long-wave equation derived by Benjamin [3] is used. To state the equation, we define the volume flux per unit span due to the wave motion by Q, the flow force per unit span and unit density by S and the energy density per unit span by R. The background current has constant vorticity $-\omega_0$. Defining the total depth from the free surface to the rigid bottom by the function $\zeta(x)$, and aiming for the description of steady long waves, the approximate governing equation for ζ found in [3] takes the form

$$\left(Q + \frac{\omega_0}{2}\zeta^2\right)^2 \zeta'^2 = -3\left(\frac{\omega_0^2}{12}\zeta^4 + g\zeta^3 - (2R - \omega_0 Q)\zeta^2 + 2S\zeta - Q^2\right). \tag{1}$$

This equation is expected to be a fair model of surface waves in the case when the waves are periodic, and the wavelength is larger than the undisturbed depth of the fluid. A more general long-wave equation allowing for arbitrary vorticity distribution was found in [3], but we present a different derivation which enables us to obtain an approximate expression for the fluid pressure below the wave.

A formulation of the steady version of the well known Korteweg de-Vries equation in terms of Q, R and S was put forward in [4], and the parametrization of periodic traveling waves of the full Euler equations in term of these quantities has has been studied extensively [7, 8, 12]. Previous work on traveling waves on background vorticity can be found in [13, 14], and evolutionary model equations similar to (1) have been derived in [5, 15]. A mathematical proof of several qualitative properties of the pressure beneath a periodic irrotational surface wave was provided in [6]. Uniqueness of the flow below a given surface profile was proved in [10]. The present study is related to the work in [1] where the dynamic pressure associated to time-dependent irrotational surface waves in the Boussinesq scaling was found asymptotically. In [9], particle paths and pressure profiles in rotational linear surface waves were studied.

The derivation of (1) is based on writing the streamfunction as an asymptotic expansion in terms of derivatives of the function $f(x)$, which represents the primitive of the horizontal velocity at the bottom of the channel. To find an approximation for the pressure for a given solution of (1), one needs to express f in terms of ζ. To this end, one may use the identity

$$Q = \frac{1}{2}\zeta^2\omega_0 + \zeta f - \zeta^3\frac{1}{6}f''$$

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which is valid to second order in the long-wave parameter $\beta = h_0^2/\lambda^2$, where h_0 is the undisturbed depth of the fluid, and λ is the wavelength.

To obtain an expression for f in terms of ζ, one has to invert the operator $1 - \frac{1}{6}\zeta^2\partial_{xx}$, leading to

$$\Big[1 - \frac{1}{6}\zeta^2\partial_{xx}\Big]^{-1}\Big(\frac{Q}{\zeta} - \frac{1}{2}\zeta\omega_0\Big) = f.$$

This identity, which is also valid to second order in β, can be used in order to find the pressure as a function of ζ. After some computations, it appears that to second order in β, the pressure formula is given by

$$P = \rho\Big\{R - gz - \frac{1}{2}\Big(\frac{Q}{\zeta^2} + \frac{\omega_0}{2}\Big)^2(z^2\zeta'^2 + \zeta^2)$$
$$+\frac{1}{2}\Big(\frac{\omega_0}{6}\zeta^3 - \frac{\omega_0}{2}z^2\zeta - \frac{2}{3}\omega_0 z^3 - \frac{Q}{3}\zeta + z^2\frac{Q}{\zeta}\Big)\Big(2Q\frac{\zeta'^2}{\zeta^3} - \zeta''\big(\tfrac{Q}{\zeta^2} + \tfrac{1}{2}\omega_0\big)\Big)\Big\}.$$

This formula can be used to find the pressure field associated to a surface profile once approximate solutions of (1) have been computed numerically. The numerical method for computing these approximations is based on writing (1) as an integral equation, and is similar to the method used in [11].

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